5. 设 $\sum a_n$ 为正项级数，证明:

(1) 若 $\exists\,\varepsilon>0$, $N\in\mathbb{N}$, s.t. $n>N$ 时 $\frac{\ln\frac{1}{a_n}}{\ln n}\geq 1+\varepsilon$，则 $\sum a_n$ 收敛;

(2) 若 $\exists\,N\in\mathbb{N}$, s.t. $\forall n>N$, 有 $\frac{\ln\frac{1}{a_n}}{\ln n}<1$，则 $\sum a_n$ 发散.

(3) 利用上述结论判断下列级数的敛散性:

① $\sum\frac{1}{3^{\ln n}}$ ② $\sum n^{\ln x}$ $(x>0)$.

解: (1) $\ln\frac{1}{a_n}/\ln n\geq 1+\varepsilon \Rightarrow \frac{1}{a_n}\geq n^{1+\varepsilon} \Rightarrow a_n\leq\frac{1}{n^{1+\varepsilon}}$. $\sum\frac{1}{n^{1+\varepsilon}}$ 收敛;

(2) $\ln\frac{1}{a_n}/\ln n<1 \Rightarrow a_n>\frac{1}{n}$, $\sum\frac{1}{n}$ 发散;

(3) ① $a_n=\frac{1}{3^{\ln n}}$ $\therefore \ln\frac{1}{a_n}=\ln(3^{\ln n})=\ln n\cdot\ln 3$

$\frac{\ln\frac{1}{a_n}}{\ln n}=\ln 3>1+\varepsilon$ $\varepsilon>0$. ①收敛.

② $a_n=n^{\ln x}$ $\ln\frac{1}{a_n}=\ln(n^{-\ln x})=-\ln x\ln n$

$\frac{\ln\frac{1}{a_n}}{\ln n}=-\ln x$

当 $0<x<\frac{1}{e}$ 时 $-\ln x>1$ 收敛;

$1>x\geq\frac{1}{e}$ 时 α $-\ln x<1$. 发散; $\Rightarrow$ $x\geq\frac{1}{e}$ 发散.

#.

6. f 在区间 I 上一致连续, $\varphi_n\rightrightarrows\varphi$, $x\in\Omega$ 且 $\varphi(\Omega)$, $\varphi_n(\Omega)\subset I$. 则 $f(\varphi_n(x))\underset{n\to\infty}{\rightrightarrows} f(\varphi(x))$, $x\in\Omega$.

证明: $\forall\varepsilon>0$, $\exists\,\delta>0$, 当 $|x_1-x_2|<\delta$ 时, $(x_{1,2}\in I)$, $|f(x_1)-f(x_2)|<\varepsilon$.

对此 δ, 由 $\varphi_n\rightrightarrows\varphi$, $\exists\,N$, 当 $n\geq N$ 时, $\forall x\in\Omega$,

$|\varphi_n(x)-\varphi_n(x)|<\delta$.

$\Rightarrow$ $|f(\varphi_n(x))-f(\varphi_n(x))|<\varepsilon$.

得证.

#.