Harmonic Analysis— Excercises（ 2 ）

1. Show there exists at most one smooth solution of this initial value problem for the telegraph equation

$$\begin{cases} u_{tt} + du_x - u_{xx} = f & \text{in } (-\infty,\infty)\times(0,T) \\ u = g,\ u_t = h & \text{on } (-\infty,\infty)\times\{t=0\}. \end{cases}$$

Here d is a constant.

2. Show there exists at most one smooth solution of this problem for the beam equation

$$\begin{cases} u_{tt} + u_{xxxx} = 0 & \text{in } (0,1)\times(0,T) \\ u = u_x = 0 & \text{on } (\{0\}\times(0,T))\cup(\{1\}\times(0,T)) \\ u = g,\ u_t = h & \text{on } (0,1)\times\{t=0\}. \end{cases}$$

3. Let $G(u) = \lambda|u|^{p-1}u$. Show that for p any odd integer and $s > \frac{1}{2}$ any real number (note that u may be complex-valued):

(1) $G: u \mapsto G(u)$ is a continuous map on $H^s(\mathbb{R}^1)$;

(2) For any $u, v \in H^s(\mathbb{R}^1)$, there is a function $L(x,y)$ which is nondecreasing for each independent variables $x \geq 0, y \geq 0$ such that

$$||G(u) - G(v)||_{H^s(\mathbb{R}^1)} \leq L(||u||_{H^s(\mathbb{R}^1)}, ||v||_{H^s(\mathbb{R}^1)})||u - v||_{H^s(\mathbb{R}^1)}.$$

4. Show that $H^1(\mathbb{R}^n)$ is dense in $H^{\frac{1}{2}}(\mathbb{R}^n)$.

5. Assume that $u \in \mathscr{S}'(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$ and $u(x) \geq 0$. Show that if $\hat{u} \in L^\infty(\mathbb{R}^n)$, then $u \in L^1(\mathbb{R}^n)$ and

$$||u||_{L^1(\mathbb{R}^n)} = ||\hat{u}||_{L^\infty(\mathbb{R}^n)}.$$

Hint: Consider $u_k(x) = \chi(x/k)u(x)$, with $\chi \in C_c^\infty(\mathbb{R}^n),\ 0 \leq \chi \leq 1, \chi(0) = 1$.