习题三. P.119. (9) 设映射 $f: R^2 \to R^2$ 定义为

$$y_1 = x_1 e^{x_2} + x_2, \quad y_2 = x_1 e^{x_2} - x_2.$$

证明f为光滑同胚，并且求切映射 f_* 和余切映射 f^* 在自然基底下的矩阵.

解. (1) f^{-1}:

$$\begin{cases} x_2 = \dfrac{y_1 - y_2}{2} \\ x_1 = \dfrac{y_1 + y_2}{2} e^{-\frac{y_1 - y_2}{2}} \end{cases}$$

显然 f, f^{-1} 均为 $R^2 \to R^2$ 的 C^∞ 映射. 从而 f 为光滑同胚.

(2)

$$\begin{cases} f_*\left(\dfrac{\partial}{\partial x^1}\right) = \dfrac{\partial y_1}{\partial x^1}\dfrac{\partial}{\partial y^1} + \dfrac{\partial y_2}{\partial x^1}\dfrac{\partial}{\partial y^2} = e^{x_2}\dfrac{\partial}{\partial y^1} + e^{x_2}\dfrac{\partial}{\partial y^2} \\ f_*\left(\dfrac{\partial}{\partial x^2}\right) = \dfrac{\partial y_1}{\partial x^2}\dfrac{\partial}{\partial y^1} + \dfrac{\partial y_2}{\partial x^2}\dfrac{\partial}{\partial y^2} = (1 + x_1 e^{x_2})\dfrac{\partial}{\partial y^1} + (x_1 e^{x_2} - 1)\dfrac{\partial}{\partial y^2} \end{cases}$$

$\Rightarrow$ f_* 在自然基底下矩阵为 $\begin{pmatrix} e^{x_2} & e^{x_2} \\ 1 + x_1 e^{x_2} & -1 + x_1 e^{x_2} \end{pmatrix}$

(3)

$$f^*(dy_1) = \frac{\partial y_1}{\partial x_1} dx_1 + \frac{\partial y_1}{\partial x_2} dx_2 = e^{x_2} dx_1 + (x_1 e^{x_2} + 1) dx_2$$

$$f^*(dy_2) = \frac{\partial y_2}{\partial x_1} dx_1 + \frac{\partial y_2}{d x^2} dx^2 = e^{x_2} dx_1 + (x_1 e^{x_2} - 1) dx_2$$

$\therefore$ f^* 相应矩阵为 $\begin{pmatrix} e^{x_2} & x_1 e^{x_2} + 1 \\ e^{x_2} & x_1 e^{x_2} - 1 \end{pmatrix}$.

利用公式 $f^*(d\varphi) = d(\varphi \circ f)$, 也有

$$f^*(dy^1) = d(y^1 \circ f) = d(x_1 e^{x_2} + x_2) = e^{x_2} dx_1 + (x_1 e^{x_2} + 1) dx_2$$

$$f^*(dy^2) = d(y^2 \circ f) = d(x_1 e^{x_2} - x_2) = e^{x_2} dx_1 + (x_1 e^{x_2} - 1) dx_2.$$

$\left\{\frac{\partial}{\partial x^i}\right\} \to \left\{\frac{\partial}{\partial y^\alpha}\right\}$ $\qquad$ $\{dy^\alpha\} \to \{dx^i\}$. □

特别注意. $f_*: T_p M \to T_{f(p)} N$; $\quad f^*: T^*_{f(p)} N \to T^*_p M$.

如果计算余切映射时写 $f^*(dx_1) = \cdots$ 就完全错了。