P.119. 习题二. (10) 设映射 $f: R^2 \longrightarrow R^3$ 和 $g: R^3 \rightarrow \mathbb{R}^2$ 分别定义为

$$f(X) = (e^{2x_1+x_2},\ 3x_2 - \cos x_1,\ x_1^2 + x_2 + 2),$$

$$g(y) = (3y_1 + 2y_2 + y_3^2,\ y_1^2 - y_3 + 1).$$

(1) 设 $F = g\circ f$，求 $F_{*0}$； (2) 设 $G = f\circ g$，求 $G_{*0}$.

解: (1) $F_{*(0)} = (g\circ f)_{*0} = g_{*(1,-1,2)} \circ f_{*0}$

而 $f_{*0}\left(\frac{\partial}{\partial x^1}\right) = e^{2x_1+x_2}\cdot 2\Big|_0 \frac{\partial}{\partial y^1} + \sin x_1\Big|_0 \frac{\partial}{\partial y^2} + 2x_1\Big|_0 \frac{\partial}{\partial y^3}$

$= 2\frac{\partial}{\partial y^1}$；

$f_{*0}\left(\frac{\partial}{\partial x^2}\right) = \frac{\partial}{\partial y^1} + 3\frac{\partial}{\partial y^2} + \frac{\partial}{\partial y^3}$

从而 $f_{*0}$ 矩阵为 $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix}$

$g_{*(1,-1,2)}\left(\frac{\partial}{\partial y^1}\right) = 3\frac{\partial}{\partial z^1} + 2\frac{\partial}{\partial z^2}$

$g_{*(1,-1,2)}\left(\frac{\partial}{\partial y^2}\right) = 2\frac{\partial}{\partial z^1}$ $g_{*(1,-1,2)}$ 矩阵为 $\begin{pmatrix} 3 & 2 \\ 2 & 0 \\ 4 & -1 \end{pmatrix}$

$g_{*(1,-1,2)}\left(\frac{\partial}{\partial y^3}\right) = 4\frac{\partial}{\partial z^1} + (-1)\frac{\partial}{\partial z^2}$

从而 $F_{*0}$ 矩阵为 $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 0 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 13 & 1 \end{pmatrix}$.

(2) 可以类似计算.

(3) 作为练习，计算 $G^* = (f\circ g)^*_0$.

由于 $(f\circ g)^* = g^* \circ f^*$. 而

$$\begin{pmatrix} f^*(dy^1) \\ f^*(dy^2) \\ f^*(dy^3) \end{pmatrix} = \begin{pmatrix} 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ +\sin x_1 & 3 \\ 2x_1 & 1 \end{pmatrix}\begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}$$

$$\begin{pmatrix} g^*(dz_1) \\ g^*(dz_2) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{pmatrix} \begin{pmatrix} dy_1 \\ dy_2 \\ dy_3 \end{pmatrix}$$

从而由

$$a^* = g^* \circ f^*,$$ ~~[illegible]~~

$$\begin{pmatrix} a^*(dx^1) \\ a^*(dx^2) \\ a^*(dx^3) \end{pmatrix} = g^* \circ \begin{pmatrix} 2e^{2z_1+z_2} & e^{2z_1+z_2} \\ \sin z_1 & 3 \\ 2z_1 & 1 \end{pmatrix} \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2z_1+z_2} & e^{2z_1+z_2} \\ \sin z_1 & 3 \\ 2z_1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{pmatrix} \begin{pmatrix} dy_1 \\ dy_2 \\ dy_3 \end{pmatrix}$$

$$= \begin{pmatrix} 6e^{2z_1+z_2} + 2y_1 e^{2z_1+z_2} & 4e^{2z_1+z_2} & 2y_3 e^{2z_1+z_2} - e^{2z_1+z_2} \\ 3\sin z_1 + 6y_1 & 2\sin z_1 & 2y_3 \sin z_1 - 3 \\ 6z_1 + 2y_1 & 4z_1 & 4y_3 z_1 - 1 \end{pmatrix} \begin{pmatrix} dy_1 \\ dy_2 \\ dy_3 \end{pmatrix}$$

这里: $g: y \to z$, $f: z \to x$. $y = (y_1, y_2, y_3)$, $z = (z_1, z_2)$, $x = (x_1, x_2, x_3)$.

$\square$.