(二).

33. (1) ①

$$(U_1,\varphi_1):\quad \underset{x^{n+1}\neq 1}{(x^1,\cdots,x^{n+1})} \xrightarrow{\varphi_1} \left(u_1=\frac{x^1}{1-x^{n+1}},\ \cdots,\ u_n=\frac{x^n}{1-x^{n+1}}\right) \quad (1)$$

$$(U_2,\varphi_2):\quad \underset{x^{n+1}\neq -1}{(x^1,\cdots,x^{n+1})} \xrightarrow{\varphi_2} \left(y_1=\frac{x^1}{1+x^{n+1}},\ \cdots,\ y_n=\frac{x^n}{1+x^{n+1}}\right) \quad (2)$$

$$(1)\Rightarrow x^i = u^i(1-x^{n+1}) \Rightarrow \sum_{i=1}^{n}(x^i)^2 = u^2(1-x^{n+1})^2 \Rightarrow 1-(x^{n+1})^2 = u^2(1-x^{n+1})^2$$

$$\Rightarrow 1+x^{n+1} = u^2(1-x^{n+1}) \Rightarrow x^{n+1} = \frac{u^2-1}{u^2+1}$$

$$\therefore \begin{cases} x^i = u^i\left(1-\frac{u^2-1}{u^2+1}\right) = u^i\left(\frac{u^2+1-u^2+1}{u^2+1}\right) = \frac{2u^i}{u^2+1} \quad i=1\cdots n \\ x^{n+1} = \frac{u^2-1}{u^2+1} \qquad 1+x^{n+1} = \frac{2u^2}{u^2+1} \end{cases}$$

$$\therefore y^i = \frac{x^i}{2u^2/(u^2+1)} = \frac{u^i}{u^2}$$

$$\therefore \frac{\partial y^i}{\partial u^j} = \frac{1}{u^2}\delta_{ij} + u^i\left(-\frac{1}{u^4}\right)(2u_j) = \frac{1}{u^4}\left(u^2\delta_{ij} - 2u_iu_j\right)$$

$$\left(\frac{\partial y}{\partial u}\right) = \frac{1}{u^4}\left(u^2 I - 2\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}(u_1\cdots u_n)\right)$$

设 $A\begin{pmatrix}|u|\\ 0\\ \vdots\\ 0\end{pmatrix} = \begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$，$A$为正交阵. 则 $2\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}(u_1\cdots u_n) = 2A\begin{pmatrix}|u|\\ 0\\ \vdots\\ 0\end{pmatrix}(|u|\cdots 0)A^T$

$$= 2A\begin{pmatrix}|u|^2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & & & \vdots\\ 0 & \cdots & 0 & 0\end{pmatrix}A^T$$

$$\therefore \left|\left(\frac{\partial y}{\partial u}\right)\right| = \frac{1}{u^4}\begin{vmatrix}-u^2 & 0 & \cdots & 0\\ \vdots & u^2 & & \\ 0 & & \ddots & u^2\end{vmatrix} = -u^{2n}/u^4 < 0.$$

从而 (U_1,φ_1) 与 (U_2,φ_2) 定向相反. 令 $\overline{\varphi}_2(x^1,\cdots x^{n+1}) = \left(\frac{-x^1}{1+x^{n+1}},\cdots \frac{x^n}{1+x^{n+1}}\right)$，

则 $\{(U_1,\varphi_1),(U_2,\overline{\varphi}_2)\}$ 为 S^n 的一个定向.

②

33.<27>: $p:(x^1,\cdots,x^{n+1})\longrightarrow(-x^1,\cdots,-x^{n+1})$ 反转定向(n为偶数)，保持定向(n为奇数)

以(U_1,φ_1)为例. 挖去$(0,\cdots,0,1)$及$(0,\cdots,0,-1)$点，则$-U_1=U_1$，

而 φ_1变为：$\tilde{\varphi}_1:(x^1,\cdots,x^{n+1})\longrightarrow\left(-\dfrac{x^1}{1+x^{n+1}},\cdots,-\dfrac{x^n}{1+x^{n+1}}\right)=(-1)^n\varphi_2$.

要证 $U_1\cap U_2$上，$(U_1,\tilde{\varphi}_1)$与(U_1,φ_1)定向相同或相反.

由(1)计算，相应Jacobi为 $(-1)^n\cdot(-1)\,u^{2n}/u^4=(-1)^{n+1}u^{2n-4}$

$$\begin{cases}>0 & n\text{为奇}\\ <0 & n\text{为偶}\end{cases}$$

类似对以$(-1,0\ldots0)$，$(1,0\ldots0)$等为极点的坐标计算，有相应结果.

从而n为奇数时p保持定向；n为偶数时p反转定向.

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