THE BATALIN-VILKOVISKY STRUCTURE OVER THE HOCHSCHILD COHOMOLOGY RING OF A GROUP ALGEBRA

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Abstract. We realize explicitly the well-known additive decomposition of the Hochschild cohomology ring of a group algebra. As a result, we describe the cup product, the Batalin-Vilkovisky operator and the Lie bracket in the Hochschild cohomology ring of a group algebra.

1. Introduction

Let \( k \) be a field and \( G \) a finite group. Then the Hochschild cohomology ring of the group algebra \( kG \) admits an additive decomposition:

\[
HH^*(kG) \cong \bigoplus_{x \in X} H^*(C_G(x), k)
\]

where \( X \) is a set of representatives of conjugacy classes of elements of \( G \) and \( C_G(x) \) is the centralizer of \( x \in G \). The proof of this isomorphism can be found in [1] or [9]. The usual proof is abstract rather than giving an explicit isomorphism. For example, one of the key steps is to use the so-called Eckmann-Shapiro Lemma, one need to construct some comparison maps between two projective resolutions in order to write it down explicitly, and this is usually difficult. In [9], Siegel and Witherspoon used techniques and notations from group representation theory to interpret the above additive decomposition explicitly. For our purpose, we need to give an explicit isomorphism in the elements level.

A priori, the additive decomposition gives an isomorphism of vector spaces. The left handed side has a graded commutative algebra structure given by the cup product, a graded Lie algebra structure given by the Gerstenhaber Lie bracket ([6]), and a Batalin-Vilkovisky (BV) algebra structure given by the \( \Delta \) operator ([10]). It would be interesting to describe these structures in terms of pieces from the right handed side.

For graded algebra structure, it was done by Holm for abelian groups using computations ([7]), then Cibils and Solotar gave a conceptual proof in ([3]). The general case was dealt with by Siegel and Witherspoon ([9]), they described the cup product formula by notations from group representation theory. Our goal in the present paper is to represent the cup product, the Lie bracket and the BV operator in the Hochschild cohomology ring in terms of the additive decomposition. This is based on the explicit construction of an isomorphism in the additive decomposition (although there is no canonical choice for such an isomorphism).

The main obstruction in realizing an isomorphism in the additive decomposition comes from the fact that, it is usually difficult to construct the comparison map between two projective resolutions of modules. There is a surprising way to simplify such construction, namely, one can reduce it to construct a set-like self-homotopy over one projective resolution, which is often much easier. This method was already used in a recent paper by the second author jointly with Le ([8]). For convenience, we shall give here a brief introduction to this idea.

This article is organized as follows. In the second section, we review the definitions of Hochschild cohomology and cup product, using normalized bar resolutions. We use the normalized bar resolution since it is easy to describe and can greatly simplify the computations. We also recall the graded

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Lie algebra structure and the BV algebra structure (in case that $A$ is a symmetric algebra) over the Hochschild cohomology ring $HH^*(A)$ of an algebra $A$. In the group algebra case, we will state the relationship of the Hochschild cohomology and the usual group cohomology.

In Section 3, we give a way to realize explicitly the additive decomposition of the Hochschild cohomology of a group algebra. The main line of our method follows from [9]. In Section 4, we shall use some idea from [3] to give another way to realize the additive decomposition.

We give the cup product formula in Section 5.

We deal with the $\Delta$ operator and the graded Lie bracket in the next two sections.

In the final section, we use our formulas to compute several concrete examples.

2. How to construct comparison morphisms

**Definition 2.1.** Let $A$ be an algebra over a field $k$. Let

$$C^\bullet : \cdots \rightarrow C^{n+1} \xrightarrow{d_{n+1}} C^n \xrightarrow{d_n} C^{n-1} \rightarrow \cdots$$

be a chain complex of $A$-modules. If there are maps (just as maps between sets) $s_n : C_n \rightarrow C_{n+1}$ such that $s_{n-1}d_n + d_{n+1}s_n = id_{C^n}$ for all $n$, then the maps $\{s_n\}$ are called a set-like self-homotopy over the complex $C^\bullet$.

**Remark 2.2.** There is a set-like self-homotopy over a complex $C^\bullet$ of $A$-modules if and only if $C^\bullet$ is an exact complex, that is, $C^\bullet$ is a zero object in the derived category $D(ModA)$. Compare this with the usual self-homotopy, which is equivalent to saying that $C^\bullet$ is a zero object in the homotopy category $K(ModA)$.

We will show how to use a set-like self-homotopy to construct a comparison map. Let $M$ and $N$ be two $A$-modules, and let $f : M \rightarrow N$ be an $A$-module homomorphism. Suppose that $P^\bullet = (P_i, \partial_i)$ is a free resolution of $M$, and that $Q^\bullet = (Q_i, d_i)$ is a projective resolution of $N$. Suppose further that there is a set-like self-homotopy $s = \{s_n\}$:

$$\cdots \rightarrow Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} N \rightarrow 0$$

For each $i \geq 0$, choose a basis $X_i$ for the free $A$-module $P_i$ (the $i$-th term of $P^\bullet$). We define inductively the maps $f_i : X_i \rightarrow Q_i$ as follows: for $x \in X_0$, $f_0(x) = s_{-1}f_0(x)$; for $i > 1$ and $x \in X_i$, $f_i(x) = s_{i-1}f_{i-1}(x)$. Extending $A$-linearly the maps $f_i$ we get $A$-homomorphisms $f_i : P_i \rightarrow Q_i$. It is easy to verify that $\{f_i\}$ gives a chain map between the complexes $P^\bullet$ and $Q^\bullet$.

We illustrate the above procedure in the following diagram:

$$\cdots \rightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \rightarrow 0$$

$$\cdots \rightarrow Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} N \rightarrow 0$$

$$\cdots \rightarrow Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} N \rightarrow 0.$$
two chain maps inducing identity maps $1_M : M \to M$, then $F\varphi : FD^* \to FC^*$ and $F\psi : FC^* \to FD^*$ are inverse homotopy equivalences.

3. REMAINDER ON HOCHSCHILD COHOMOLOGY

In this section, we recall the definitions of various structures over Hochschild cohomology. For the cup product and the Lie bracket in the Hochschild cohomology ring, we refer to Gerstenhaber’s original paper [6]; for the Batalin-Vilkovisky algebra structure, we refer to [10].

Let $k$ be a field and $A$ an associative $k$-algebra with identity $1_A$. Denote by $\overline{A}$ the quotient space $A/(k \cdot 1_A)$. We shall write $\otimes$ for $\otimes_k$ and $A \otimes_k^n$ for the $n$-fold tensor product $A \otimes \cdots \otimes A$. The normalized bar resolution $(\text{Bar}_n(A), d_n)$ of $A$ is a free resolution of $A$ as $A$-$A$-bimodules, where

$$
\text{Bar}_{-1}(A) = A, \quad \text{for } n \geq 0, \quad \text{Bar}_n(A) = A \otimes \overline{A} \otimes \cdots \otimes A,
$$

$$
d_0 : \text{Bar}_0(A) = A \otimes A \to A, \quad a_0 \otimes a_1 \mapsto a_0 a_1 \text{(multiplication map)}, \quad \text{and for } n \geq 1,
$$

$$
d_n : \text{Bar}_n(A) \to \text{Bar}_{n-1}(A), \quad a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \mapsto \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.
$$

The exactness of the normalized bar resolution is an easy consequence of the following fact: there is a set-like self-homotopy $s_n : \text{Bar}_n(A) \to \text{Bar}_{n+1}(A)$ over $\text{Bar}_n(A)$ given by

$$
s_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}.
$$

Notice that here each $s_n$ is just a right $A$-module homomorphism.

Let $\gamma M_A$ be an $A$-$A$-bimodule. Remember that any $A$-$A$-bimodule can be identified with a left module over the enveloping algebra $A^e = A \otimes A^{op}$. We have the Hochschild cohomology complex $(C^*(A, M), \delta^*)$:

$$
C^n(A, M) = \text{Hom}_A(\text{Bar}_n(A), M) \cong \text{Hom}_k(\overline{A} \otimes^n, M), \quad \text{for } n \geq 0,
$$

$$
\delta^n : C^n(A, M) \to C^{n-1}(A, M), \quad f \mapsto \delta^n(f), \quad \text{where } \delta^n(f) \text{ sends } a_1 \otimes \cdots \otimes a_{n+1} \text{ to }
$$

$$
a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^n (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n)a_{n+1}.
$$

For $n \geq 0$, the degree-$n$ Hochschild cohomology group of the algebra $A$ with coefficients in $M$ is defined to be

$$
HH^n(A, M) = H^n(C^*(A, M)) \cong \text{Ext}^n_{A^e}(A, M).
$$

If in particular, $A = kG$ the group algebra of a finite group $G$, then the Hochschild cohomology complex $(C^*(A, M), \delta^*)$ has the following form:

$$
C^n(kG, M) = \text{Hom}_k(kG \otimes^n, M) \cong \text{Map}(\overline{G}^n, M), \quad \text{for } n \geq 0,
$$

where $\overline{G} = G - \{1\}$ and $\text{Map}(\overline{G}^n, M)$ denotes all the maps between the sets $\overline{G}^n$ and $M$, and the differential is given by

$$
\delta^n : C^n(kG, M) \to C^{n+1}(kG, M), \quad f \mapsto \delta^n(f), \quad \text{where } \delta^n(f) \text{ sends } (g_1, \ldots, g_{n+1}) \in \overline{G}^{n+1} \text{ to }
$$

$$
g_1 f(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n+1} (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1} f(g_1, \cdots, g_n) g_{n+1}.
$$

When $M = A$ with the obvious $A$-$A$-bimodule structure, we write $C^n(A)$ (resp. $HH^n(A)$) for $C^n(A, A)$ (resp. $HH^n(A, A)$). Let $f \in C^n(A)$, $g \in C^m(A)$. Then the cup product $f \cup g \in C^{n+m}(A)$ is defined as follows:

$$
f \cup g : \overline{A}^{(n+m)} \to A, \quad a_1 \otimes \cdots \otimes a_{n+m} \mapsto f(a_1 \otimes \cdots \otimes a_n)g(a_{n+1} \otimes \cdots \otimes a_{n+m}).
$$

This cup product is associative and induces a well-defined product over

$$
HH^*(A) = \bigoplus_{n \geq 0} HH^n(A) = \bigoplus_{n \geq 0} \text{Ext}^n_{A^e}(A, A),
$$

which is called the Hochschild cohomology ring of $A$. Moreover, $HH^*(A)$ is graded commutative, that is, $\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha$ for $\alpha \in HH^n(A)$ and $\beta \in HH^m(A)$. 
The Lie bracket is defined as follows. Let \( f \in C^n(A,M) \), \( g \in C^m(A) \). If \( n, m \geq 1 \), then for \( 1 \leq i \leq n \), the so-called brace operation \( f \circ_i g \in C^{n+m-1}(A,M) \) is defined by

\[
f \circ_i g(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes \cdots \otimes a_{i+m-1} \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});
\]

if \( n \geq 1 \) and \( m = 0 \), then \( g \in A \) and for \( 1 \leq i \leq n \), set

\[
f \circ_i g(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i \otimes \cdots \otimes a_{n-1});
\]

for any other case, set \( f \circ_i g \) to be zero. Define

\[
f \circ g = \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} f \circ_i g \in C^{n+m-1}(A,M)
\]

and for \( f \in C^n(A) \), \( g \in C^m(A) \), define

\[
[f,g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f \in C^{n+m-1}(A).
\]

The above \([\cdot,\cdot]\) induces a well-defined (graded) Lie bracket in Hochschild cohomology

\[
H^n(A) \times H^m(A) \rightarrow H^{n+m-1}(A)
\]

such that \((H^*(A),[\cdot,\cdot])\) is a Gerstenhaber algebra, that is, the following three conditions hold:

- \((H^*(A), \cup, [\cdot,\cdot])\) is a graded Lie algebra;
- Possion rule: \([f \cup g, h] = [f, h] \cup g + (-1)^{|f||h|-1} f \cup [g, h]\), where \(|\cdot|\) denotes the degree.

We now assume that \(A\) is a symmetric \(k\)-algebra, that is, \(A\) is isomorphic to its dual \(D(A) = \text{Hom}_k(A,k)\) as \(A^*\)-modules, or equivalently, if there exists a symmetric associative non-degenerate bilinear form \(\langle \cdot,\cdot \rangle : A \times A \rightarrow k\). This bilinear form induces a duality between the Hochschild cohomology and the Hochschild homology. In fact, for any \(n \geq 0\) there is an isomorphism between \(H^n(A)\) and \(H_n(A)\) induced by the following canonical isomorphisms

\[
\text{Hom}_k(A \otimes A^*, \text{Bar}_n(A), k) \cong \text{Hom}_{A^*}(\text{Bar}_n(A), D(A)) \cong \text{Hom}_{A^*}(\text{Bar}_n(A), A).
\]

Via this duality, we have, for \(n \geq 1\), an operator \(\triangle : H^n(A) \rightarrow H^{n-1}(A)\) which corresponds to the Connes’ \(B\)-operator on the Hochschild homology. More precisely, for any \(f \in C^n(A)\), \(\triangle(f) \in C^{n-1}(A)\) is given by the equation

\[
(\triangle(f)(a_1 \otimes \cdots a_{n-1}), a_n) = \sum_{i=1}^{n} (-1)^{(n-1)(i-1)}(f(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_n \otimes a_i \otimes \cdots \otimes a_{n-1}), 1).
\]

Then the Gerstenhaber algebra \((H^*(A), \cup, [\cdot,\cdot])\) together with the operator \(\triangle\) is a Batalin-Vilkovisky algebra (BV-algebra), that is, \((H^*(A), \triangle)\) is a complex and

\[
\left[\alpha, \beta\right] = (-1)^{(|\alpha|-1)|\beta|} \triangle(\alpha \cup \beta) - \triangle(\alpha) \cup \beta - (-1)^{|\alpha|} \alpha \cup \triangle(\beta)
\]

for all homogeneous elements \(\alpha, \beta \in H^*(A)\).

4. Remainder on group cohomology

Let \(G\) be a finite group and \(U\) a left \(kG\)-module. The group cohomology of \(G\) with coefficient in \(U\) is defined to be \(H^n(G,U) = \text{Ext}_{kG}^n(k,U)\). The complex \(\text{Bar}_*(kG) \otimes_{kG} k\) is the standard resolution of the trivial module \(k\). In fact, as the set-like homotopy \(s_n\) over \(\text{Bar}_*(kG)\) are right module homomorphisms, \(\text{Bar}_*(kG) \otimes_{kG} k\) is exact and thus a projective resolution of \(kG \otimes_{kG} k \cong k\). We write the complex \(C^*(G,U) = \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, U)\). Therefore, for \(n \geq 0\),

\[
C^n(G,U) \cong \text{Hom}_{kG}((kG \otimes \overline{G}^n \otimes kG) \otimes_{kG} k, U) \cong \text{Map}_{G}(\overline{G}^n, U),
\]

where \(\overline{G} = G - \{1\}\) and \(\text{Map}_{G}(\overline{G}^n, U)\) denotes all the maps between the sets \(\overline{G}^n\) and \(U\), and the differential is given by

\[
d_0(x)(g) = gx - x \quad (\text{for } x \in U \text{ and } g \in \overline{G})
\]
and (for $\varphi: G^x \rightarrow U$ and $g_1, \ldots, g_{n+1} \in G$)

$$d_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1 \varphi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_{gi}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \ldots, g_n).$$

Of particular interest to us are the following two cases which relate group cohomology to Hochschild cohomology in fact which underly our two realisations of the additive decomposition of the Hochschild cohomology of a group algebra.

Let $kG$ be a group algebra and $U$ a $kG$-$kG$-bimodule. Note that we have an algebra isomorphism $(kG)^e \cong k(G \times G)$ given by $g_1 \otimes g_2 \mapsto (g_1, g_2^{-1})$, for $g_1, g_2 \in G$. Thus we can also identify each $kG$-$kG$-bimodule $M$ as a left $k(G \times G)$-module: $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$. In the sequel, we shall write the Hochschild cohomology complex for the group algebra $kG$ in terms of $k(G \times G)$-modules.

Case 1. $M = kG$, the module $kG$ with the obvious $kG$-$kG$-bimodule, or equivalently, the $k(G \times G)$-module $kG$ with action: $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$ for $g_1, g_2 \in G$. It is easy to verify that there is a $k(G \times G)$-module isomorphism $Ind_{G}^{G \times G} k = k(G \times G) \otimes_{kG} k \cong kG ((g_1, g_2) \otimes 1 \mapsto g_1 g_2^{-1})$, where the right $kG$-module structure over $k(G \times G)$ is given by diagonal action. So we have

$$HH^n(kG, kG) \cong Ext^n_{k(G \times G)}(kG, kG) \cong Ext^n_{kG}(Ind_{G}^{G \times G} k, kG) \cong Ext^n_{kG}(k, Res_{G}^{G \times G} kG) = Ext^n_{kG}(k, kG^e),$$

where the third isomorphism is given by the adjoint equivalence and $kG^e$ is considered as a left $kG$-module by conjugation: $g \cdot x = g x g^{-1}$ for $g, x \in G$. This verifies a well-known fact observed by Eilenberg and Mac Lane ([4]): the Hochschild cohomology $HH^n(kG, kG)$ of $kG$ with coefficients in $kG$ is isomorphic to the ordinary group cohomology $H^n(G, kG)$ of $G$ with coefficients in $kG$ under the conjugation.

Case 2. $M = k$, the trivial $kG$-$kG$-bimodule, or equivalently, the $k(G \times G)$-module $k$ with action: $(g_1, g_2) \cdot 1 = 1$ for $g_1, g_2 \in G$. Since we have

$$HH^n(kG, k) \cong Ext^n_{k(G \times G)}(kG, k) \cong Ext^n_{k(G \times G)}(k(G \times G) \otimes_{kG} k, k) \cong Ext^n_{kG}(k, k) = H^n(G, k),$$

the Hochschild cohomology $HH^n(kG, k)$ of $kG$ with coefficients in $k$ is isomorphic to the ordinary group cohomology $H^n(G, k)$. Another way to see this lies in the fact that the two complexes $C^*(kG, k)$ and $C^*(G, k)$ coincide.

Let $kG$ be a group algebra. Then $kG$ is a symmetric algebra with the bilinear form

$$\langle g, h \rangle = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for $g, h \in G$. So there is a well-defined BV-algebra structure on $HH^*(kG)$. On the other hand, since the ordinary group cohomology $H^*(G, k)$ is isomorphic to the Hochschild cohomology $HH^n(kG, k)$ of $kG$ with coefficients in the trivial module $k$, we can define a cup product on the group cohomology

$$H^*(G, k) = \bigoplus_{n \geq 0} H^n(G, k)$$

in a similar way:

$$\varphi_1 \cup \varphi_2 : G^{x \times n+m} \rightarrow k, \quad (g_1, \ldots, g_{n+m}) \mapsto \varphi_1(g_1, \ldots, g_n) \varphi_2(g_{n+1}, \ldots, g_{n+m}),$$

where $\varphi_1 \in \text{Map}(G^{x \times n})$, and $\varphi_2 \in \text{Map}(G^{x \times m})$. By [5, Corollary 2.2], $H^*(G, k)$ is a Gerstenhaber subalgebra of $HH^*(kG)$. In fact, as in [5, Proof of Theorem 1.8], there is a chain map at the cohomology complex level:

$$\text{Hom}_{kG}(Bar_*(kG) \otimes_{kG} k, k) \hookrightarrow \text{Hom}_{k(G \times G)}(Bar_*(kG), kG),$$

$$\langle \varphi : G^{x \times n} \rightarrow k \rangle \mapsto \langle \psi : G^{x \times n} \rightarrow kG \rangle, \quad \psi(g_1, \ldots, g_n) = \varphi(g_1, \ldots, g_n) g_1 \cdots g_n.$$

This inclusion map preserves the brace operations in the following sense:

Let $\varphi_1, \varphi_2 \in C^n(kG, k) \cong \text{Map}(G^{x \times n}, k)$, and $\varphi_{11}, \varphi_{22} \in C^m(kG, k)$ be the corresponding elements under the above inclusion map. Then $\varphi_1 \circ \varphi_2 = \varphi_1 \circ \varphi_2 \in C^{n+m-1}(kG)$. 


Notice that if $G$ is an abelian group, then there is an isomorphism of graded rings (see [7] or [3]):

$$HH^*(kG, kG) \cong kG \otimes_k H^*(G, k).$$

5. An realization of the additive decomposition

Let $k$ be a field and $G$ a finite group. Then the Hochschild cohomology ring of the group algebra $kG$ admits an additive decomposition:

$$HH^*(kG) \cong \bigoplus_{x \in X} H^*(C_G(x), k)$$

where $X$ is a set of representatives of conjugacy classes of elements of $G$ and $C_G(x)$ is the centralizer subgroup of $G$. In this section, we give an explicit construction of the additive decomposition. The main technique we used here is to construct comparison maps based on some set-like self-homotopies. (cf. Section 1: Introduction.)

By definition, $HH^*(kG) = H^*(\text{Hom}_{kG}(\text{Bar}_*, kG), kG)$). Since there is an algebra isomorphism $kG)^* \cong k(G \times G)$ by $g_1 \otimes g_2 \mapsto (g_1, g_2^{-1})$, we can identify $H^*(\text{Hom}_{kG}(\text{Bar}_*, kG), kG)$ with $H^*(\text{Hom}_{kG}(G \times G; \text{Bar}_*, kG), kG)$, where the $(G \times G)$-module structure over $kG$ is given by $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$ (the same applies to the terms in $\text{Bar}_*(kG)$). Since we always use the normalized bar resolutions, we have

$$\text{Hom}_{kG}(G \times G; \text{Bar}_n(kG), kG) \cong \text{Hom}_k(\overline{C}_G^n, kG) \cong \text{Map}(\overline{G}^n, kG).$$

It follows that $HH^*(kG)$ is the cohomology of the following cochain complex:

$$0 \to kG \xrightarrow{d_0} \text{Map}(\overline{G}, kG) \xrightarrow{d_1} \cdots \xrightarrow{d_n} \text{Map}(\overline{G}^n, kG) \xrightarrow{d_{n+1}} \cdots,$$

where the differential is given by

$$d_0(x)(g) = gx - xg \quad (\text{for } x \in kG \text{ and } g \in \overline{G})$$

and (for $\varphi: \overline{G}^n \to kG$ and $g_1, \ldots, g_{n+1} \in \overline{G}$)

$$d_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1 \varphi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \ldots, g_n g_{n+1}).$$

The main line of our method follows from [9], there the formula is given between cohomology groups using standard operations like restriction, induction, conjugation, etc., while we choose some special projective resolutions and construct maps directly on the cohomology complex level. We will divide our construction into six steps.

The first step. Let $kG$ be the $(G \times G)$-module in $HH^*(kG) = \text{Hom}_{k(G \times G)}(\text{Bar}_*, kG, kG)$. Then we have a $(G \times G)$-module isomorphism $k(G \times G) \otimes_{kG} k \cong kG$ given by $(g_1, g_2) \otimes 1 \mapsto g_1 g_2^{-1}$, where $kG$ operates diagonally on $k(G \times G)$ from the right. Moreover, $k(G \times G) \otimes_{kG} \text{Bar}_* kG) \otimes_{kG} k$ is also a free resolution of the above $(G \times G)$-module $kG$. Notice that here and in the following we still view the terms in $\text{Bar}_*(kG)$ as the usual $kG$-module bimodules. If we identify $k(G \times G) \otimes_{kG} \text{Bar}_n(kG) \otimes_{kG} k$ with $kG \otimes kG \otimes \overline{G}_G^n$, then the differential is as follows:

$$kG \otimes kG \to kG, \quad x \otimes y \mapsto x y^{-1};$$

$$kG \otimes kG \otimes \overline{G} \to kG \otimes kG, \quad x \otimes y \otimes g_1 \mapsto x g_1 \otimes y g_1 - x \otimes y;$$

$$
\quad \cdot \cdot \cdot ;
$$

$$kG \otimes kG \otimes \overline{G}_G^n \to kG \otimes kG \otimes \overline{G}_G^{n-1}, \quad x \otimes y \otimes g_1 \otimes \cdots \otimes g_n \mapsto x g_1 \otimes y g_1 \otimes g_2 \otimes \cdots \otimes g_n + \sum_{i=1}^{n-1} (-1)^i x \otimes y \otimes g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n + (-1)^{n} x \otimes y \otimes g_1 \otimes \cdots \otimes g_n.$$

We also have

$$\text{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \text{Bar}_n(kG) \otimes_{kG} k, kG) \cong \text{Hom}_k(\overline{G}_G^n, kG) \cong \text{Map}(\overline{G}^n, kG).$$
Using this identification, \( H^*(\text{Hom}_{k(G \times G)}(k(G \times G) \otimes_k G \text{Bar}_\ast(kG) \otimes_k k, kG)) \) is given by the following cochain complex:

\[
0 \to kG \xrightarrow{d_0^G} \text{Map}((-), kG) \xrightarrow{d_1^n} \cdots \to \text{Map}((-)^n, kG) \xrightarrow{d_n^n} \cdots ,
\]

where the differential is given by

\[
d_n^G(x)(g) = gxy^{-1}x \quad (\text{for } x \in kG \text{ and } g \in G)
\]

and (for \( \varphi : G^n \to kG \))

\[
d_n^G(\varphi)(g_1, \cdots, g_{n+1}) = g_1 \varphi(g_2, \cdots, g_{n+1})g_1^{-1} + \sum_{i=1}^n (-1)^i \varphi(g_1, \cdots, g_ig_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \cdots, g_n).
\]

By standard homological algebra, we know that

\[
(1) \quad H^*(\text{Hom}_{k(G \times G)}(\text{Bar}_\ast(kG), kG)) \cong H^*(\text{Hom}_{k(G \times G)}(k(G \times G) \otimes_k G \text{Bar}_\ast(kG) \otimes_k k, kG)).
\]

Notice that there is no canonical choice for the isomorphism in (1). However, we can always choose a “natural” one and fix it in this paper. We will give an explicit isomorphism in (1) and its inverse, based on the construction of the comparison maps between the two free resolutions \( \text{Bar}_\ast(kG) \) and \( k(G \times G) \otimes_k G \text{Bar}_\ast(kG) \otimes_k k \) of the above \( k(G \times G) \)-module \( kG \). As explained in Introduction, this is reduced to construct set-like self-homotopies over these resolutions. Our principle here is to choose those set-like self-homotopies so that the computations and results are simple.

We choose a set-like self-homotopy over \( \text{Bar}_\ast(kG) \) as

\[
u_n : kG \otimes G^n \otimes kG \to kG \otimes \overline{kG}^n \otimes kG,
\]

\[
g_0 \otimes g_1 \otimes \cdots \otimes g_{n+1} \mapsto (-1)^{n+1} g_0 \otimes g_1 \otimes \cdots \otimes g_{n+1} \otimes 1.
\]

Using \( \{\nu_n\} \) we can construct a comparison map

\[
\alpha : k(G \times G) \otimes_k G \text{Bar}_\ast(kG) \otimes_k k \to \text{Bar}_\ast(kG)
\]

as follows (we only write down the maps on basis vectors):

\[
\alpha_{-1} : kG \to kG, x \mapsto x,
\]

\[
\alpha_0 : kG \otimes kG \to kG \otimes kG, x \otimes y \mapsto x \otimes y^{-1},
\]

\[
\alpha_1 : kG \otimes kG \otimes \overline{kG} \to kG \otimes \overline{kG} \otimes kG, x \otimes y \otimes g_1 \mapsto -xg_1 \otimes g_1^{-1} \otimes y^{-1},
\]

\[
\alpha_n : kG \otimes kG \otimes kG^\otimes \to kG \otimes kG \otimes kG^\otimes, x \otimes y \otimes g_1 \otimes \cdots \otimes g_n \mapsto (-1)^{n(n+1)} xg_1 \cdots g_n g_1^{-1} \cdots g_n^{-1} y^{-1}.
\]

Similarly, we choose a set-like self-homotopy over \( k(G \times G) \otimes_k G \text{Bar}_\ast(kG) \otimes_k k \) as

\[
u_n : kG \otimes kG \otimes kG^\otimes \to kG \otimes kG \otimes kG^\otimes,
\]

\[
x \otimes y \otimes g_1 \otimes \cdots \otimes g_n \mapsto xy^{-1} \otimes 1 \otimes g_1 \otimes \cdots \otimes g_n.
\]

Using \( \{\nu_n\} \) we can construct a comparison map

\[
\beta : \text{Bar}_\ast(kG) \to k(G \times G) \otimes_k G \text{Bar}_\ast(kG) \otimes_k k
\]

as follows (we only write down the maps on basis vectors):

\[
\beta_{-1} : kG \to kG, x \mapsto x,
\]

\[
\beta_0 : kG \otimes kG \to kG \otimes kG, x \otimes y \mapsto x \otimes y^{-1},
\]

\[
\beta_1 : kG \otimes \overline{kG} \otimes kG \to kG \otimes kG \otimes \overline{kG}, x \otimes g_1 \otimes y \mapsto -xg_1 \otimes y^{-1} \otimes g_1^{-1},
\]

\[
\beta_n : kG \otimes \overline{kG} \otimes kG \to kG \otimes \overline{kG} \otimes kG^\otimes, x \otimes g_1 \otimes \cdots \otimes g_n \otimes y \mapsto (-1)^{n(n+1)} xg_1 \cdots g_n \otimes y^{-1} \otimes g_1^{-1} \cdots g_n^{-1}.
\]

It is easy to check that the chain maps \( \{\alpha_n\} \) and \( \{\beta_n\} \) are inverse to each other, and therefore we get an isomorphism

\[
\text{Hom}_{k(G \times G)}(\text{Bar}_\ast(kG), kG) \cong \text{Hom}_{k(G \times G)}(k(G \times G) \otimes_k G \text{Bar}_\ast(kG) \otimes_k k, kG),
\]

\[
(\varphi : G^n \to kG) \mapsto (\varphi : G^n \to kG), \varphi(g_1, \cdots, g_n) = (-1)^{n(n+1)} g_1 \cdots g_n \varphi(g_1^{-1}, \cdots, g_n^{-1}).
\]
Its inverse is given by 
\[ \text{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \text{Bar}_r(kG) \otimes_{kG} k, kG) \rightarrow \text{Hom}_{k(G \times G)}(\text{Bar}_r(kG), kG), \]
\[ (\varphi_1 : \bar{G}^n \rightarrow kG) \mapsto (\varphi : G^n \rightarrow kG), \varphi(g_1, \ldots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \varphi_1(g_1^{-1}, \ldots, g_n^{-1}). \]

Passing to the cohomology, we realize an isomorphism in (1) and its inverse.

The second step. Since \( (k(G \times G) \otimes_{kG} -, \text{Hom}_{k(G \times G)}(k(G \times G), -)) \) is an adjoint pair, we have an isomorphism (Note that \( k(G \times G) \) is viewed as a right \( kG \)-module by diagonal action)
\[ \text{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \text{Bar}_r(kG) \otimes_{kG} k, kG) \cong \text{Hom}_{kG}(\text{Bar}_r(kG) \otimes_{kG} k, kG), \]
where the \( kG \)-module structure on the last \( kG \) is given by conjugation: \( g : x = gxg^{-1} \) for \( g \in G \).

Passing to the cohomology, we get an isomorphism
\[ H^*(\text{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \text{Bar}_r(kG) \otimes_{kG} k, kG)) \cong H^*(\text{Hom}_{kG}(\text{Bar}_r(kG) \otimes_{kG} k, kG)). \]
Remind that the right hand side is just the ordinary group cohomology \( H^*(G, kG) \) of \( G \) with coefficients in \( kG \) under the conjugation (cf. Case 1 of Example ??). We also have
\[ \text{Hom}_{kG}(\text{Bar}_r(kG) \otimes_{kG} k, kG) \cong \text{Hom}_{kG}(kG \otimes_{kG} \bar{G}^n, kG) \cong \text{Hom}_{kG}(\bar{G}^n, kG) \cong \text{Map}(G^n, kG). \]
Using this identification, \( H^*(G, kG) = H^*(\text{Hom}_{kG}(\text{Bar}_r(kG) \otimes_{kG} k, kG)) \) is given by the following cochain complex:
\[ 0 \rightarrow kG \xrightarrow{d_0} \text{Map}(G, kG) \xrightarrow{d_1} \cdots \xrightarrow{d_n} \text{Map}(G^n, kG) \xrightarrow{d_{n+1}} \cdots, \]
where the differential is given by
\[ d_0(x)(g) = gxg^{-1} - x \quad (\text{for } x \in kG \text{ and } g \in \bar{G}) \]
and (for \( \varphi : \bar{G}^n \rightarrow kG \) and \( g_1, \ldots, g_{n+1} \in \bar{G} \))
\[ d_n(\varphi)(g_1, \ldots, g_{n+1}) = \varphi(g_2, \ldots, g_{n+1})g_1^{-1} + \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_1g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \ldots, g_n). \]
So formally the left hand side and the right hand side in (2) are identical, though they have different meaning. It is also easy to check that under the above identifications, the adjoint isomorphisms are identity maps:
\[ \text{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \text{Bar}_r(kG) \otimes_{kG} k, kG) \rightarrow \text{Hom}_{kG}(\text{Bar}_r(kG) \otimes_{kG} k, kG), \]
\[ (\varphi_1 : \bar{G}^n \rightarrow kG) \mapsto (\varphi_2 : \bar{G}^n \rightarrow kG), \varphi_2(g_1, \ldots, g_n) = \varphi_1(g_1, \ldots, g_n). \]
Its inverse is given by
\[ \text{Hom}_{kG}(\text{Bar}_r(kG) \otimes_{kG} k, kG) \rightarrow \text{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \text{Bar}_r(kG) \otimes_{kG} k, kG), \]
\[ (\varphi_2 : \bar{G}^n \rightarrow kG) \mapsto (\varphi_1 : \bar{G}^n \rightarrow kG), \varphi_1(g_1, \ldots, g_n) = \varphi_2(g_1, \ldots, g_n). \]
Passing to the cohomology, we realize an isomorphism in (2) and its inverse.

The third step. We choose a complete set \( X \) of representatives of the conjugacy classes in the finite group \( G \). Take \( x \in X \). Then \( C_x = \{gxg^{-1} | g \in G\} \) is the conjugacy class corresponding to \( x \) and \( C_G(x) = \{g \in G | gxg^{-1} = x\} \) is the centralizer subgroup. Clearly the \( k \)-space \( kC_x \) generated by the elements in \( C_x \) is a left \( kG \)-module under the conjugation action. We choose a right coset decomposition of \( C_G(x) \) in \( G \): \( G = C_G(x) \gamma_{1,x} \cup C_G(x) \gamma_{2,x} \cup \cdots \cup C_G(x) \gamma_{n_x,x} \) (equivalently, \( G = \gamma_{1,x}^{-1}C_G(x) \cup \gamma_{2,x}^{-1}C_G(x) \cup \cdots \cup \gamma_{n_x,x}^{-1}C_G(x) \) is a left coset decomposition of \( C_G(x) \) in \( G \)), and such that \( C_x = \{x = \gamma_{1,x}x, \gamma_{2,x}x, \gamma_{3,x}x, \ldots, \gamma_{n_x,x}x\} \). (We will always take \( \gamma_{1,x} = 1 \), and we write \( x_1 \) for \( \gamma_{1,x}^{-1} \).) Then we have the following \( kG \)-module isomorphisms:
\[ kC_x \cong kG \otimes_{kC_G(x)} k, \quad x_i \mapsto \gamma_{i,x}^{-1} \otimes_{kC_G(x)} 1, \]
\[ kC_x \cong \text{Hom}_{kC_G(x)}(kG, k), \quad x_i \mapsto \gamma_i : kG \rightarrow k, \gamma_i(\gamma_{j,x}) = \delta_{ij}. \]
where in the first isomorphism, the left $kG$-module structure on $kG$ is the usual left multiplication and the right $kC_G(x)$-module structure on $kG$ is given by restriction, and $k$ is the trivial $kC_G(x)$-module, and the similar as in the second isomorphism.

In the second step, we have arrived at the ordinary group cohomology $H^*(G, kG)$ of $G$ with coefficients in $kG$, where the $kG$-module structure over $kG$ is given by the conjugation. This $kG$ has a $kG$-module decomposition:

$$kG = \bigoplus_{x \in X} kC_x.$$ 

Denote by $\pi_x : kG \to kC_x$ and $i_x : kC_x \to kG$ the canonical projection and the canonical injection, respectively. Then we have the following isomorphism

$$\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kG) \to \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \bigoplus_{x \in X} kC_x),$$ 

$$(\varphi_2 : \widetilde{G}^x \to kG) \mapsto \varphi_3 = \{\varphi_{3,x} | x \in X\},$$

where if we write $\varphi_3, x \to \pi_x \varphi_2 \to \widetilde{G}^x \to kC_x$.

Its inverse is given by

$$\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \bigoplus_{x \in X} kC_x) \to \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kG),$$

$$\varphi_3 = \{\varphi_{3,x} : \widetilde{G}^x \to kC_x | x \in X\} \mapsto (\varphi_2 = \sum_{x \in X} i_x \varphi_{3,x} : \widetilde{G}^x \to kG).$$

Passing to the cohomology, we realize an isomorphism:

$$(3) \quad H^*(G, kG) \cong \bigoplus_{x \in X} H^*(G, kC_x),$$

where the $kG$-module structure over $kG$ is given by the conjugation.

The fourth step. We have stated in the third step the following $kG$-module isomorphism

$$kC_x \cong \text{Hom}_{kC_G(x)}(kG, k), \quad x_i \mapsto \gamma_i : kG \to k, \gamma_i(\gamma_j x) = \delta_{ij}.$$ 

Therefore we have the following isomorphism

$$\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kC_x) \to \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \text{Hom}_{kC_G(x)}(kG, k)),$$

$$(\varphi_{3,x} : \widetilde{G}^x \to kC_x) \mapsto (\varphi_{4,x} : \widetilde{G}^x \to \text{Hom}_{kC_G(x)}(kG, k)),$$

where if we write $\varphi_{3,x}(y_1, y_2, \cdots, y_n) = \sum_{i=1}^{n} a_{i,x} y_i$, then $\varphi_{4,x}(y_1, y_2, \cdots, y_n)$ maps $\gamma_i x$ to $a_{i,x}$ for any $i$. The inverse isomorphism is given by

$$\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \text{Hom}_{kC_G(x)}(kG, k)) \to \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kC_x),$$

$$(\varphi_{4,x} : \widetilde{G}^x \to \text{Hom}_{kC_G(x)}(kG, k)) \mapsto (\varphi_{3,x} : \widetilde{G}^x \to kC_x),$$

where if $\varphi_{4,x}(g_1, g_2, \cdots, g_n)$ maps $\gamma_i x$ to $a_{i,x}$ for any $i$, then $\varphi_{3,x}(g_1, g_2, \cdots, g_n) = \sum_{i=1}^{n} a_{i,x} g_i$.

Passing to the cohomology, we realize an isomorphism:

$$(4) \quad H^*(G, kC_x) \cong H^*(\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \text{Hom}_{kC_G(x)}(kG, k))).$$

The fifth step. Since $(kG \otimes_{kG} -, \text{Hom}_{kC_G(x)}(kG, -))$ is an adjoint pair, we have the following isomorphism

$$\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \text{Hom}_{kC_G(x)}(kG, k)) \to \text{Hom}_{kC_G(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k).$$

Passing to the cohomology, we get an isomorphism

$$(5) \quad H^*(\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \text{Hom}_{kC_G(x)}(kG, k))) \cong H^*(\text{Hom}_{kC_G(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k),$$

where the right hand side is isomorphic to the ordinary group cohomology $H^*(C_G(x), k)$ of $C_G(x)$ with coefficients in the trivial module $k$. Since there are $kC_G(x)$-module isomorphisms

$$\text{Bar}_*(kG) \otimes_{kG} k \cong \bigoplus_{i=1}^{n} kC_G(x) \gamma_{i,x} \otimes kG^\otimes n,$$
we have
\[ \text{Hom}_{kG(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k) \cong \text{Hom}_{k}(\bigoplus_{i=1}^{n_x} k\gamma_{i,x} \otimes kG^{\otimes n}, k) \cong \text{Map}(S_x \times \overline{\mathbb{G}}^{\otimes n}, k), \]
where \( S_x = \{ \gamma_{1,x}, \ldots, \gamma_{n_x,x} \} \) (cf. The third step). Using this identification, the adjoint isomorphism is given by
\[ \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \text{Hom}_{kG(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k), \]
where \( \varphi \in kG \) maps \( \gamma_{i,x} \) to \( a_{i,x} \) for any \( i \), then \( \varphi \cdot a_{i,x} = a_{i,x} \) for any \( i \). The inverse isomorphism is given by
\[ \text{Hom}_{kG(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k) \rightarrow \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, \text{Hom}_{kG(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k), \]
where \( \varphi \cdot a_{i,x} = a_{i,x} \) for any \( i \), then \( \varphi \cdot a_{i,x} = a_{i,x} \) for any \( i \). Passing to the cohomology, we realize an isomorphism in (5) and its inverse.

The sixth step. In the fifth step, we have arrived at the ordinary group cohomology \( H^*(C_G(x), k) \) of \( C_G(x) \) with coefficients in the trivial module \( k \), where \( H^*(C_G(x), k) \) is computed by the cochain complex \( \text{Hom}_{kG(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k) \). By the identification in fifth step, this is given by the following cochain complex:
\[ 0 \rightarrow k^{\times n_x} \overset{d_0}{\rightarrow} \text{Map}(S_x \times \overline{\mathbb{G}}^n, k) \overset{d_1}{\rightarrow} \cdots \overset{d_n}{\rightarrow} \text{Map}(S_x \times \overline{\mathbb{G}}^{\otimes n}, k) \overset{d_{n+1}}{\rightarrow} \cdots, \]
where the differential is given by \( d_0([a_{i,x}])((\gamma_{j,x}, g_1)) = a_{s_j,x} - a_{j,x} \), and \( a_{s_j,x} \) is determined as follows: for \( \{a_{i,x} \} \in k^{\times n_x}, \gamma_{j,x} \in S_x, g_1 \in \overline{\mathbb{G}} \), we have
\[ \gamma_{j,x} g_1 = h_{j,1} \gamma_{s_{j,x}} \text{ for some } h_{j,1} \in C_G(x) \text{ and for some } 1 \leq s_j \leq n_x. \]
and by \( \varphi : \overline{\mathbb{G}}^{\otimes n} \rightarrow k, \gamma_{j,x} \in S_x, g_1, \ldots, g_{n+1} \in \overline{\mathbb{G}} \) such that \( \gamma_{j,x} g_1 = h_{j,1} \gamma_{s_{j,x}} \)
\[ d_n(\varphi)(\gamma_{j,x}, g_1, \ldots, g_{n+1}) = \varphi(\gamma_{s_{j,x}}, g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(\gamma_{j,x} g_1, \ldots, g_{i+1}, \ldots, g_{n+1} + (-1)^{n+1} \varphi(\gamma_{j,x}, g_1, \ldots, g_{n+1}). \]
(Remark that for a fixed \( g_1 \in \overline{\mathbb{G}}, \{s_1, s_2, \ldots, s_{n_x}\} \) is a permutation of \( \{1, 2, \ldots, n_x\} \).

The above computation for \( H^*(C_G(x), k) \) uses the projective resolution \( \text{Bar}_*(kG) \otimes_{kG} k \) of the trivial \( kC_G(x) \)-module \( k \), which is identified as the following complex (It is in fact a projective resolution of the trivial \( k \)-module, but we view it as a complex of \( kC_G(x) \)-modules by restriction)
\[ \cdots \rightarrow kG \otimes \overline{\mathbb{G}}^{\otimes n} \overset{d_n}{\rightarrow} \cdots \overset{d_1}{\rightarrow} kG \otimes \overline{\mathbb{G}} \overset{d_0}{\rightarrow} kG \overset{d_0}{\rightarrow} k \rightarrow 0, \]
where the differential is given by \( d_0(g_0) = 1 \) (for \( g_0 \in G \)) and (for \( g_0 \in G, g_1, \ldots, g_n \in \overline{\mathbb{G}} \))
\[ d_n(g_0, g_1, \ldots, g_n) = g_0 g_1 \otimes \cdots \otimes g_n + \sum_{i=1}^{n-1} (-1)^i g_0 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n + (-1)^n g_0 \otimes g_1 \otimes \cdots \otimes g_{n-1}. \]
We now use another projective resolution \( \text{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k \) of the trivial \( kC_G(x) \)-module \( k \), which is identified as the following complex
\[ \cdots \rightarrow kC_G(x) \otimes kC_G(x) \overset{d_n}{\rightarrow} \cdots \overset{d_1}{\rightarrow} kC_G(x) \otimes kC_G(x) \overset{d_0}{\rightarrow} kC_G(x) \overset{d_0}{\rightarrow} k \rightarrow 0, \]
where the differential is given by \( d_0(h_0) = 1 \) (for \( h_0 \in \overline{\mathbb{G}}(x) \))
and (for $h_0 \in G, h_1, \cdots, h_n \in C_G(x)$)
\[
d_n(h_0, h_1, \cdots, h_n) = h_0 h_1 \otimes h_2 \otimes \cdots \otimes h_n + \\
\sum_{i=1}^{n-1} (-1)^i h_0 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n + (-1)^n h_0 \otimes h_1 \otimes \cdots \otimes h_{n-1}.
\]

We have
\[
\text{Hom}_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k) \cong \text{Hom}_k(kC_G(x) \otimes \overline{kC_G(x)}^\otimes, k) \cong \text{Map}(\overline{C_G(x)}^\times, k),
\]
so $H^*(G(x), k)$ can also be computed by the following cochain complex
\[
0 \longrightarrow k \overset{d_0}{\longrightarrow} \text{Map}(\overline{C_G(x)}, k) \overset{d_1}{\longrightarrow} \cdots \longrightarrow \text{Map}(\overline{C_G(x)}^\times, k) \overset{d_n}{\longrightarrow} \cdots,
\]
where the differential is given by
\[
d_0(a)(h_1) = 0 \quad \text{(for } a \in k, h_1 \in \overline{C_G(x)})
\]
and (for $\varphi : \overline{C_G(x)}^\times \longrightarrow k, h_1, \cdots, h_{n+1} \in C_G(x)$)
\[
d_n(\varphi)(h_1, \cdots, h_{n+1}) = \varphi(h_2, \cdots, h_{n+1}) + \\
\sum_{i=1}^{n} (-1)^i \varphi(h_1, \cdots, h_i h_{i+1}, \cdots, h_{n+1}) + (-1)^{n+1} \varphi(h_1, \cdots, h_n).
\]

Clearly, we have
\[
(6) \quad H^*(\text{Hom}_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k)) \cong H^*(\text{Hom}_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k)).
\]
To give an explicit isomorphism in (6), we need to construct the comparison maps between two projective resolutions $\text{Bar}_*(kG) \otimes_{kG} k$ and $\text{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k$ of the trivial $kC_G(x)$-module $k$.

The comparison map from $\text{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k$ to $\text{Bar}_*(kG) \otimes_{kG} k$ is just the inclusion map
\[
i : kC_G(x) \otimes \overline{kC_G(x)}^\otimes \hookrightarrow kG \otimes \overline{kC_G(x)}^\otimes.
\]

To construct the comparison map on the reverse direction, we use a set-like self-homotopy over $kC_G(x) \otimes \overline{kC_G(x)}^\otimes$ as follows (for $h_0 \in C_G(x), h_1, \cdots, h_n \in C_G(x)$)
\[
kC_G(x) \otimes \overline{kC_G(x)}^\otimes \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^\otimes +1,
\]
\[
h_0 \otimes h_1 \otimes \cdots \otimes h_n \longmapsto 1 \otimes h_0 \otimes h_1 \otimes \cdots \otimes h_n.
\]

Then we get a comparison map
\[
\rho : \text{Bar}_*(kG) \otimes_{kG} k \longrightarrow \text{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k
\]
as follows (we only write down the maps on basis vectors):
\[
\rho_{-1} : k \longrightarrow k, 1 \longmapsto 1,
\]
\[
\rho_{-1} : k \longrightarrow kC_G(x), h_\gamma x \longmapsto h, \quad \text{where } h_\gamma x \text{ belongs to the right coset } C_G(x) x l \gamma x,
\]
\[
\rho_{1} : kG \otimes \overline{kC_G(x)}^\otimes \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^\otimes, h_\gamma x \otimes g_1 \longmapsto h \otimes h_\gamma x, \quad \text{where } g_1 \gamma x g_1 = h_\gamma x \gamma x, \text{ for } h_\gamma x \in C_G(x),
\]
\[
\rho_{n} : kG \otimes \overline{kC_G(x)}^\otimes \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^\otimes, h_\gamma x \otimes g_1 \otimes \cdots \otimes g_n \longmapsto h \otimes h_\gamma x \otimes \cdots \otimes h_n,
\]
where for $x \in X, h_\gamma x, \cdots, h_n \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \cdots, g_n\}$ as follows:
\[
\gamma x g_1 = h_\gamma x \gamma x g_2 = h_\gamma x \gamma x g_3, \cdots, \gamma x g_{n-1} = h_\gamma x \gamma x g_n = h_\gamma x \gamma x.
\]

It follows that we have two homomorphisms:
\[
\text{Hom}_{kC_G(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k) \longrightarrow \text{Hom}_{kC_G(x)}(\text{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k),
\]
\[
(\varphi_{5,x} : S_x \times \overline{C_G(x)}^\otimes \longrightarrow k) \longmapsto (\varphi_{6,x} : \overline{C_G(x)}^\otimes \longrightarrow k), \varphi_{6,x}(h_1, \cdots, h_n) = \varphi_{5,x}(1, h_1, \cdots, h_n) = a_{1,x},
\]
where $a_{1,x}$ is the coefficients of $x$ in $\varphi_{3,x}(h_1, \cdots, h_n) = \sum_{i=1}^{n} a_{i,x} \gamma x^{-1} x \gamma x$;

and
\[
\text{Hom}_{kC_G(x)}(\text{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k) \longrightarrow \text{Hom}_{kC_G(x)}(\text{Bar}_*(kG) \otimes_{kG} k, k),
\]
(φ₆ₓ : \overline{G}(x) → k) → (φ₅ₓ : Sₓ × \overline{G}^{\times n} → k), \varphi₅ₓ(γ₁,x,g₁,\cdots,gₙ) = φ₆ₓ(h₁,\cdots,hₙ),
where for \( x \in X, h₁,\cdots, hₙ \in \overline{G}(x) \) are determined by the sequence \( \{g₁,\cdots,gₙ\} \) as follows:

\[ γ₁,xg₁ = h₁,1γ₁,x₁,γ₁,x₂g₂ = h₁,2γ₁,x₂,\cdots, γ₁,xₙ⁻¹,xgₙ = h₁,nγ₁,xₙ. \]

Since both \( \iota \) and \( ρ \) induce the identity map \( 1 : k \to k \), by Lemma 2.3, we have inverse isomorphisms between \( H^*(\text{Hom}_{kG}(x)(\text{Bar}_{k}(kG)\otimes kG(k), k)) \) and \( H^*(\text{Hom}_{kG}(x)(\text{Bar}_{k}(kG(x))\otimes kG(x), k), k) \). The correspondence is induced by \( φ₅ₓ \leftrightarrow φ₆ₓ \), as we stated above. So we realize an isomorphism in (6) and its inverse.

Summarizing the above six steps, we get the following main result in this section.

**Theorem 5.1.** Let \( k \) be a field and \( G \) a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra \( kG \):

\[ HH^*(kG) ≅ \bigoplus_{x \in X} HH^*(C_G(x), k) \]

where \( X \) is a set of representatives of conjugacy classes of elements of \( G \) and \( C_G(x) \) is the centralizer subgroup of \( G \). We compute the Hochschild cohomology \( HH^*(kG) = H^*(\text{Hom}_{kG}(x)(\text{Bar}_{k}(kG), k)) \) by the classical normalized bar resolution, and we compute the group cohomology \( H^*(C_G(x), k) \) by \( H^*(\text{Hom}_{kG}(x)(\text{Bar}_{k}(kG(x))\otimes kG(x), k), k) \). Then, we can realize an isomorphism in additive decomposition as follows:

\[ HH^*(kG) \xrightarrow{\sim} \bigoplus_{x \in X} HH^*(C_G(x), k) \]

where \( [φ : \overline{G}^{\times n} \to kG] \mapsto [\bar{φ}] = \bigoplus_{x \in X} [\bar{φ}_x : \overline{G}(x)^{\times n} \to k] \),

\[ \bar{φ}_x(h₁,\cdots,hₙ) = a₁,x, \quad \text{where} \quad \pi_x((-1)^{n(n+1)}h₁\cdots hₙφ(h₁⁻¹,\cdots,hₙ⁻¹)) = \sum_{i=1}^{n} a_i,x. \]

In other word, \( \bar{φ}_x(h₁,\cdots,hₙ) \) is just the coefficient of \( x \) in \((-1)^{n(n+1)}h₁\cdots hₙφ(h₁⁻¹,\cdots,hₙ⁻¹) \) in \( kG \).

The inverse of the above isomorphism is given as follows:

\[ \bigoplus_{x \in X} HH^*(C_G(x), k) \xrightarrow{\sim} HH^*(kG) \]

\[ [\bar{φ}] = \bigoplus_{x \in X} [\bar{φ}_x : \overline{G}(x)^{\times n} \to k] \mapsto [φ : \overline{G}^{\times n} \to kG], \]

\[ φ(g₁,\cdots,gₙ) = (-1)^{n(n+1)} g₁\cdots gₙ \sum_{x \in X} \sum_{i=1}^{n} \bar{φ}_x(h₁,i,…,hₙ,i)x_i, \]

where for \( x \in X, h₁,i,\cdots,hₙ,i \in \overline{G}(x) \) are determined by the sequence \( \{g₁⁻¹,\cdots,gₙ⁻¹\} \) as follows:

\[ γ₁,xg₁⁻¹ = h₁,i₁γ₁,x₁, \quad γ₁,x₂g₂⁻¹ = h₁,i₂γ₁,x₂, \cdots, \quad γ₁,xₙ⁻¹,xgₙ⁻¹ = h₁,iₙγ₁,xₙ. \]

**Proof** This is a direct consequence by applying the above isomorphisms from (1) to (6) and their inverses. For an element \( φ : \overline{G}^{\times n} \to kG \) in the \( n \)-th term \( C^n(kG) \) of the Hochschild cohomology complex, \( [φ] \) denotes the corresponding element in the Hochschild cohomology group \( HH^n(kG) \). Note that the elements \( h₁,i,\cdots,hₙ,i \) depend on \( x \in X \) and the sequence \( \{g₁⁻¹,\cdots,gₙ⁻¹\} \). For the simplicity of notations, we avoid to write them down explicitly.

\[ \square \]

**Remark 5.2.** (a) The correspondence in Theorem 5.1 can be illustrated as follows:

\[ HH^*(kG) \overset{(1)}{\cong} H^*(G, kG) \overset{(2)}{\cong} \bigoplus_{x \in X} H^*(G, kC_x) \overset{(3)}{\cong} \bigoplus_{x \in X} H^*(kC_G(x), k). \]

This is just the same line used by Siegel and Witherspoon in [9]. The difference is: they realize each step between cohomology groups using standard operations like restriction, induction, conjugation, etc., while we construct maps directly in each step on the cohomology complex level.
Since we have
\[ HH^*(kG) \cong H^*(\text{Hom}_{k(G\times G)}(\text{Bar}_*(kG), kG)) \]
\[ \cong H^*(\text{Hom}_{k(G\times G)}(k(G \times G) \otimes_{kG} \text{Bar}_*(kG) \otimes_{kG} k, kG)), \]
we can choose either cohomology complex when discuss the ring structure of \( HH^*(kG) \), it is not harm to the result up to isomorphism. If we compute \( HH^*(kG) \) by the projective resolution \( (k(G \times G) \otimes_{kG} \text{Bar}_*(kG) \otimes_{kG} k, kG) \), then the correspondence in Theorem 5.1 become simpler:

\[ HH^*(kG) \sim \bigoplus_{x \in X} H^*(C_G(x), k), \]
\[ [\varphi : \mathcal{C}^x \rightarrow kG] \mapsto [\bar{\varphi}] = \bigoplus_{x \in X} [\tilde{\varphi}_x] : \mathcal{C}_G(x)^{\times n} \rightarrow k, \]
\[ \tilde{\varphi}_x(h_1, \ldots, h_n) = a_{i,x}, \text{ the coefficient of } x \text{ in } \varphi(h_1, \ldots, h_n) \in kG; \]
\[ \bigoplus_{x \in X} H^*(C_G(x), k) \sim HH^*(kG), \]
\[ [\bar{\varphi}] = \bigoplus_{x \in X} [\tilde{\varphi}_x] : \mathcal{C}_G(x)^{\times n} \rightarrow k \mapsto [\varphi : \mathcal{C}^x \rightarrow kG], \]
\[ \varphi(g_1, \ldots, g_n) = \sum_{x \in X} \sum_{i=1}^{n_x} \tilde{\varphi}_x(h_{i,1}, \ldots, h_{i,n}) x_i, \]
where for \( x \in X, h_{i,1}, \ldots, h_{i,n} \in C_G(x) \) are determined by the sequence \( \{g_1, \ldots, g_n\} \) as follows:
\[ \gamma_{i,x} g_1 = h_{i,1}\gamma_{i,1,x} \gamma_{i,1,x} \cdots \gamma_{i,1,x} g_2 = h_{i,2}\gamma_{i,2,x} \cdots \cdots \gamma_{i,1,x} g_n = h_{i,n}\gamma_{i,n,x}. \]

6. Another way to realize the additive decomposition

In [3], Cibils and Solotar constructed a subcomplex of the Hochschild cohomology complex for each conjugacy class, and then they showed that for a finite abelian group, the subcomplex is isomorphic to the complex computing group cohomology. We will generalize this to any finite group: for each conjugacy class, and then they showed that for a finite abelian group, the subcomplex is isomorphic to the complex computing group cohomology. As a result, we give a second way to realize the additive decomposition.

As before, let \( k \) be a field and \( G \) a finite group. Recall that the Hochschild cohomology \( HH^*(kG) \) of the group algebra \( kG \) can be computed by the following (cochain) complex:

\[ (\mathcal{H}^*) \quad 0 \rightarrow kG \xrightarrow{d_0} \text{Map}(\mathcal{G}, kG) \xrightarrow{d_1} \cdots \rightarrow \text{Map}(\mathcal{G}^{\times n}, kG) \xrightarrow{d_n} \cdots, \]
where the differential is given by
\[ d_0(x)(g) = gx - xg \quad (\text{for } x \in kG \text{ and } g \in \mathcal{G}) \]
and (for \( \varphi : \mathcal{G}^{\times n} \rightarrow kG \) and \( g_1, \ldots, g_{n+1} \in \mathcal{G} \))
\[ d_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1 \varphi(g_2, \ldots, \varphi(g_{n+1}, g_{n+1})), \]
\[ \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_{i}, g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \ldots, g_n g_{n+1}). \]

We keep the following notations in Section 3: \( X \) is a complete set of representatives of the conjugacy classes in the finite group \( G \). For \( x \in X, C_x = \{g x g^{-1} | g \in G \} \) is the conjugacy class corresponding to \( x \) and \( C_G(x) = \{g \in G | g x g^{-1} = x \} \) is the centralizer subgroup. Now take a conjugacy class \( x \) and define
\[ \mathcal{H}^x_0 = kC_x, \]
and for \( n \geq 1, \]
\[ \mathcal{H}^x_n = \{ \varphi : \mathcal{G}^{\times n} \rightarrow kG | \varphi(g_1, \ldots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \ldots, g_n \in \mathcal{G} \}, \]
where \( g_1 \cdots g_n C_x \) denotes the subset of \( G \) by multiplying \( g_1 \cdots g_n \) on \( C_x \) and \( k[g_1 \cdots g_n C_x] \) is the k-subspace of \( kG \) generated by this set. Let \( \mathcal{H}^x_+ = \bigoplus_{n=0} \mathcal{H}^x_n \). Cibils and Solotar ([3, Page 20, Proof of the theorem]) observed that \( \mathcal{H}^x_+ \) is a subcomplex of \( \mathcal{H}^* \) and \( \mathcal{H}^* = \bigoplus_{x \in X} \mathcal{H}^x_+ \).
Lemma 6.1. $H^*_x$ is canonically isomorphic to the complex $\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kC_x)$, which computes the group cohomology $H^*(G, kC_x)$ of $G$ with coefficients in $kC_x$, where $kC_x$ is a left $kG$-module under conjugation.

Proof. We know from Section 3 that the complex $\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kC_x)$ is identified as the following complex:

$$0 \rightarrow kC_x \xrightarrow{d_0} \text{Map}(G, kC_x) \xrightarrow{d_1} \cdots \rightarrow \text{Map}(G^{n+1}, kC_x) \xrightarrow{d_n} \cdots,$$

where the differential is given by

$$d_0(x)(g) = gxg^{-1} - x \quad (x \in kC_x \text{ and } g \in G)$$

and (for $\varphi : G^n \rightarrow kC_x$ and $g_1, \cdots, g_{n+1} \in G$)

$$d_n(\varphi)(g_1, \cdots, g_{n+1}) = g_1 \varphi(g_2, \cdots, g_{n+1})g_1^{-1} + \sum_{i=1}^{n} (-1)^i \varphi(g_1, \cdots, g_{gi+1}, \cdots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \cdots, g_n).$$

A direct computation shows that the following map is an isomorphism of complexes:

$$H^*_x \rightarrow \text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kC_x),$$

$$(\varphi_1 : G^n \rightarrow kG) \mapsto (\varphi_2 : G^n \rightarrow kC_x), \varphi_2(g_1, \cdots, g_n) = \varphi_1(g_1, \cdots, g_n)g_1^{-1} \cdots g_1^{-1}.$$ 

Its inverse is given by

$$\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kC_x) \rightarrow H^*_x,$$

$$(\varphi_2 : G^n \rightarrow kC_x) \mapsto (\varphi_1 : G^n \rightarrow kG), \varphi_1(g_1, \cdots, g_n) = \varphi_2(g_1, \cdots, g_n)g_1 \cdots g_n.$$ 

Passing to the cohomology, we have $H^*(H^*_x) \cong H^*(G, kC_x)$. 

□

On the other hand, we have shown that the complex $\text{Hom}_{kG}(\text{Bar}_*(kG) \otimes_{kG} k, kC_x)$ is isomorphic to the complex $\text{Hom}_{kC_G(x)}(\text{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, kC_x)$, which computes the group cohomology $H^*(C_G(x), k)$ of the centralizer subgroup $C_G(x)$ with coefficients in the trivial module $k$. (cf. Section 3, from the fourth step to the sixth step.) Therefore we get another realization to the additive decomposition:

Theorem 6.2. Let $k$ be a field and $G$ a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra $kG$:

$$HH^*(kG) \cong \bigoplus_{x \in X} H^*(C_G(x), k)$$

where $X$ is a set of representatives of conjugacy classes of elements of $G$ and $C_G(x)$ is the centralizer subgroup of $G$. We compute the Hochschild cohomology $HH^*(kG) = H^*(\text{Hom}_{kG}([G, G]), kG)$ by the classical normalized bar resolution, and we compute the group cohomology $H^*(C_G(x), k)$ by $H^*(\text{Hom}_{kC_G(x)}([C_G(x), C_G(x)] \otimes_{kC_G(x)} k, kC_x))$. Then, we can realize an isomorphism in additive decomposition as follows:

$$HH^*(kG) \rightarrow \bigoplus_{x \in X} H^*(C_G(x), k),$$

$$[\varphi_x : G^n \rightarrow kG, \varphi_x \in H^*_x] \mapsto [\tilde{\varphi}_x : C_G(x)^n \rightarrow k],$$

$$\tilde{\varphi}_x(h_1, \cdots, h_n) = a_{i,x}, \text{ where } \varphi_x(h_1, \cdots, h_n)h_n^{-1} \cdots h_1^{-1} = \sum_{i=1}^{n} a_{i,x} x_i \in kC_x.$$ 

In other word, $\tilde{\varphi}_x(h_1, \cdots, h_n)$ is just the coefficient of $x$ in $\varphi_x(h_1, \cdots, h_n)h_n^{-1} \cdots h_1^{-1} \in kC_x$. The inverse of the above isomorphism is given as follows:

$$\bigoplus_{x \in X} H^*(C_G(x), k) \rightarrow HH^*(kG),$$

$$[\tilde{\varphi}_x : C_G(x)^n \rightarrow k] \mapsto [\varphi_x : G^n \rightarrow kG], \varphi_x \in H^*_x.$$
This is obvious from the correspondence in Theorem 6.2. Notice that \( \gamma_{i,n}g_i = h_i, \gamma_{l_1, x}, \gamma_{l_2, x}g_2 = h_{i,2} \gamma_{n} x, \ldots, \gamma_{l_{n-1}, x}g_{n-1} = h_{i,n} \gamma_{l_{n-1}, x} \).

**Proof.** This is a combination of Lemma 6.1 and the correspondence from the fourth step to the six step in Section 3.

\[ \square \]

**Remark 6.3.** Comparing Theorem 5.1 with Theorem 6.2 we see that the two realizations of the additive decomposition are very close to each other. In the sequel, we prefer to the second realization since it is simpler.

### 7. The Cup Product Formula

We keep the notations of the previous sections: \( k \) is a field, and \( G \) is a finite group, and so on. We describe the cup product formula for the Hochschild cohomology ring \( HH^*(kG) \) in terms of the additive decomposition.

**Theorem 7.1.** With the notations in Theorem 6.2, the cup product in the Hochschild cohomology ring \( HH^*(kG) \cong H^*(Hom_{k(G\times G)}(Bar_*(kG), kG)) \) is given as follows. Let \([\varphi_x] : \mathcal{C}_G(x) \rightarrow k\) and \([\varphi_y] : \mathcal{C}_G(y) \rightarrow k\) be two elements in \( H^n(C_G(x), k) \) and in \( H^m(C_G(y), k) \), respectively. Denote by \([\varphi_x] : \mathcal{C}_x \rightarrow kG[\varphi_x \in H_n^0] \) and \([\varphi_y] : \mathcal{C}_y \rightarrow kG[\varphi_y \in H_m^0] \) be the corresponding elements in \( HH^*(kG) \), and denote by \( \varphi_x \cup \varphi_y : \mathcal{C}_x \times \mathcal{C}_y \rightarrow kG \) the cup product. Then, for any \( z \in X \), we have the following cup product formula:

\[
(\varphi_x \cup \varphi_y)z : \mathcal{C}_x \times \mathcal{C}_y \rightarrow kG,
\]

\[
(g_1, \ldots, g_n, \ldots, g_n, \ldots, g_{n+m}) = \sum_{k=1}^{n+m} \varphi_x(h_{i,1}, \ldots, h_{i,n}) \varphi_y(h_{j,1}, \ldots, h_{j,m})z_k,
\]

where the second sum takes over all pairs \((i, j)\) such that \( x_i g_1 \cdots g_n y_{n+1} \cdots g_{n+m} = z_k \). In particular, we have the following formula:

\[
(\varphi_x \cup \varphi_y)z_2 : \mathcal{C}_G(z) \rightarrow k,
\]

\[
(\varphi_x \cup \varphi_y)z_2(h_{1,1}, \ldots, h_{n,n}, \ldots, h_{n+n+m} = \sum_{(i,j)} a_{i,x}a_{j,y},
\]

where \( \varphi_x(h_{1,1}, \ldots, h_{n,n}) = \sum_{i=1}^{n} a_{i,x}x_i, \varphi_y(h_{n+1,1}, \ldots, h_{n+n+m}) = \sum_{j=1}^{n} a_{j,y}y_j, \) and where the sum takes over all pairs \((i, j)\) such that \( x_i(h_{1,1} \cdots h_{n,n})y_j(h_{1,1} \cdots h_{n,n})^{-1} = z_1 = z \).

**Proof.** This is obvious from the correspondence in Theorem 6.2. Notice that \( h_{i,1}, \ldots, h_{i,n} \in \mathcal{C}_G(x) \) depend on the sequence \( \{g_1, \ldots, g_n\} \) by the following steps:

\[
\gamma_{i,n}g_i = h_{i,1} \gamma_{n} x, \gamma_{i,2} x g_2 = h_{i,2} \gamma_{n} x, \ldots, \gamma_{i,n-1} x g_{n-1} = h_{i,n} \gamma_{i,n} x,
\]

and that \( h_{j,1}, \ldots, h_{j,m} \in \mathcal{C}_G(y) \) depend on the sequence \( \{g_{n+1}, \ldots, g_{n+m}\} \) by the following steps:

\[
\gamma_{j,n}g_{n+1} = h_{j,1} \gamma_{n} y, \gamma_{j,2} y g_{n+2} = h_{j,2} \gamma_{n} y, \ldots, \gamma_{j,m} y g_{n+m} = h_{j,m} \gamma_{j,m} y,
\]

\[ \square \]
Remark 7.3. For $\varphi$ is induced by the following equation:

$$\langle \varphi \rangle \in H^*(C_G(x), k)$$

with the cup product

$$\cup : H^*(C_G(x), k) \otimes H^*(C_G(y), k) \rightarrow H^*(C_G(x \times y), k)$$

and denote by $\varphi \cup \varphi : C_G(x) \times C_G(y) \rightarrow C_G(x \times y)$ be the corresponding elements in $H^*(C_G(x \times y), k)$, and denote by $\varphi_x \cup \varphi_y : C_G((x \times n) \times (m \times n)) \rightarrow kG$ the cup product. Then, for any $z \in X$, we have the following cup product formula:

$$(\varphi_x \cup \varphi_y)_z : C_G((x \times n) \times (m \times n)) \rightarrow kG_z,$$

$$(g_1, \cdots, g_n, \cdots, g_{n+m}) \mapsto (-1)^{\frac{n(n+1)}{2}} \sum_{k=1}^{n} \overline{\varphi}_x(h_{i,1}, \cdots, h_{i,k-1}) \overline{\varphi}_y(h_{i,k}, \cdots, h_{i,m-1}) z_k,$$

where the second sum takes over all pairs $(i, j)$ such that $g_1 \cdots g_n g_{n+1} \cdots g_{n+m} y_j = z_k$. In particular, we have the following formula:

$$(\varphi_x \cup \varphi_y)_z : C_G((x \times n) \times (m \times n)) \rightarrow k,$$

$$(h_1, \cdots, h_n, \cdots, h_{n+m}) \mapsto (-1)^{\frac{n(n+m+1)}{2}(n+m)} \sum_{(i,j)} a_i a_j z,$$

where $\varphi_x(h_{1,n+1}, \cdots, h_{1,n+m}) = \sum_{i=1}^{n} a_i x_i$, $\varphi_y(h_{n+1}, \cdots, h_1, 1) = \sum_{j=1}^{n} a_j y_j$, and where the sum takes over all pairs $(i, j)$ such that $h_1 \cdots h_n \cdots h_{n+m} x_i y_j = z_i = z$.

Proof This is obvious from the correspondence in Theorem 5.1. Notice that $h'_{i,1}, \cdots, h'_{i,n} \in C_G(x)$ depend on the sequence $\{g_n^{-1}, \cdots, g_1^{-1}\}$ by the following steps:

$$\gamma_{i,x} g_n^{-1} = h'_{i,1} \gamma_{i,x}^1, \quad \gamma_{i,x} g_{n-1}^{-1} = h'_{i,2} \gamma_{i,x}^2, \quad \cdots, \quad \gamma_{i,x} g_{n-1,1} = h'_{i,n} \gamma_{i,x}^n,$$

and that $h'_{j,1}, \cdots, h'_{j,m} \in C_G(y)$ depend on the sequence $\{g_{n+m}, \cdots, g_{n+1}\}$ by the following steps:

$$\gamma_{j,y} g_{n+m}^{-1} = h'_{j,1} \gamma_{j,y}^1, \quad \gamma_{j,y} g_{n+m-1}^{-1} = h'_{j,2} \gamma_{j,y}^2, \quad \cdots, \quad \gamma_{j,y} g_{n+1}^{-1} = h'_{j,m} \gamma_{j,y}^m.$$

Remark 7.3. (1) We prefer to the cup product formula in Theorem 7.1 since it is simpler.

(2) By Remark 5.2 (a), our cup product formula in Theorem 7.2 is consistent with Siegel and Witherspoon’s formula in [9, Theorem 5.1].

8. The $\Delta$ operator

Let $k$ be a field and $G$ a finite group. Recall that the group algebra $kG$ is a symmetric algebra with the bilinear form

$$(, ) : kG \times kG \rightarrow k,$$

$$(g, h) = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for $g, h \in G$. For $n \geq 1$, the operator $\Delta : HH^n(kG) \rightarrow HH^{n-1}(kG)$ on the Hochschild cohomology is induced by the following equation:

$$\langle \Delta(\varphi)(g_1, \cdots, g_{n-1}, g_n) = \sum_{i=1}^{n} (-1)^{(n-1)i} \langle \varphi(g_i, \cdots, g_{n-1}, g_{n+1}, \cdots, g_1, \cdots, g_i), 1 \rangle$$

for $\varphi \in C^n(kG) \cong Map(G^\times n, kG)$, $\Delta(\varphi) \in C^{n-1}(kG) \cong Map(G^\times n-1, kG)$. This operator together with the cup product $\cup$ and the Lie bracket $[, ]$ define a BV algebra structure on $HH^*(kG)$.

We know from Section 4 that, for a conjugacy class $C_x$ of $G$, $H^*_x = \bigoplus_{n \geq 0} H^*_x$ is a subcomplex of the Hochschild cohomology complex $H^*$, where

$$H^*_x = \{ \varphi : G^\times n \rightarrow kG | \varphi(g_1, \cdots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \cdots, g_n \in G \}. $$
Lemma 8.1. The operator $\triangle : \mathcal{H}^n \to \mathcal{H}^{n-1}$ restricts to $\triangle_x : \mathcal{H}^n_x \to \mathcal{H}^{n-1}_x$ for each conjugacy class $C_x$.

Proof We need to show that $\triangle(\varphi) \in \mathcal{H}^{n-1}_x$ for each $\varphi \in \mathcal{H}^n_x$. Suppose that $\triangle(\varphi)(g_1, \cdots, g_{n-1}) \neq 0$ for some $g_1, \cdots, g_{n-1} \in G$. Notice that $g_n \in G$ with $(\triangle(\varphi)(g_1, \cdots, g_{n-1}), g_n) \neq 0$ if and only if the coefficient of $g_n^{-1}$ in $\triangle(\varphi)(g_1, \cdots, g_{n-1})$ is nonzero. For each such $g_n$, there exists some $i$ such that $(\varphi(g_1, \cdots, g_{n-1}, g_i, \cdots, g_{n-1}, 1)) \neq 0$. This implies that $1 \in g_1 \cdots g_{n-1}g_1 \cdots g_{n-1}C_x$, or equivalently, $g_n^{-1} \in g_1 \cdots g_1g_1 \cdots g_{n-1}C_x$. It follows that $\triangle(\varphi)(g_1, \cdots, g_{n-1}) \in k[g_1 \cdots g_{n-1}C_x]$.

Now we can determine the behavior of the operator $\triangle$ under the additive decomposition.

Theorem 8.2. Let $k$ be a field and $G$ a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra $kG$:

$$HH^*(kG) \cong \bigoplus_{x \in X} H^*(C_G(x), k)$$

where $X$ is a set of representatives of conjugacy classes of elements of $G$ and $C_G(x)$ is the centralizer subgroup of $G$. Let $\widehat{\triangle}_x : H^n(C_G(x), k) \to H^{n-1}(C_G(x), k)$ be the map induced by the operator $\triangle_x : HH^n(kG) \to HH^{n-1}(kG)$. Then $\widehat{\triangle}_x$ is defined as follows:

$$\widehat{\triangle}_x(\psi)(h_1, \cdots, h_{n-1}) = \sum_{i=1}^{n} (-1)^{(n-1)}\psi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_{n}^{-1} \cdots h_{1}^{-1}x^{-1}, h_1, \cdots, h_{i-1})$$

for $\psi : C_G(x)^{\times n} \to k$ and for $h_1, \cdots, h_{n-1} \in C_G(x)$.

Proof We have the following commutative diagram

$$\begin{array}{ccc}
H^n(H^*_x) & \xrightarrow{\triangle_x} & H^{n-1}(H^*_x) \\
\downarrow & & \downarrow \\
H^n(C_G(x), k) & \xrightarrow{\widehat{\triangle}_x} & H^{n-1}(C_G(x), k).
\end{array}$$

Take an element $\psi : C_G(x)^{\times n} \to k$ in $\text{Hom}_{kC_G(x)}(Bar_n(kC_G(x)) \otimes C_G(x), k, k)$ and denote by $\varphi : G^{\times n} \to kG$ the corresponding element in $\mathcal{H}_n^x$. By Theorem 6.2, for any $h_1, \cdots, h_n \in C_G(x)$, $\psi(h_1, \cdots, h_n)$ is equal to the coefficient of $x$ in $\varphi(h_1, \cdots, h_n)h_n^{-1} \cdots h_1^{-1} \in kC_x$. On the other hand, $\triangle_x(\varphi)(g_1, \cdots, g_{n-1}) \in k[g_1 \cdots g_{n-1}C_x]$ is defined by the following equation:

$$\langle \triangle_x(\varphi)(g_1, \cdots, g_{n-1}), g_n \rangle = \sum_{i=1}^{n} (-1)^{(n-1)}\langle \varphi(g_i, \cdots, g_{n-1}, g_1, \cdots, g_{i-1}, 1) \rangle,$$

where $g_n \in G$. Under the vertical isomorphism, $\triangle_x(\varphi)$ corresponds to $\widehat{\triangle}_x(\psi)$. For any $h_1, \cdots, h_{n-1} \in C_G(x)$, $\widehat{\triangle}_x(\psi)(h_1, \cdots, h_{n-1})$ is the coefficient of $x$ in $\triangle_x(\varphi)(h_1, \cdots, h_{n-1})h_{n-1}^{-1} \cdots h_1^{-1} \in kC_x$, or equivalently, the coefficient of $xh_1 \cdots h_{n-1}$ in $\triangle_x(\varphi)(h_1, \cdots, h_{n-1}) \in k[h_1 \cdots h_{n-1}C_x]$. This coefficient is equal to

$$\langle \triangle_x(\varphi)(h_1, \cdots, h_{n-1}), h_{n-1}^{-1} \cdots h_1^{-1}x^{-1} \rangle$$

$$= \sum_{i=1}^{n} (-1)^{(n-1)}\langle \varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1}x^{-1}, h_1, \cdots, h_{i-1}, 1) \rangle.$$

We also know that $\psi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1}x^{-1}, h_1, \cdots, h_{i-1})$ is equal to the coefficient of $x$ in

$$\varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1}x^{-1}, h_1, \cdots, h_{i-1})h_{i-1}^{-1} \cdots h_1^{-1}x_1 \cdots h_{n-1} \cdots h_1^{-1} \cdots h_1^{-1}$$

$$= \varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1}x^{-1}, h_1, \cdots, h_{i-1})x \in kC_x,$$

which is again equal to $\langle \varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1}x^{-1}, h_1, \cdots, h_{i-1}) \rangle$. It follows that

$$\widehat{\triangle}_x(\psi)(h_1, \cdots, h_{n-1}) = \sum_{i=1}^{n} (-1)^{(n-1)}\psi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1}x^{-1}, h_1, \cdots, h_{i-1}).$$
Remark 8.3. By [5, Corollary 2.2], we know that $H^n(G, k)$ is a Gerstenhaber subalgebra of $HH^*(kG)$ under the inclusion map (cf. Remark ??):

$$Hom_{kG}(Bar_x(kG) \otimes_{kG} k, k) \hookrightarrow Hom_{k(G \otimes G)}(Bar_x(kG), kG),$$

$(\varphi : G^\times \rightarrow k) \mapsto (\psi : G^\times \rightarrow kG)$, where $\psi(g_1, \ldots, g_n) = \varphi(g_1, \ldots, g_n)g_1 \cdots g_n$.

Notice that by notations in Section 4, we can similarly define an operator $\Delta_1 : H^n(G, k) \rightarrow H^{n-1}(G, k)$ in the group cohomology $H^*(G, k)$ as follows:

$$\Delta_1(\varphi)(g_1, \ldots, g_{n-1}) = \sum_{i=1}^n (-1)^{i(n-1)}\varphi(g_1, \ldots, g_{n-1}, g_n^{-1} \cdots g_{i+1}^{-1}g_i, \ldots, g_{n-1})$$

for $\varphi : G^\times \rightarrow k$ and $g_1, \ldots, g_{n-1} \in G$.

We prove that $H^n(G, k)$ is in fact a BV subalgebra of $HH^*(kG)$.

Proposition 8.4. Let $k$ be a field and $G$ a finite group. Then $H^n(G, k) \hookrightarrow HH^*(kG)$ is a BV subalgebra.

Proof Suppose that under the above inclusion map $H^n(G, k) \hookrightarrow HH^*(kG)$, $\varphi : G^\times \rightarrow k$ corresponds to $\tilde{\varphi} : G^\times \rightarrow kG$, where $\tilde{\varphi}(g_1, \ldots, g_n) = \varphi(g_1, \ldots, g_n)g_1 \cdots g_n$. We need to show that $\Delta_1(\tilde{\varphi}) = \Delta(\tilde{\varphi})$. For $g_1, \ldots, g_{n-1} \in G$, $\Delta(\tilde{\varphi})(g_1, \ldots, g_{n-1})$

$$= \sum_{i=1}^n (-1)^{i(n-1)}\tilde{\varphi}(g_1, \ldots, g_{n-1}, g_n^{-1} \cdots g_{i+1}^{-1}g_i, \ldots, g_{n-1})$$

$$= \sum_{i=1}^n (-1)^{i(n-1)}\varphi(g_1, \ldots, g_{n-1}, g_n^{-1} \cdots g_{i+1}^{-1}g_i, \ldots, g_{n-1})$$

$$= \Delta_1(\varphi)(g_1, \ldots, g_{n-1}).$$

Now let $G$ be an abelian group. In this case, the Hochschild cohomology ring $HH^*(kG)$ of the group algebra $kG$ is isomorphic to the tensor product algebra of $kG$ and the group cohomology ring $H^*(G, k)$: $HH^*(kG) \cong kG \otimes_{k} H^*(G, k)$. According to [3], this isomorphism is given as follows.

For $G$ an abelian group, conjugacy classes are elements of $G$, hence a cochain $\varphi_x$ of $H^n_x$ for $x \in G$ attributes a scalar multiple of $g_1 \cdots g_n x$ for each $(g_1, \ldots, g_n) \in G^\times$; we denote by $\varphi_x(g_1, \ldots, g_n)$ the corresponding scalar and we obtain in this way a map $\varphi_x : G^\times \rightarrow k$. It is not difficult to verify that the map $\varphi \mapsto \varphi_x : G^\times \rightarrow kG$ defines a ring isomorphism $C^\ast(kG) \rightarrow kG \otimes C^\ast(kG, k)$ compatible with the differentials, and therefore it induces the above isomorphism.

Proposition 8.5. Let $k$ be a field and $G$ a finite abelian group. Under the above isomorphism $HH^*(kG) \cong kG \otimes_{k} H^*(G, k)$, the operator $\Delta : HH^*(kG) \rightarrow HH^{n-1}(kG)$ corresponds to the sum of operators $x \otimes \Delta_x : x \otimes H^n(G, k) \rightarrow x \otimes H^{n-1}(G, k)$, where $x \in G$ and $\Delta_x : H^n(G, k) \rightarrow H^{n-1}(G, k)$ is defined as follows:

$$\Delta_x(\varphi)(g_1, \ldots, g_{n-1}) = \sum_{i=1}^n (-1)^{i(n-1)}\varphi(g_1, \ldots, g_{n-1}, g_n^{-1} \cdots g_{i+1}^{-1}g_i, \ldots, g_{n-1})$$

for $\varphi : G^\times \rightarrow k$ and $g_1, \ldots, g_{n-1} \in G$.

Proof The proof is similar to Theorem 8.2. According to Lemma 8.1, the operator $\Delta : H^n \rightarrow H^{n-1}$ restricts to $\Delta_x : H^n_x \rightarrow H^{n-1}_x$ for each $x \in G$. Let $\varphi \in H^n(G, k)$ corresponds to $\varphi \in H^n_x$. It suffices to show that $\Delta_x(\varphi) = \Delta_x(\varphi)$. For $g_1, \ldots, g_{n-1} \in G$, $\Delta_x(\varphi)(g_1, \ldots, g_{n-1})$ is the coefficient of $g_1 \cdots g_{n-1} x$ in $\Delta_x(\varphi)(g_1, \ldots, g_{n-1})$, and it is equal to $
\langle \Delta_x(\varphi)(g_1, \ldots, g_{n-1}), x^{-1}g_{n-1}^{-1} \cdots g_1^{-1} \rangle$

\[\Delta(\varphi)(g_1, \ldots, g_{n-1}) = \Delta_x(\varphi)(g_1, \ldots, g_{n-1}) = \langle \Delta_x(\varphi)(g_1, \ldots, g_{n-1}), x^{-1}g_{n-1}^{-1} \cdots g_1^{-1} \rangle\]
\[ = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \ldots, g_{n-1}, x^{-1}g_{n-1}^{-1} \cdots g_1^{-1}, g_1, \ldots, g_{i-1}), 1 \rangle \]

\[ = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \ldots, g_{n-1}, x^{-1}g_{n-1}^{-1} \cdots g_1^{-1}, g_1, \ldots, g_{i-1}, x), x^{-1} \rangle \]

\[ = \sum_{i=1}^{n} (-1)^{i(n-1)} \varphi(g_i, \ldots, g_{n-1}, x^{-1}g_{n-1}^{-1} \cdots g_1^{-1}, g_1, \ldots, g_{i-1})
\]

\[ = \sum_{i=1}^{n} (-1)^{i(n-1)} \varphi(g_i, \ldots, g_{n-1}, g_{n-1}^{-1} \cdots g_1^{-1} x^{-1}, g_1, \ldots, g_{i-1})
\]

\[ = \Delta_x(\varphi)(g_1, \ldots, g_{n-1}). \]

9. THE LIE BRACKET

Then the Gerstenhaber algebra \((HH^*(A), [\ , \ ])\) together with the operator \(\Delta\) is a Batalin-Vilkovisky algebra (BV-algebra), that is, \((HH^*(A), \Delta)\) is a complex and

\[ [\alpha, \beta] = -(-1)^{|\alpha|-1}|\beta| (\Delta(\alpha \cup \beta) - \Delta(\alpha) \cup \beta - (-1)^{|\alpha|} \alpha \cup \Delta(\beta)) \]

for all homogeneous elements \(\alpha, \beta \in HH^*(A)\).

10. SOME EXAMPLES

References


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