1. 证明: $2\int_a^b |\sin x|dx \cdot \int_a^b |\cos x| dx \le (b-a)^2$.

证明. $2\int_a^b |\sin x| dx \cdot \int_a^b |\cos x| dx$

$$\le \left(\int_a^b |\sin x| dx\right)^2 + \left(\int_a^b |\cos x| dx\right)^2$$

$$\le (b-a)\int_a^b \sin^2 x dx + (b-a)\int_a^b \cos^2 x dx$$

$$= (b-a)^2.$$

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2. 设 $f\in C'[a,b]$ 且 $f(a)=0$, 则 $\int_a^b f(x)^2dx \le \frac{(b-a)^2}{2}\int_0^b f'(x)^2dx$.

证明: $f(x)=\int_a^x f'(s)ds$

$$\Rightarrow f(x)^2 = \left(\int_a^x f'(s)ds\right)^2 \le (x-a)\int_a^x f'(s)^2ds$$

$$\therefore \int_a^b f(x)^2dx \le \int_a^b (x-a)\int_a^x f'(s)^2ds\, dx$$

$$\le \int_a^b f'(s)^2ds \cdot \frac{1}{2}(b-a)^2.$$

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3. 设 $f\in C[a,b]$, 且 $\forall x\in[a,b]$, $f(x)>0$, 则 $\int_a^b f(x)dx \int_a^b \frac{dx}{f(x)} \ge (b-a)^2$.

证明: 记 $D=[a,b)\times[a,b)$, 则

$$2\int_a^b f(x)dx\int_a^b \frac{dx}{f(x)} = \int_a^b f(x)dx\int_a^b \frac{dy}{f(y)} + \int_a^b f(y)dy\int_a^b \frac{dx}{f(x)}$$

$$= \iint_D \frac{f(x)}{f(y)}dxdy + \iint_D \frac{f(y)}{f(x)}dxdy \ge 2\iint_D dxdy = 2(b-a)^2.$$

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4. 设 $f\in C[0,1]$ 且 $f(x)>0$, $x\in[0,1]$. 则 $1\le\int_0^1 f(x)dx\int_0^1\frac{dx}{f(x)} \le \frac{(m+M)^2}{4mM}$.

其中 $m=\inf\limits_{x\in[0,1]} f(x)$. $M=\sup\limits_{x\in[0,1]} f(x)$.

证明: 左边已证.

右边.

$$0 \le \int_0^1 (f(x)-m)\left(\frac{1}{f(x)} - \frac{1}{M}\right) dx = 1 + \frac{m}{M} - m\int_0^1 \frac{1}{f(x)} dx - \frac{1}{M}\int_0^1 f(x) dx$$

$$\Rightarrow \quad 1 + \frac{m}{M} \geq m\int_0^1 \frac{1}{f(x)} dx + \frac{1}{M}\int_0^1 f(x)dx \geq 2\sqrt{\frac{m}{M}\int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)}dx}$$

$$\Rightarrow \quad \int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)}dx \le \left(\frac{m+M}{2M}\right)^2 \frac{M}{m} = \frac{(M+m)^2}{4Mm}.$$

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5, 证明: $\left(\iint_D e^{x^2+y^2} dxdy\right)^{\frac{1}{2}} \le \sqrt{\pi} \int_0^1 e^{\frac{\pi t^2}{4}} dt.$

其中 $D = \{ (x,y) \in \mathbb{R}^2 : |x^2+y^2| \le 1 \}$.

<u>证明</u>: 令 $Q = \{ (x,y) \in \mathbb{R}^2 : |x| \le \frac{\sqrt{\pi}}{2}, |y| \le \frac{\sqrt{\pi}}{2} \}$. 注意. 面积

$S_{Q-D} = S_{D-Q}$. 从而

$$\iint_D e^{x^2+y^2} dxdy = \iint_{D\cap Q} e^{x^2+y^2} dxdy + \iint_{D-Q} e^{x^2+y^2} dxdy$$

$$\le \iint_{D\cap Q} e^{x^2+y^2} dxdy + \iint_{D-Q} e^1 dxdy$$

$$\le \iint_{D\cap Q} e^{x^2+y^2} dxdy + \iint_{Q-D} e^{x^2+y^2} dxdy$$

$$= \iint_Q e^{x^2+y^2} dxdy = 4\int_0^{\frac{\sqrt{\pi}}{2}} \int_0^{\frac{\sqrt{\pi}}{2}} e^{x^2+y^2} dxdy$$

$$= 4\left(\int_0^{\frac{\sqrt{\pi}}{2}} e^{x^2} dx\right)^2 \overset{x = \frac{\sqrt{\pi}}{2}t}{=\!=} \sqrt{\pi}\int_0^1 e^{\frac{\pi}{4}t^2} dt$$

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