6. 设 $\{a_n\}$ 为正的单调递减的数列，证明：$\sum_{n=1}^{\infty} a_n \sin nx$ 在任何闭区间上都一致收敛的充要条件为 $\lim_{n\to\infty} n a_n = 0$.

证明：$\Longrightarrow$ $\sum_{n=1}^{\infty} a_n \sin nx$ 一致收敛 $\Longleftrightarrow$ $\forall \varepsilon>0$, $\exists N_0 \in \mathbb{N}$, $\forall$ $n, m > N_0$ s.t.

$$\left|\sum_{k=n+1}^{n+m} a_k \sin kx\right| < \varepsilon. \qquad \forall x \in \mathbb{R}.$$

(i) 取 $m=n$, $x=\frac{\pi}{4n}$, 则

$$(n+1)x, \cdots, 2nx \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

从而

$$\sin kx \geq \sin\frac{\pi}{4} = \sqrt{2}/2, \quad k=n+1, \cdots, 2n.$$

于是

$$\varepsilon > \left|\sum_{k=n+1}^{2n} a_k \sin kx\right| = \sum_{k=n+1}^{2n} a_k \sin kx \geq \frac{\sqrt{2}}{2}\sum_{k=n+1}^{2n} a_k$$

$$\geq \frac{\sqrt{2}}{2}\sum_{k=n+1}^{2n} a_{2n} = \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\cdot 2n\cdot a_{2n} \Rightarrow 2n a_{2n} \to 0 \text{ as } n\to\infty.$$

(ii) 取 $m=n+1$, $x=\frac{\pi}{4(n+1)}$, 则

$$(n+1)x, \cdots, (2n+1)x \in \left[\frac{\pi}{4}, \frac{3}{4}\pi\right].$$

事实上，$(2n+1)\frac{\pi}{4(n+1)} < \frac{3}{4}\pi \Longleftrightarrow 2n+1 < 3(n+1) \Longleftrightarrow n+2>0.$

从而

$$\varepsilon > \frac{\sqrt{2}}{2}\sum_{k=n+1}^{2n+1} a_k \geq \frac{\sqrt{2}}{2}\sum_{k=n+1}^{2n+1} a_{2n+1} = \frac{\sqrt{2}}{2}\cdot(2n+1)\, a_{2n+1}\cdot\frac{n+1}{2n+1}$$

$$\therefore (2n+1)\, a_{2n+1} < \frac{2n+1}{n+1}\cdot\frac{2}{\sqrt{2}}\varepsilon < \frac{4}{\sqrt{2}}\varepsilon \Rightarrow (2n+1)\, a_{2n+1} \to 0 \text{ as } n\to\infty.$$

从而 $\lim_{n\to\infty} n a_n = 0.$