THE BATALIN-VILKOVISKY STRUCTURE OVER THE HOCHSCHILD COHOMOLOGY RING OF A GROUP ALGEBRA

YUMING LIU AND GUODONG ZHOU*

Abstract. We realize explicitly the well-known additive decomposition of the Hochschild cohomology ring of a group algebra. As a result, we describe the cup product, the Batalin-Vilkovisky operator and the Lie bracket in the Hochschild cohomology ring of a group algebra.

1. Introduction

Let k be a field and G a finite group. Then the Hochschild cohomology ring of the group algebra kG admits an additive decomposition:

$$HH^*(kG) \cong \bigoplus_{x \in X} H^*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer of $x \in G$. The proof of this isomorphism can be found in [1] or [9]. The usual proof is abstract rather than giving an explicit isomorphism. For example, one of the key steps is to use the so-called Eckmann-Shapiro Lemma, one need to construct some comparison maps between two projective resolutions in order to write it down explicitly, and this is usually difficult. In [9], Siegel and Witherspoon used techniques and notations from group representation theory to interpret the above additive decomposition explicitly. For our purpose, we need to give an explicit isomorphism in the elements level.

A priori, the additive decomposition gives an isomorphism of vector spaces. The left handed side has a graded commutative algebra structure given by the cup product, a graded Lie algebra structure given by the Gerstenhaber Lie bracket ([6]), and a Batalin-Vilkovisky (BV) algebra structure given by the $\triangle$ operator ([10]). It would be interesting to describe these structures in terms of pieces from the right handed sided.

For graded algebra structure, it was done by Holm for abelian groups using computations ([7]), then Cibils and Solotar gave a conceptual proof in ([3]). The general case was dealt with by Siegel and Witherspoon ([9]), they described the cup product formula by notations from group representation theory. Our goal in the present paper is to represent the cup product, the Lie bracket and the BV operator in the Hochschild cohomology ring in terms of the additive decomposition. This is based on the explicit construction of an isomorphism in the additive decomposition (although there is no canonical choice for such an isomorphism).

The main obstruction in realizing an isomorphism in the additive decomposition comes from the fact that, it is usually difficult to construct the comparison map between two projective resolutions of modules. There is a surprising way to simplify such construction, namely, one can reduce it to construct a set-like self-homotopy over one projective resolution, which is often much easier. This method was already used in a recent paper by the second author jointly with Le ([8]). For convenience, we shall give here a brief introduction to this idea.

This article is organized as follows. In the second section, we review the definitions of Hochschild cohomology and cup product, using normalized bar resolutions. We use the normalized bar resolution since it is easy to describe and can greatly simplify the computations. We also recall the graded

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* Corresponding author.

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Lie algebra structure and the BV algebra structure (in case that A is a symmetric algebra) over the Hochschild cohomology ring $HH^*(A)$ of an algebra A. In the group algebra case, we will state the relationship of the Hochschild cohomology and the usual group cohomology.

In Section 3, we give a way to realize explicitly the additive decomposition of the Hochschild cohomology of a group algebra. The main line of our method follows from [9]. In Section 4, we shall use some idea from [3] to give another way to realize the additive decomposition.

We give the cup product formula in Section 5.

We deal with the $\triangle$ operator and the graded Lie bracket in the next two sections.

In the final section, we use our formulas to compute several concrete examples.

2. How to construct comparison morphisms

Definition 2.1. *Let A be an algebra over a field k. Let*

$$C^\bullet : \cdots \longrightarrow C^{n+1} \stackrel{d_{n+1}}{\longrightarrow} C^n \stackrel{d_n}{\longrightarrow} C^{n-1} \longrightarrow \cdots$$

be a chain complex of A-modules. If there are maps (just as maps between sets) $s_n : C_n \longrightarrow C_{n+1}$ such that $s_{n-1}d_n + d_{n+1}s_n = id_{C^n}$ for all n, then the maps $\{s_n\}$ are called a set-like self-homotopy over the complex $C^\bullet$.

Remark 2.2. There is a set-like self-homotopy over a complex $C^\bullet$ of A-modules if and only if $C^\bullet$ is an exact complex, that is, $C^\bullet$ is a zero object in the derived category $D(ModA)$. Compare this with the usual self-homotopy, which is equivalent to saying that $C^\bullet$ is a zero object in the homotopy category $K(ModA)$.

We will show how to use a set-like self-homotopy to construct a comparison map. Let M and N be two A-modules, and let $f : M \longrightarrow N$ be an A-module homomorphism. Suppose that $P^\bullet = (P_i, \partial_i)$ is a free resolution of M, and that $Q^\bullet = (Q_i, d_i)$ is a projective resolution of N. Suppose further that there is a set-like self-homotopy $s = \{s_n\}$:

$$\begin{tikzcd}
\cdots \arrow[r] & Q_2 \arrow[r, "d_2"] \arrow[d, "1"] \arrow[dl, "s_2"'] & Q_1 \arrow[r, "d_1"] \arrow[d, "1"] \arrow[dl, "s_1"'] & Q_0 \arrow[r, "d_0"] \arrow[d, "1"] \arrow[dl, "s_0"'] & N \arrow[r] \arrow[d, "1"] \arrow[dl, "s_{-1}"'] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2"] & Q_1 \arrow[r, "d_1"] & Q_0 \arrow[r, "d_0"] & N \arrow[r] & 0.
\end{tikzcd}$$

For each $i \geq 0$, choose a basis X_i for the free A-module P_i (the i-th term of $P^\bullet$). We define inductively the maps $f_i : X_i \longrightarrow Q_i$ as follows: for $x \in X_0, f_0(x) = s_{-1}f\partial_0(x)$; for $i > 1$ and for $x \in X_i, f_i(x) = s_{i-1}f_{i-1}\partial_i(x)$. Extending A-linearly the maps f_i we get A-homomorphisms $f_i : P_i \longrightarrow Q_i$. It is easy to verify that $\{f_i\}$ gives a chain map between the complexes $P^\bullet$ and $Q^\bullet$. We illustrate the above procedure in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2"] \arrow[d, "1"] & Q_1 \arrow[r, "d_1"] \arrow[d, "1"] \arrow[dl, "s_1"'] & Q_0 \arrow[r, "d_0"] \arrow[d, "1"] \arrow[dl, "s_0"'] & N \arrow[r] \arrow[d, "1"] \arrow[dl, "s_{-1}"'] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2"] & Q_1 \arrow[r, "d_1"] & Q_0 \arrow[r, "d_0"] & N \arrow[r] & 0.
\end{tikzcd}$$

We shall use the following standard homological fact.

Lemma 2.3. *Let A and B be two rings and let $F : ModA \longrightarrow ModB$ be an additive contravariant functor. If $C^\bullet$ and $D^\bullet$ are two projective resolutions of an A-module M, then the cochain complexes $FC^\bullet$ and $FD^\bullet$ of B-modules are homotopic. In particular, if $\varphi : C^\bullet \longrightarrow D^\bullet$ and $\psi : D^\bullet \longrightarrow C^\bullet$ are* *two chain maps inducing identity maps $1_M : M \longrightarrow M$, then $F\varphi : FD^\bullet \longrightarrow FC^\bullet$ and $F\psi : FC^\bullet \longrightarrow FD^\bullet$ are inverse homotopy equivalences.*

3. Remainder on Hochschild cohomology

In this section, we recall the definitions of various structures over Hochschild cohomology. For the cup product and the Lie bracket in the Hochschild cohomology ring, we refer to Gerstenhaber's original paper [6]; for the Batalin-Vilkovisky algebra structure, we refer to [10].

Let k be a field and A an associative k-algebra with identity 1_A. Denote by $\overline{A}$ the quotient space $A/(k \cdot 1_A)$. We shall write $\otimes$ for $\otimes_k$ and $A^{\otimes n}$ for the n-fold tensor product $A \otimes \cdots \otimes A$. The *normalized bar resolution* $(Bar_*(A), d_*)$ of A is a free resolution of A as A-A-bimodules, where

$$Bar_{-1}(A) = A, \text{ and for } n \geq 0, \quad Bar_n(A) = A \otimes \overline{A}^{\otimes n} \otimes A,$$

$$d_0 : Bar_0(A) = A \otimes A \longrightarrow A, \quad a_0 \otimes a_1 \longmapsto a_0 a_1 (\text{multiplication map}), \text{ and for } n \geq 1,$$

$$d_n : Bar_n(A) \longrightarrow Bar_{n-1}(A), \quad a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \longmapsto \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

The exactness of the normalized bar resolution is an easy consequence of the following fact: there is a set-like self-homotopy $s_n : Bar_n(A) \longrightarrow Bar_{n+1}(A)$ over $Bar_*(A)$ given by

$$s_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}.$$

Notice that here each s_n is just a right A-module homomorphism.

Let ${}_AM_A$ be an A-A-bimodule. Remember that any A-A-bimodule can be identified with a left module over the enveloping algebra $A^e = A \otimes A^{op}$. We have the *Hochschild cohomology complex* $(C^*(A, M), \delta^*)$:

$$C^n(A, M) = Hom_{A^e}(Bar_n(A), M) \cong Hom_k(\overline{A}^{\otimes n}, M), \quad \text{for } n \geq 0,$$

$$\delta^n : C^n(A, M) \longrightarrow C^{n+1}(A, M), \quad f \longmapsto \delta^n(f), \quad \text{where } \delta^n(f) \text{ sends } a_1 \otimes \cdots \otimes a_{n+1} \text{ to}$$

$$a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}.$$

For $n \geq 0$, the *degree-n Hochschild cohomology group* of the algebra A with coefficients in M is defined to be

$$HH^n(A, M) = H^n(C^*(A, M)) \cong Ext^n_{A^e}(A, M).$$

If in particular, $A = kG$ the group algebra of a finite group G, then the Hochschild cohomology complex $(C^*(A, M), \delta^*)$ has the following form:

$$C^n(kG, M) \simeq Hom_k(\overline{kG}^{\otimes n}, M) \simeq Map(\overline{G}^n, M), \quad \text{for } n \geq 0,$$

where $\overline{G} = G - \{1\}$ and $Map(\overline{G}^{\times n}, M)$ denotes all the maps between the sets $\overline{G}^{\times n}$ and M, and the differential is given by

$$\delta^n : C^n(kG, M) \longrightarrow C^{n+1}(kG, M), \quad f \longmapsto \delta^n(f), \quad \text{where } \delta^n(f) \text{ sends } (g_1, \cdots, g_{n+1}) \in \overline{G}^{n+1} \text{ to}$$

$$g_1 f(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1} f(g_1, \cdots, g_n) g_{n+1}.$$

When $M = A$ with the obvious A-A-bimodule structure, we write $C^n(A)$ (resp. $HH^n(A)$) for $C^n(A, A)$ (resp. $HH^n(A, A)$). Let $f \in C^n(A)$, $g \in C^m(A)$. Then the *cup product* $f \cup g \in C^{n+m}(A)$ is defined as follows:

$$f \cup g : \overline{A}^{\otimes(n+m)} \longrightarrow A, \quad a_1 \otimes \cdots \otimes a_{n+m} \longmapsto f(a_1 \otimes \cdots a_n) g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

This cup product is associative and induces a well-defined product over

$$HH^*(A) = \bigoplus_{n \geq 0} HH^n(A) = \bigoplus_{n \geq 0} Ext^n_{A^e}(A, A),$$

which is called the *Hochschild cohomology ring* of A. Moreover, $HH^*(A)$ is graded commutative, that is, $\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha$ for $\alpha \in HH^n(A)$ and $\beta \in HH^m(A)$.

The Lie bracket is defined as follows. Let $f \in C^n(A, M)$, $g \in C^m(A)$. If $n, m \geq 1$, then for $1 \leq i \leq n$, the so-called *brace operation* $f \circ_i g \in C^{n+m-1}(A, M)$ is defined by

$$f \circ_i g(a_1 \otimes \cdots a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});$$

if $n \geq 1$ and $m = 0$, then $g \in A$ and for $1 \leq i \leq n$, set

$$f \circ_i g(a_1 \otimes \cdots a_{n-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g \otimes a_i \otimes \cdots \otimes a_{n-1});$$

for any other case, set $f \circ_i g$ to be zero. Define

$$f \circ g = \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} f \circ_i g \in C^{n+m-1}(A, M)$$

and for $f \in C^n(A)$, $g \in C^m(A)$, define

$$[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f \in C^{n+m-1}(A).$$

The above $[\ ,\ ]$ induces a well-defined *(graded) Lie bracket* in Hochschild cohomology

$$[\ ,\ ] : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)$$

such that $(HH^*(A), \cup, [\ ,\ ])$ is a *Gerstenhaber algebra*, that is, the following three conditions hold:

- $(HH^*(A), \cup)$ is an associative algebra and it is graded commutative;
- $(HH^*(A), [\ ,\ ])$ is a graded Lie algebra;
- Possion rule: $[f \cup g, h] = [f, h] \cup g + (-1)^{|f|(|h|-1)} f \cup [g, h]$, where $|\cdot|$ denotes the degree.

We now assume that A is a *symmetric k-algebra*, that is, A is isomorphic to its dual $D(A) = Hom_k(A, k)$ as A^e-modules, or equivalently, if there exists a symmetric associative non-degenerate bilinear form $\langle\ ,\ \rangle : A \times A \longrightarrow k$. This bilinear form induces a duality between the Hochschild cohomology and the Hochschild homology. In fact, for any $n \geq 0$ there is an isomorphism between $HH^n(A)$ and $HH_n(A)$ induced by the following canonical isomorphisms

$$\mathrm{Hom}_k(A \otimes_{A^e} \mathrm{Bar}_n(A), k) \cong \mathrm{Hom}_{A^e}(\mathrm{Bar}_n(A), D(A)) \cong \mathrm{Hom}_{A^e}(\mathrm{Bar}_n(A), A).$$

Via this duality, we have, for $n \geq 1$, an operator $\triangle : HH^n(A) \longrightarrow HH^{n-1}(A)$ which corresponds to the *Connes' B-operator* on the Hochschild homology. More precisely, for any $f \in C^n(A)$, $\triangle(f) \in C^{n-1}(A)$ is given by the equation

$$\langle \triangle(f)(a_1 \otimes \cdots \otimes a_{n-1}), a_n \rangle = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle f(a_i \otimes \cdots \otimes a_{n-1} \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1}), 1 \rangle.$$

Then the Gerstenhaber algebra $(HH^*(A), \cup, [\ ,\ ])$ together with the operator $\triangle$ is a *Batalin-Vilkovsky algebra* (BV-algebra), that is, $(HH^*(A), \triangle)$ is a complex and

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)|\beta|}(\triangle(\alpha \cup \beta) - \triangle(\alpha) \cup \beta - (-1)^{|\alpha|} \alpha \cup \triangle(\beta))$$

for all homogeneous elements $\alpha, \beta \in HH^*(A)$.

4. Remainder on group cohomology

Let G be a finite group and U a left kG-module. The group cohomology of G with coefficient in U is defined to be $H^n(G, U) = \mathrm{Ext}^n_{kG}(k, U)$. The complex $Bar_*(kG) \otimes_{kG} \otimes k$ is the standard resolution of the trivial module k. In fact, as the set-like homotopy s_n over $Bar_*(kG)$ are right module homomorphisms, $Bar_*(kG) \otimes_{kG} k$ is exact and thus a projective resolution of $kG \otimes_{kG} k \simeq k$. We write the complex $C^*(G, U) = Hom_{kG}(Bar_*(kG) \otimes_{kG} k, U)$. Therefore, for $n \geq 0$,

$$\begin{aligned} C^n(G, U) &\simeq Hom_{kG}((kG \otimes \overline{kG}^{\otimes n} \otimes kG) \otimes_{kG} k, U) \simeq Hom_{kG}(kG \otimes \overline{kG}^{\otimes n}, U) \\ &\simeq Hom_k(\overline{kG}^{\otimes n}, U) \simeq Map(\overline{G}^{\times n}, U), \end{aligned}$$

where $\overline{G} = G - \{1\}$ and $Map(\overline{G}^{\times n}, U)$ denotes all the maps between the sets $\overline{G}^{\times n}$ and U, and the differential is given by

$$d_0(x)(g) = gx - x \quad (\text{for } x \in U \text{ and } g \in \overline{G})$$

and (for $\varphi : \overline{G}^{\times n} \longrightarrow U$ and $g_1, \cdots, g_{n+1} \in \overline{G}$)

$$d_n(\varphi)(g_1, \cdots, g_{n+1}) = g_1\varphi(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n}(-1)^i\varphi(g_1, \cdots, g_ig_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \cdots, g_n).$$

Of particular interest to us are the following two cases which relate group cohomology to Hochschild cohomology and in fact which underly our two realisations of the additive decomposition of the Hochschild cohomology of a group algebra.

Let kG be a group algebra and U a kG-kG-bimodule. Note that we have an algebra isomorphism $(kG)^e \cong k(G \times G)$ given by $g_1 \otimes g_2 \longmapsto (g_1, g_2^{-1})$, for $g_1, g_2 \in G$. Thus we can also identify each kG-kG-bimodule M as a left $k(G \times G)$-module: $(g_1, g_2) \cdot x = g_1xg_2^{-1}$. In the sequel, we shall write the Hochschild cohomology complex for the group algebra kG in terms of $k(G \times G)$-modules.

Case 1. $M = kG$, the module kG with the obvious kG-kG-bimodule, or equivalently, the $k(G\times G)$-module kG with action: $(g_1, g_2) \cdot x = g_1xg_2^{-1}$ for $g_1, g_2 \in G$. It is easy to verify that there is a $k(G \times G)$-module isomorphism $Ind_G^{G\times G}k = k(G \times G) \otimes_{kG} k \cong kG$ $((g_1, g_2) \otimes 1 \longmapsto g_1g_2^{-1})$, where the right kG-module structure over $k(G \times G)$ is given by diagonal action. So we have

$$\begin{aligned} HH^n(kG, kG) &\simeq Ext^n_{k(G\times G)}(kG, kG) \cong Ext^n_{k(G\times G)}(Ind_G^{G\times G}k, kG) \\ &\cong Ext^n_{kG}(k, Res_G^{G\times G}kG) = Ext^n_{kG}(k, kG^c) \\ &= H^n(G, kG^c), \end{aligned}$$

where the third isomorphism is given by the adjoint equivalence and kG^c is considered as a left kG-module by conjugation: $g \cdot x = gxg^{-1}$ for $g, x \in G$. This verifies a well-known fact observed by Eilenberg and Mac Lane ([4]): the Hochschild cohomology $HH^n(kG, kG)$ of kG with coefficients in kG is isomorphic to the ordinary group cohomology $H^n(G, kG)$ of G with coefficients in kG under the conjugation.

Case 2. $M = k$, the trivial kG-kG-bimodule, or equivalently, the $k(G \times G)$-module k with action: $(g_1, g_2) \cdot 1 = 1$ for $g_1, g_2 \in G$. Since we have

$$HH^n(kG, k) \simeq Ext^n_{k(G\times G)}(kG, k) \cong Ext^n_{k(G\times G)}(k(G \times G) \otimes_{kG} k, k) \cong Ext^n_{kG}(k, k) = H^n(G, k),$$

the Hochschild cohomology $HH^n(kG, k)$ of kG with coefficients in k is isomorphic to the ordinary group cohomology $H^n(G, k)$. Another way to see this iies in the fact that the two complexes $C^*(kG, k)$ and $C^*(G, k)$ coincide.

et kG be a group algebra. Then kG is a symmetric algebra with the bilinear form

$$\langle\ ,\ \rangle : kG \times kG \longrightarrow k,$$

$$\langle g, h\rangle = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for $g, h \in G$. So there is a well-defined BV-algebra structure on $HH^*(kG)$. On the other hand, since the ordinary group cohomology $H^n(G, k)$ is isomorphic to the Hochschild cohomology $HH^n(kG, k)$ of kG with coefficients in the trivial module k, we can define a cup product on the group cohomology

$$H^*(G, k) = \bigoplus_{n\geq 0} H^n(G, k)$$

in a similar way:

$$\varphi_1 \cup \varphi_2 : \overline{G}^{\times n+m} \longrightarrow k, \quad (g_1, \cdots, g_{n+m}) \longmapsto \varphi_1(g_1, \cdots, g_n)\varphi_2(g_{n+1}, \cdots, g_{n+m}),$$

where $\varphi_1 \in Map(\overline{G}^{\times n}, k)$, and $\varphi_2 \in Map(\overline{G}^{\times m}, k)$. By [5, Corollary 2.2], $H^n(G, k)$ is a Gerstenhaber subalgebra of $HH^*(kG)$. In fact, as in [5, Proof of Theorem 1.8], there is a chain map at the cohomology complex level:

$$Hom_{kG}(Bar_*(kG) \otimes_{kG} k, k) \hookrightarrow Hom_{k(G\times G)}(Bar_*(kG), kG),$$

$$(\varphi : \overline{G}^{\times n} \longrightarrow k) \longmapsto (\psi : \overline{G}^{\times n} \longrightarrow kG), \psi(g_1, \cdots, g_n) = \varphi(g_1, \cdots, g_n)g_1 \cdots g_n.$$

This inclusion map preserves the brace operations in the following sense:

Let $\varphi_1, \varphi_2 \in C^n(kG, k) \cong Map(\overline{G}^{\times n}, k)$, and $\widehat{\varphi}_1, \widehat{\varphi}_2 \in C^m(kG)$ be the corresponding elements under the above inclusion map. Then $\widehat{\varphi}_1 \circ_i \widehat{\varphi}_2 = \widehat{\varphi_1 \circ_i \widehat{\varphi}_2} \in C^{n+m-1}(kG)$.

Notice that if G is an abelian group, then there is an isomorphism of graded rings (see [7] or [3]):

$$HH^*(kG,kG)\cong kG\otimes_k H^*(G,k).$$

5. An realization of the additive decomposition

Let k be a field and G a finite group. Then the Hochschild cohomology ring of the group algebra kG admits an additive decomposition:

$$HH^*(kG)\cong \bigoplus_{x\in X} H^*(C_G(x),k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of G. In this section, we give an explicit construction of the additive decomposition. The main technique we used here is to construct comparison maps based some set-like self-homotopys. (cf. Section 1: Introduction.)

By definition, $HH^*(kG)=H^*(Hom_{(kG)^e}(Bar_*(kG),kG))$. Since there is an algebra isomorphism $(kG)^e\cong k(G\times G)$ by $g_1\otimes g_2\longmapsto (g_1,g_2^{-1})$, we can identify $H^*(Hom_{(kG)^e}(Bar_*(kG),kG))$ with $H^*(Hom_{k(G\times G)}(Bar_*(kG),kG))$, where the $k(G\times G)$-module structure over kG is given by $(g_1,g_2)\cdot x=g_1xg_2^{-1}$ (the same applies to the terms in $Bar_*(kG)$). Since we always use the normalized bar resolutions, we have

$$Hom_{k(G\times G)}(Bar_n(kG),kG)\cong Hom_k(\overline{kG}^{\otimes n},kG)\cong Map(\overline{G}^{\times n},kG).$$

It follows that $HH^*(kG)$ is the cohomology of the following cochain complex:

$$0\longrightarrow kG\xrightarrow{d_0} Map(\overline{G},kG)\xrightarrow{d_1}\cdots\longrightarrow Map(\overline{G}^{\times n},kG)\xrightarrow{d_n}\cdots,$$

where the differential is given by

$$d_0(x)(g)=gx-xg\quad (\text{for } x\in kG \text{ and } g\in\overline{G})$$

and (for $\varphi:\overline{G}^{\times n}\longrightarrow kG$ and $g_1,\cdots,g_{n+1}\in\overline{G}$)

$$\begin{aligned}d_n(\varphi)(g_1,\cdots,g_{n+1})=g_1\varphi(g_2,\cdots,g_{n+1})+&\\ \sum_{i=1}^{n}(-1)^i\varphi(g_1,\cdots,g_ig_{i+1},\cdots,g_{n+1})&+(-1)^{n+1}\varphi(g_1,\cdots,g_n)g_{n+1}.\end{aligned}$$

The main line of our method follows from [9], there the formula is given between cohomology groups using standard operations like restriction, induction, conjugation, etc., while we choose some special projective resolutions and construct maps directly on the cohomology complex level. We will divide our construction into six steps.

The first step. Let kG be the $k(G\times G)$-module in $HH^*(kG)=Hom_{k(G\times G)}(Bar_*(kG),kG)$. Then we have a $k(G\times G)$-module isomorphism $k(G\times G)\otimes_{kG}k\cong kG$ given by $(g_1,g_2)\otimes 1\longmapsto g_1g_2^{-1}$, where kG operates diagonally on $k(G\times G)$ from the right. Moreover, $k(G\times G)\otimes_{kG}Bar_*(kG)\otimes_{kG}k$ is also a free resolution of the above $k(G\times G)$-module kG. Notice that here and in the following we still view the terms in $Bar_*(kG)$ as the usual kG-kG-bimodules. If we identify $k(G\times G)\otimes_{kG}Bar_n(kG)\otimes_{kG}k$ with $kG\otimes kG\otimes\overline{kG}^{\otimes n}$, then the differential is as follows:

$$kG\otimes kG\longrightarrow kG,\quad x\otimes y\longmapsto xy^{-1};$$

$$kG\otimes kG\otimes\overline{kG}\longrightarrow kG\otimes kG,\quad x\otimes y\otimes g_1\longmapsto xg_1\otimes yg_1-x\otimes y;$$

$$\cdots\cdots\cdots$$

$$kG\otimes kG\otimes\overline{kG}^{\otimes n}\longrightarrow kG\otimes kG\otimes\overline{kG}^{\otimes n-1},\quad x\otimes y\otimes g_1\otimes\cdots\otimes g_n\longmapsto$$

$$xg_1\otimes yg_1\otimes g_2\otimes\cdots\otimes g_n+\sum_{i=1}^{n-1}(-1)^i x\otimes y\otimes g_1\otimes\cdots\otimes g_ig_{i+1}\otimes\cdots\otimes g_n+(-1)^n x\otimes y\otimes g_1\otimes\cdots\otimes g_{n-1}.$$

We also have

$$Hom_{k(G\times G)}(k(G\times G)\otimes_{kG}Bar_n(kG)\otimes_{kG}k,kG)\cong Hom_k(\overline{kG}^{\otimes n},kG)\cong Map(\overline{G}^{\times n},kG).$$

Using this identification, $H^*(Hom_{k(G\times G)}(k(G\times G)\otimes_{kG}Bar_*(kG)\otimes_{kG}k, kG))$ is given by the following cochain complex:

$$0 \longrightarrow kG \xrightarrow{d_0'} Map(\overline{G}, kG) \xrightarrow{d_1'} \cdots \longrightarrow Map(\overline{G}^{\times n}, kG) \xrightarrow{d_n'} \cdots,$$

where the differential is given by

$$d_0'(x)(g) = gxg^{-1} - x \quad (\text{for } x \in kG \text{ and } g \in \overline{G})$$

and (for $\varphi : \overline{G}^{\times n} \longrightarrow kG$ and $g_1, \cdots, g_{n+1} \in \overline{G}$)

$$d_n'(\varphi)(g_1, \cdots, g_{n+1}) = g_1\varphi(g_2, \cdots, g_{n+1})g_1^{-1} + \sum_{i=1}^{n}(-1)^i\varphi(g_1, \cdots, g_ig_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \cdots, g_n).$$

By standard homological algebra, we know that

$$H^*(Hom_{k(G\times G)}(Bar_*(kG), kG)) \cong H^*(Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG) \otimes_{kG} k, kG)). \tag{1}$$

Notice that there is no canonical choice for the isomorphism in (1). However, we can always choose a "natural" one and fix it in this paper. We will give an explicit isomorphism in (1) and its inverse, based on the construction of the comparison maps between the two free resolutions $Bar_*(kG)$ and $k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k$ of the above $k(G\times G)$-module kG. As explained in Introduction, this is reduced to construct set-like self-homotopys over these resolutions. Our principle here is to choose those set-like self-homotopys so that the computations and results are simple.

We choose a set-like self-homotopy over $Bar_*(kG)$ as

$$u_n : kG \otimes \overline{kG}^{\otimes n} \otimes kG \longrightarrow kG \otimes \overline{kG}^{\otimes n} \otimes kG,$$
$$g_0 \otimes g_1 \otimes \cdots \otimes g_{n+1} \longmapsto (-1)^{n+1} g_0 \otimes g_1 \otimes \cdots \otimes g_{n+1} \otimes 1.$$

Using $\{u_n\}$ we can construct a comparison map

$$\alpha : k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k \longrightarrow Bar_*(kG)$$

as follows (we only write down the maps on basis vectors):

$$\alpha_{-1} : kG \longrightarrow kG, x \longmapsto x,$$
$$\alpha_0 : kG\otimes kG \longrightarrow kG\otimes kG, x\otimes y \longmapsto x\otimes y^{-1},$$
$$\alpha_1 : kG\otimes kG\otimes \overline{kG} \longrightarrow kG\otimes\overline{kG}\otimes kG, x\otimes y\otimes g_1 \longmapsto -xg_1\otimes g_1^{-1}\otimes y^{-1},$$
$$\cdots\cdots\cdots$$
$$\alpha_n : kG\otimes kG\otimes\overline{kG}^{\otimes n} \longrightarrow kG\otimes\overline{kG}^{\otimes n}\otimes kG, x\otimes y\otimes g_1\otimes\cdots\otimes g_n \longmapsto (-1)^{\frac{n(n+1)}{2}} xg_1\cdots g_n\otimes g_n^{-1}\otimes\cdots\otimes g_1^{-1}\otimes y^{-1}.$$

Similarly, we choose a set-like self-homotopy over $k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k$ as

$$v_n : kG\otimes kG\otimes\overline{kG}^{\otimes n} \longrightarrow kG\otimes kG\otimes\overline{kG}^{\otimes n},$$
$$x\otimes y\otimes g_1\otimes\cdots\otimes g_n \longmapsto xy^{-1}\otimes 1\otimes y\otimes g_1\otimes\cdots\otimes g_n.$$

Using $\{v_n\}$ we can construct a comparison map

$$\beta : Bar_*(kG) \longrightarrow k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k$$

as follows (we only write down the maps on basis vectors):

$$\beta_{-1} : kG \longrightarrow kG, x \longmapsto x,$$
$$\beta_0 : kG\otimes kG \longrightarrow kG\otimes kG, x\otimes y \longmapsto x\otimes y^{-1},$$
$$\beta_1 : kG\otimes\overline{kG}\otimes kG \longrightarrow kG\otimes kG\otimes\overline{kG}, x\otimes g_1\otimes y \longmapsto -xg_1\otimes y^{-1}\otimes g_1^{-1},$$
$$\cdots\cdots\cdots$$
$$\beta_n : kG\otimes\overline{kG}^{\otimes n}\otimes kG \longrightarrow kG\otimes kG\otimes\overline{kG}^{\otimes n}, x\otimes g_1\otimes\cdots\otimes g_n\otimes y \longmapsto (-1)^{\frac{n(n+1)}{2}} xg_1\cdots g_n\otimes y^{-1}\otimes g_n^{-1}\otimes\cdots\otimes g_1^{-1}.$$

It is easy to check that the chain maps $\{\alpha_n\}$ and $\{\beta_n\}$ are inverse to each other, and therefore we get an isomorphism

$$Hom_{k(G\times G)}(Bar_*(kG), kG) \longrightarrow Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k, kG),$$
$$(\varphi : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_1 : \overline{G}^{\times n} \longrightarrow kG), \varphi_1(g_1, \cdots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1\cdots g_n\varphi(g_n^{-1}, \cdots, g_1^{-1}).$$

Its inverse is given by

$$Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k, kG) \longrightarrow Hom_{k(G\times G)}(Bar_*(kG), kG),$$

$$(\varphi_1 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi : \overline{G}^{\times n} \longrightarrow kG), \varphi(g_1, \cdots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1\cdots g_n\varphi_1(g_n^{-1}, \cdots, g_1^{-1}).$$

Passing to the cohomology, we realize an isomorphism in (1) and its inverse.

The second step. Since $(k(G\times G)\otimes_{kG} -, Hom_{k(G\times G)}(k(G\times G), -))$ is an adjoint pair, we have an isomorphism (Note that $k(G\times G)$ is viewed as a right kG-module by diagonal action)

$$Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k, kG) \cong Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kG),$$

where the kG-module structure on the last kG is given by conjugation: $g\cdot x = gxg^{-1}$ for $g\in G$. Passing to the cohomology, we get an isomorphism

$$H^*(Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k, kG)) \cong H^*(Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kG)). \tag{2}$$

Remind that the right hand side is just the ordinary group cohomology $H^*(G, kG)$ of G with coefficients in kG under the conjugation (cf. Case 1 of Example **??**). We also have

$$Hom_{kG}(Bar_n(kG)\otimes_{kG} k, kG) \cong Hom_{kG}(kG\otimes \overline{kG}^{\otimes n}, kG) \cong Hom_k(\overline{kG}^{\otimes n}, kG) \cong Map(\overline{G}^{\times n}, kG).$$

Using this identification, $H^*(G, kG) = H^*(Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kG))$ is given by the following cochain complex:

$$0 \longrightarrow kG \xrightarrow{d_0} Map(\overline{G}, kG) \xrightarrow{d_1} \cdots \longrightarrow Map(\overline{G}^{\times n}, kG) \xrightarrow{d_n} \cdots,$$

where the differential is given by

$$d_0(x)(g) = gxg^{-1} - x \quad (\text{for } x\in kG \text{ and } g\in \overline{G})$$

and (for $\varphi : \overline{G}^{\times n} \longrightarrow kG$ and $g_1, \cdots, g_{n+1} \in \overline{G}$)

$$d_n(\varphi)(g_1, \cdots, g_{n+1}) = g_1\varphi(g_2, \cdots, g_{n+1})g_1^{-1} + \sum_{i=1}^{n}(-1)^i\varphi(g_1, \cdots, g_ig_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \cdots, g_n).$$

So formally the left hand side and the right hand side in (2) are identical,though they have different meaning. It is also easy to check that under the above identifications, the adjoint isomorphisms are identity maps:

$$Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k, kG) \longrightarrow Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kG),$$

$$(\varphi_1 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_2 : \overline{G}^{\times n} \longrightarrow kG), \varphi_2(g_1, \cdots, g_n) = \varphi_1(g_1, \cdots, g_n).$$

Its inverse is given by

$$Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kG) \longrightarrow Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k, kG),$$

$$(\varphi_2 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_1 : \overline{G}^{\times n} \longrightarrow kG), \varphi_1(g_1, \cdots, g_n) = \varphi_2(g_1, \cdots, g_n).$$

Passing to the cohomology, we realize an isomorphism in (2) and its inverse.

The third step. We choose a complete set X of representatives of the conjugacy classes in the finite group G. Take $x\in X$. Then $C_x = \{gxg^{-1} | g\in G\}$ is the conjugacy class corresponding to x and $C_G(x) = \{g\in G | gxg^{-1} = x\}$ is the centralizer subgroup. Clearly the k-space kC_x generated by the elements in C_x is a left kG-module under the conjugation action. We choose a right coset decomposition of $C_G(x)$ in G: $G = C_G(x)\gamma_{1,x} \cup C_G(x)\gamma_{2,x} \cup \cdots \cup C_G(x)\gamma_{n_x,x}$ (equivalently, $G = \gamma_{1,x}^{-1}C_G(x) \cup \gamma_{2,x}^{-1}C_G(x) \cup \cdots \cup \gamma_{n_x,x}^{-1}C_G(x)$ is a left coset decomposition of $C_G(x)$ in G), and such that $C_x = \{x = \gamma_{1,x}^{-1}x\gamma_{1,x}, \gamma_{2,x}^{-1}x\gamma_{2,x}, \cdots, \gamma_{n_x,x}^{-1}x\gamma_{n_x,x}\}$. (We will always take $\gamma_{1,x} = 1$, and we write x_i for $\gamma_{i,x}^{-1}x\gamma_{i,x}$.) Then we have the following kG-module isomorphisms:

$$kC_x \cong kG\otimes_{kC_G(x)} k, \quad x_i \longmapsto \gamma_{i,x}^{-1}\otimes_{kC_G(x)} 1,$$

$$kC_x \cong Hom_{kC_G(x)}(kG, k), \quad x_i \longmapsto \gamma_i : kG \longrightarrow k, \gamma_i(\gamma_{j,x}) = \delta_{ij},$$

where in the first isomorphism, the left kG-module structure on kG is the usual left multiplication and the right $kC_G(x)$-module structure on kG is given by restriction, and k is the trivial $kC_G(x)$-module, and the similar as in the second isomorphism.

In the second step, we have arrived at the ordinary group cohomology $H^*(G, kG)$ of G with coefficients in kG, where the kG-module structure over kG is given by the conjugation. This kG has a kG-module decomposition:

$$kG = \bigoplus_{x\in X} kC_x.$$

Denote by $\pi_x : kG \longrightarrow kC_x$ and $i_x : kC_x \longrightarrow kG$ the canonical projection and the canonical injection, respectively. Then we have the following isomorphism

$$Hom_{kG}(Bar_*(kG) \otimes_{kG} k, kG) \longrightarrow Hom_{kG}(Bar_*(kG) \otimes_{kG} k, \bigoplus_{x\in X} kC_x),$$

$$(\varphi_2 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto \varphi_3 = \{\varphi_{3,x} | x \in X\}, \text{ where } \varphi_{3,x} = \pi_x\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x.$$

Its inverse is given by

$$Hom_{kG}(Bar_*(kG) \otimes_{kG} k, \bigoplus_{x\in X} kC_x) \longrightarrow Hom_{kG}(Bar_*(kG) \otimes_{kG} k, kG),$$

$$\varphi_3 = \{\varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kC_x | x \in X\} \longmapsto (\varphi_2 = \sum_{x\in X} i_x\varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kG).$$

Passing to the cohomology, we realize an isomorphism:

$$(3) \quad H^*(G, kG) \cong \bigoplus_{x\in X} H^*(G, kC_x),$$

where the kG-module structure over kG is given by the conjugation.

The fourth step. We have stated in the third step the following kG-module isomorphism

$$kC_x \cong Hom_{kC_G(x)}(kG, k), \quad x_i \longmapsto \gamma_i : kG \longrightarrow k, \gamma_i(\gamma_{j,x}) = \delta_{ij}.$$

Therefore we have the following isomorphism

$$Hom_{kG}(Bar_*(kG) \otimes_{kG} k, kC_x) \longrightarrow Hom_{kG}(Bar_*(kG) \otimes_{kG} k, Hom_{kC_G(x)}(kG, k)),$$

$$(\varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kC_x) \longmapsto (\varphi_{4,x} : \overline{G}^{\times n} \longrightarrow Hom_{kC_G(x)}(kG, k)),$$

where if we write $\varphi_{3,x}(g_1, g_2, \cdots, g_n) = \sum_{i=1}^{n_x} a_{i,x}x_i$, then $\varphi_{4,x}(g_1, g_2, \cdots, g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i. The inverse isomorphism is given by

$$Hom_{kG}(Bar_*(kG) \otimes_{kG} k, Hom_{kC_G(x)}(kG, k)) \longrightarrow Hom_{kG}(Bar_*(kG) \otimes_{kG} k, kC_x),$$

$$(\varphi_{4,x} : \overline{G}^{\times n} \longrightarrow Hom_{kC_G(x)}(kG, k)) \longmapsto (\varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kC_x),$$

where if $\varphi_{4,x}(g_1, g_2, \cdots, g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i, then $\varphi_{3,x}(g_1, g_2, \cdots, g_n) = \sum_{i=1}^{n_x} a_{i,x}x_i$. Passing to the cohomology, we realize an isomorphism:

$$(4) \quad H^*(G, kC_x) \cong H^*(Hom_{kG}(Bar_*(kG) \otimes_{kG} k, Hom_{kC_G(x)}(kG, k))).$$

The fifth step. Since $(kG \otimes_{kG} -, Hom_{kC_G(x)}(kG, -))$ is an adjoint pair, we have the following isomorphism

$$Hom_{kG}(Bar_*(kG) \otimes_{kG} k, Hom_{kC_G(x)}(kG, k)) \longrightarrow Hom_{kC_G(x)}(Bar_*(kG) \otimes_{kG} k, k).$$

Passing to the cohomology, we get an isomorphism

$$(5) \quad H^*(Hom_{kG}(Bar_*(kG) \otimes_{kG} k, Hom_{kC_G(x)}(kG, k))) \cong H^*(Hom_{kC_G(x)}(Bar_*(kG) \otimes_{kG} k, k)),$$

where the right hand side is isomorphic to the ordinary group cohomology $H^*(C_G(x), k)$ of $C_G(x)$ with coefficients in the trivial module k. Since there are $kC_G(x)$-module isomorphisms

$$Bar_*(kG) \otimes_{kG} k \cong \bigoplus_{i=1}^{n_x} kC_G(x)\gamma_{i,x} \otimes \overline{kG}^{\otimes n},$$

we have

$$Hom_{kC_G(x)}(Bar_*(kG)\otimes_{kG}k,k)\cong Hom_k(\bigoplus_{i=1}^{n_x}k\gamma_{i,x}\otimes\overline{kG}^{\otimes n},k)\cong Map(S_x\times\overline{G}^{\times n},k),$$

where $S_x=\{\gamma_{1,x},\cdots,\gamma_{n_x,x}\}$ (cf. The third step). Using this identification, the adjoint isomorphism is given by

$$Hom_{kG}(Bar_*(kG)\otimes_{kG}k,Hom_{kC_G(x)}(kG,k))\longrightarrow Hom_{kC_G(x)}(Bar_*(kG)\otimes_{kG}k,k),$$

$$(\varphi_{4,x}:\overline{G}^{\times n}\longrightarrow Hom_{kC_G(x)}(kG,k))\longmapsto(\varphi_{5,x}:S_x\times\overline{G}^{\times n}\longrightarrow k),$$

where if $\varphi_{4,x}(g_1,g_2,\cdots,g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i, then $\varphi_{5,x}(\gamma_{i,x},g_1,g_2,\cdots,g_n)=a_{i,x}$ for any i. The inverse isomorphism is given by

$$Hom_{kC_G(x)}(Bar_*(kG)\otimes_{kG}k,k)\longrightarrow Hom_{kG}(Bar_*(kG)\otimes_{kG}k,Hom_{kC_G(x)}(kG,k)),$$

$$(\varphi_{5,x}:S_x\times\overline{G}^{\times n}\longrightarrow k)\longmapsto(\varphi_{4,x}:\overline{G}^{\times n}\longrightarrow Hom_{kC_G(x)}(kG,k)),$$

where if $\varphi_{5,x}(\gamma_{i,x},g_1,g_2,\cdots,g_n)=a_{i,x}$ for any i, then $\varphi_{4,x}(g_1,g_2,\cdots,g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i. Passing to the cohomology, we realize an isomorphism in (5) and its inverse.

The sixth step. In the fifth step, we have arrived at the ordinary group cohomology $H^*(C_G(x),k)$ of $C_G(x)$ with coefficients in the trivial module k, where $H^*(C_G(x),k)$ is computed by the cochain complex $Hom_{kC_G(x)}(Bar_*(kG)\otimes_{kG}k,k)$. By the identification in fifth step, this is given by the following cochain complex:

$$0\longrightarrow k^{\times n_x}\xrightarrow{d_0}Map(S_x\times\overline{G},k)\xrightarrow{d_1}\cdots\longrightarrow Map(S_x\times\overline{G}^{\times n},k)\xrightarrow{d_n}\cdots,$$

where the differential is given by $d_0(\{a_{i,x}\})((\gamma_{j,x},g_1))=a_{s_j,x}-a_{j,x}$, and $a_{s_j,x}$ is determined as follows: for $\{a_{i,x}\}\in k^{\times n_x},\gamma_{j,x}\in S_x,g_1\in\overline{G}$, we have

$$\gamma_{j,x}g_1=h_{j,1}\gamma_{s_j,x}\text{ for some }h_{j,1}\in C_G(x)\text{ and for some }1\le s_j\le n_x.$$

and by (for $\varphi:S_x\times\overline{G}^{\times n}\longrightarrow k$, $\gamma_{j,x}\in S_x,g_1,\cdots,g_{n+1}\in\overline{G}$ such that $\gamma_{j,x}g_1=h_{j,1}\gamma_{s_j,x}$)

$$d_n(\varphi)(\gamma_{j,x},g_1,\cdots,g_{n+1})=\varphi(\gamma_{s_j,x},g_2,\cdots,g_{n+1})+\sum_{i=1}^{n}(-1)^i\varphi(\gamma_{j,x},g_1,\cdots,g_ig_{i+1},\cdots,g_{n+1})+(-1)^{n+1}\varphi(\gamma_{j,x},g_1,\cdots,g_n).$$

(Remark that for a fixed $g_1\in\overline{G}$, $\{s_1,s_2,\cdots,s_{n_x}\}$ is a permutation of $\{1,2,\cdots,n_x\}$.)

The above computation for $H^*(C_G(x),k)$ uses the projective resolution $Bar_*(kG)\otimes_{kG}k$ of the trivial $kC_G(x)$-module k, which is identified as the following complex (It is in fact a projective resolution of the trivial kG-module k, but we view it as a complex of $kC_G(x)$-modules by restriction)

$$\cdots\longrightarrow kG\otimes\overline{kG}^{\otimes n}\xrightarrow{d_n}\cdots\longrightarrow kG\otimes\overline{kG}\xrightarrow{d_1}kG\xrightarrow{d_0}k\longrightarrow 0,$$

where the differential is given by

$$d_0(g_0)=1\quad(\text{for }g_0\in G)$$

and (for $g_0\in G,g_1,\cdots,g_n\in\overline{G}$)

$$d_n(g_0,g_1,\cdots,g_n)=g_0g_1\otimes g_2\otimes\cdots\otimes g_n+\sum_{i=1}^{n-1}(-1)^ig_0\otimes\cdots\otimes g_ig_{i+1}\otimes\cdots\otimes g_n+(-1)^ng_0\otimes g_1\otimes\cdots\otimes g_{n-1}.$$

We now use another projective resolution $Bar_*(kC_G(x))\otimes_{kC_G(x)}k$ of the trivial $kC_G(x)$-module k, which is identified as the following complex

$$\cdots\longrightarrow kC_G(x)\otimes\overline{kC_G(x)}^{\otimes n}\xrightarrow{d_n}\cdots\longrightarrow kC_G(x)\otimes\overline{kC_G(x)}\xrightarrow{d_1}kC_G(x)\xrightarrow{d_0}k\longrightarrow 0,$$

where the differential is given by

$$d_0(h_0)=1\quad(\text{for }h_0\in\overline{C_G(x)})$$

and (for $h_0 \in G, h_1, \cdots, h_n \in \overline{C_G(x)}$)

$$d_n(h_0, h_1, \cdots, h_n) = h_0h_1 \otimes h_2 \otimes \cdots \otimes h_n + \sum_{i=1}^{n-1}(-1)^i h_0 \otimes \cdots \otimes h_ih_{i+1} \otimes \cdots \otimes h_n + (-1)^n h_0 \otimes h_1 \otimes \cdots \otimes h_{n-1}.$$

We have

$$Hom_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k) \cong Hom_k(kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n}, k) \cong Map(\overline{C_G(x)}^{\times n}, k),$$

so $H^*(C_G(x), k)$ can also be computed by the following cochain complex

$$0 \longrightarrow k \xrightarrow{d_0} Map(\overline{C_G(x)}, k) \xrightarrow{d_1} \cdots \longrightarrow Map(\overline{C_G(x)}^{\times n}, k) \xrightarrow{d_n} \cdots,$$

where the differential is given by

$$d_0(a)(h_1) = 0 \qquad (\text{for } a \in k, h_1 \in \overline{C_G(x)})$$

and (for $\varphi : \overline{C_G(x)}^{\times n} \longrightarrow k,\ h_1, \cdots, h_{n+1} \in \overline{C_G(x)}$)

$$d_n(\varphi)(h_1, \cdots, h_{n+1}) = \varphi(h_2, \cdots, h_{n+1}) + \sum_{i=1}^{n}(-1)^i \varphi(h_1, \cdots, h_ih_{i+1}, \cdots, h_{n+1}) + (-1)^{n+1}\varphi(h_1, \cdots, h_n).$$

Clearly, we have

$$(6) \quad H^*(Hom_{kC_G(x)}(Bar_*(kG) \otimes_{kG} k, k)) \cong H^*(Hom_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k)).$$

To give an explicit isomorphism in (6), we need to construct the comparison maps between two projective resolutions $Bar_*(kG) \otimes_{kG} k$ and $Bar_*(kC_G(x)) \otimes_{kC_G(x)} k$ of the trivial $kC_G(x)$-module k. The comparison map from $Bar_*(kC_G(x)) \otimes_{kC_G(x)} k$ to $Bar_*(kG) \otimes_{kG} k$ is just the inclusion map

$$\iota : kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n} \hookrightarrow kG \otimes \overline{kG}^{\otimes n}.$$

To construct the comparison map on the reverse direction, we use a set-like self-homotopy over $kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n}$ as follows (for $h_0 \in C_G(x), h_1, \cdots, h_n \in \overline{C_G(x)}$)

$$kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n} \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n+1},$$
$$h_0 \otimes h_1 \otimes \cdots \otimes h_n \longmapsto 1 \otimes h_0 \otimes h_1 \otimes \cdots \otimes h_n.$$

Then we get a comparison map

$$\rho : Bar_*(kG) \otimes_{kG} k \longrightarrow Bar_*(kC_G(x)) \otimes_{kC_G(x)} k$$

as follows (we only write down the maps on basis vectors):

$$\rho_{-1} : k \longrightarrow k, 1 \longmapsto 1,$$

$$\rho_0 : kG \longrightarrow kC_G(x), h\gamma_{i,x} \longmapsto h, \text{ where } h\gamma_{i,x} \text{ belongs to the right coset } C_G(x)\gamma_{i,x},$$

$$\rho_1 : kG \otimes \overline{kG} \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}, h\gamma_{i,x} \otimes g_1 \longmapsto h \otimes h_{i,1}, \text{ where } \gamma_{i,x}g_1 = h_{i,1}\gamma_{s_i,x} \text{ for } h_{i,1} \in \overline{C_G(x)},$$

$$\cdots\cdots\cdots$$

$$\rho_n : kG \otimes \overline{kG}^{\otimes n} \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n}, h\gamma_{i,x} \otimes g_1 \otimes \cdots \otimes g_n \longmapsto h \otimes h_{i,1} \otimes \cdots \otimes h_{i,n},$$

where for $x \in X, h_{i,1}, \cdots, h_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \cdots, g_n\}$ as follows:

$$\gamma_{i,x}g_1 = h_{i,1}\gamma_{s_i^1,x}, \gamma_{s_i^1,x}g_2 = h_{i,2}\gamma_{s_i^2,x}, \cdots, \gamma_{s_i^{n-1},x}g_n = h_{i,n}\gamma_{s_i^n,x}.$$

It follows that we have two homomorphisms:

$$Hom_{kC_G(x)}(Bar_*(kG) \otimes_{kG} k, k) \longrightarrow Hom_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k),$$

$$(\varphi_{5,x} : S_x \times \overline{G}^{\times n} \longrightarrow k) \longmapsto (\varphi_{6,x} : \overline{C_G(x)}^{\times n} \longrightarrow k), \varphi_{6,x}(h_1, \cdots, h_n) = \varphi_{5,x}(1, h_1, \cdots, h_n) = a_{1,x},$$

$$\text{where } a_{1,x} \text{ is the cofficients of } x \text{ in } \varphi_{3,x}(h_1, \cdots, h_n) = \sum_{i=1}^{n_x} a_{i,x}\gamma_{i,x}^{-1}x\gamma_{i,x};$$

and

$$Hom_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k) \longrightarrow Hom_{kC_G(x)}(Bar_*(kG) \otimes_{kG} k, k),$$

$$(\varphi_{6,x} : \overline{C_G(x)}^{\times n} \longrightarrow k) \longmapsto (\varphi_{5,x} : S_x \times \overline{G}^{\times n} \longrightarrow k), \varphi_{5,x}(\gamma_{i,x}, g_1, \cdots, g_n) = \varphi_{6,x}(h_{i,1}, \cdots, h_{i,n}),$$

where for $x \in X, h_{i,1}, \cdots, h_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \cdots, g_n\}$ as follows:

$$\gamma_{i,x} g_1 = h_{i,1} \gamma_{s_i^1,x}, \gamma_{s_i^1,x} g_2 = h_{i,2} \gamma_{s_i^2,x}, \cdots, \gamma_{s_i^{n-1},x} g_n = h_{i,n} \gamma_{s_i^n,x}.$$

Since both ι and ρ induce the identity map $1 : k \longrightarrow k$, by Lemma 2.3, we have inverse isomorphisms between $H^*(Hom_{kC_G(x)}(Bar_*(kG) \otimes_{kG} k, k))$ and $H^*(Hom_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k))$. The correspondence is induced by $\varphi_{5,x} \longleftrightarrow \varphi_{6,x}$, as we stated above. So we realize an isomorphism in (6) and its inverse.

Summarizing the above six steps, we get the following main result in this section.

Theorem 5.1. *Let k be a field and G a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra kG:*

$$HH^*(kG) \cong \bigoplus_{x \in X} H^*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of G. We compute the Hochschild cohomology $HH^*(kG) = H^*(Hom_{k(G\times G)}(Bar_*(kG), kG))$ by the classical normalized bar resolution, and we compute the group cohomology $H^*(C_G(x), k)$ by $H^*(Hom_{kC_G(x)}(Bar_*(kC_G(x)) \otimes_{kC_G(x)} k, k))$. Then, we can realize an isomorphism in additive decomposition as follows:

$$HH^*(kG) \xrightarrow{\sim} \bigoplus_{x \in X} H^*(C_G(x), k),$$

$$[\varphi : \overline{G}^{\times n} \longrightarrow kG] \longmapsto [\widehat{\varphi}] = \bigoplus_{x \in X} [\widehat{\varphi}_x], \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k,$$

$$\widehat{\varphi}_x(h_1, \cdots, h_n) = a_{1,x}, \text{ where } \pi_x((-1)^{\frac{n(n+1)}{2}} h_1 \cdots h_n \varphi(h_n^{-1}, \cdots, h_1^{-1})) = \sum_{i=1}^{n_x} a_{i,x} x_i.$$

In other word, $\widehat{\varphi}_x(h_1, \cdots, h_n)$ is just the coefficient of x in $(-1)^{\frac{n(n+1)}{2}} h_1 \cdots h_n \varphi(h_n^{-1}, \cdots, h_1^{-1}) \in kG$. The inverse of the above isomorphism is given as follows:

$$\bigoplus_{x \in X} H^*(C_G(x), k) \xrightarrow{\sim} HH^*(kG),$$

$$[\widehat{\varphi}] = \bigoplus_{x \in X} [\widehat{\varphi}_x], \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k \longmapsto [\varphi : \overline{G}^{\times n} \longrightarrow kG],$$

$$\varphi(g_1, \cdots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \sum_{x \in X} \sum_{i=1}^{n_x} \widehat{\varphi}_x(h'_{i,1}, \cdots, h'_{i,n}) x_i,$$

where for $x \in X, h'_{i,1}, \cdots, h'_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_n^{-1}, \cdots, g_1^{-1}\}$ as follows:

$$\gamma_{i,x} g_n^{-1} = h'_{i,1} \gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x} g_{n-1}^{-1} = h'_{i,2} \gamma_{s_i^2,x}, \quad \cdots, \quad \gamma_{s_i^{n-1},x} g_1^{-1} = h'_{i,n} \gamma_{s_i^n,x}.$$

Proof This is a direct consequence by applying the above isomorphisms from (1) to (6) and their inverses. For an element $\varphi : \overline{G}^{\times n} \longrightarrow kG$ in the n-th term $C^n(kG) \cong Map(\overline{G}^{\times n}, kG)$ of the Hochschild cohomology complex, $[\varphi]$ denotes the corresponding element in the Hochschild cohomology group $HH^n(kG)$. Note that the elements $h'_{i,1}, \cdots, h'_{i,n}$ depend on $x \in X$ and the sequence $(g_1, \cdots, g_n)$. For the simplicity of notations, we avoid to write them down explicitly.

□

Remark 5.2. (a) The correspondence in Theorem 5.1 can be illustrated as follows:

$$HH^*(kG) \overset{(1)\sim(2)}{\cong} H^*(G, kG) \overset{(3)}{\cong} \bigoplus_{x \in X} H^*(G, kC_x) \overset{(4)\sim(6)}{\cong} \bigoplus_{x \in X} H^*(kC_G(x), k).$$

This is just the same line used by Siegel and Witherspoon in [9]. The difference is: they realize each step between cohomology groups using standard operations like restriction, induction, conjugation, etc., while we construct maps directly in each step on the cohomology complex level.

(b) Since we have

$$\begin{aligned} HH^*(kG) &\cong H^*(Hom_{k(G\times G)}(Bar_*(kG), kG)) \\ &\cong H^*(Hom_{k(G\times G)}(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k, kG)), \end{aligned}$$

we can choose either cohomology complex when discuss the ring structure of $HH^*(kG)$, it is not harm to the result up to isomorphism. If we compute $HH^*(kG)$ by the projective resolution $(k(G\times G)\otimes_{kG} Bar_*(kG)\otimes_{kG} k$, then the correspondence in Theorem 5.1 become simpler:

$$HH^*(kG) \xrightarrow{\sim} \bigoplus_{x\in X} H^*(C_G(x), k),$$

$$[\varphi : \overline{G}^{\times n} \longrightarrow kG] \longmapsto [\widehat{\varphi}] = \bigoplus_{x\in X} [\widehat{\varphi}_x], \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k,$$

$$\widehat{\varphi}_x(h_1, \cdots, h_n) = a_{1,x}, \text{ the coefficient of } x \text{ in } \varphi(h_1, \cdots, h_n) \in kG;$$

$$\bigoplus_{x\in X} H^*(C_G(x), k) \xrightarrow{\sim} HH^*(kG),$$

$$[\widehat{\varphi}] = \bigoplus_{x\in X} [\widehat{\varphi}_x], \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k \longmapsto [\varphi : \overline{G}^{\times n} \longrightarrow kG],$$

$$\varphi(g_1, \cdots, g_n) = \sum_{x\in X} \sum_{i=1}^{n_x} \widehat{\varphi}_x(h_{i,1}, \cdots, h_{i,n}) x_i,$$

where for $x \in X, h_{i,1}, \cdots, h_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \cdots, g_n\}$ as follows:

$$\gamma_{i,x} g_1 = h_{i,1}\gamma_{s_i^1,x}, \gamma_{s_i^1,x} g_2 = h_{i,2}\gamma_{s_i^2,x}, \cdots, \gamma_{s_i^{n-1},x} g_n = h_{i,n}\gamma_{s_i^n,x}.$$

6. Another way to realize the additive decomposition

In [3], Cibils and Solotar constructed a subcomplex of the Hochschild cohomology complex for each conjugacy class, and then they showed that for a finite abelian group, the subcomplex is isomorphic to the complex computing group cohomology. We will generalize this to any finite group: for each conjugacy class, this complex computes the cohomology of the corresponding centralizer subgroup. As a result, we give a second way to realize the additive decomposition.

As before, let k be a field and G a finite group. Recall that the Hochschild cohomology $HH^*(kG)$ of the group algebra kG can be computed by the following (cochain) complex:

$$(\mathcal{H}^*) \qquad 0 \longrightarrow kG \xrightarrow{d_0} Map(\overline{G}, kG) \xrightarrow{d_1} \cdots \longrightarrow Map(\overline{G}^{\times n}, kG) \xrightarrow{d_n} \cdots,$$

where the differential is given by

$$d_0(x)(g) = gx - xg \quad (\text{for } x \in kG \text{ and } g \in \overline{G})$$

and (for $\varphi : \overline{G}^{\times n} \longrightarrow kG$ and $g_1, \cdots, g_{n+1} \in \overline{G}$)

$$\begin{aligned} d_n(\varphi)(g_1, \cdots, g_{n+1}) &= g_1\varphi(g_2, \cdots, g_{n+1}) + \\ &\sum_{i=1}^{n} (-1)^i \varphi(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \cdots, g_n) g_{n+1}. \end{aligned}$$

We keep the following notations in Section 3: X is a complete set of representatives of the conjugacy classes in the finite group G. For $x \in X$, $C_x = \{gxg^{-1} | g \in G\}$ is the conjugacy class corresponding to x and $C_G(x) = \{g \in G | gxg^{-1} = x\}$ is the centralizer subgroup. Now take a conjugacy class C_x and define

$$\mathcal{H}_x^0 = kC_x, \text{ and for } n \geq 1,$$

$$\mathcal{H}_x^n = \{\varphi : \overline{G}^{\times n} \longrightarrow kG | \varphi(g_1, \cdots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \cdots, g_n \in \overline{G}\},$$

where $g_1 \cdots g_n C_x$ denotes the subset of G by multiplying $g_1 \cdots g_n$ on C_x and $k[g_1 \cdots g_n C_x]$ is the k-subspace of kG generated by this set. Let $\mathcal{H}_x^* = \bigoplus_{n\geq 0} \mathcal{H}_x^n$. Cibils and Solotar ([3, Page 20, Proof of the theorem]) observed that $\mathcal{H}_x^*$ is a subcomplex of $\mathcal{H}^*$ and $\mathcal{H}^* = \bigoplus_{x\in X} \mathcal{H}_x^*$.

Lemma 6.1. *$\mathcal{H}_x^*$ is canonically isomorphic to the complex $Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kC_x)$, which computes the group cohomology $H^*(G, kC_x)$ of G with coefficients in kC_x, where kC_x is a left kG-module under conjugation.*

Proof We know from Section 3 that the complex $Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kC_x)$ is identified as the following complex:

$$0 \longrightarrow kC_x \xrightarrow{d_0} Map(\overline{G}, kC_x) \xrightarrow{d_1} \cdots \longrightarrow Map(\overline{G}^{\times n}, kC_x) \xrightarrow{d_n} \cdots,$$

where the differential is given by

$$d_0(x)(g) = gxg^{-1} - x \quad (\text{for } x \in kC_x \text{ and } g \in \overline{G})$$

and (for $\varphi : \overline{G}^{\times n} \longrightarrow kC_x$ and $g_1, \cdots, g_{n+1} \in \overline{G}$)

$$d_n(\varphi)(g_1, \cdots, g_{n+1}) = g_1\varphi(g_2, \cdots, g_{n+1})g_1^{-1} + \sum_{i=1}^{n}(-1)^i\varphi(g_1, \cdots, g_ig_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \cdots, g_n).$$

A direct computation shows that the following map is an isomorphism of complexes:

$$\mathcal{H}_x^* \longrightarrow Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kC_x),$$

$$(\varphi_1 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x), \varphi_2(g_1, \cdots, g_n) = \varphi_1(g_1, \cdots, g_n)g_n^{-1}\cdots g_1^{-1}.$$

Its inverse is given by

$$Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kC_x) \longrightarrow \mathcal{H}_x^*,$$

$$(\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x) \longmapsto (\varphi_1 : \overline{G}^{\times n} \longrightarrow kG), \varphi_1(g_1, \cdots, g_n) = \varphi_2(g_1, \cdots, g_n)g_1\cdots g_n.$$

Passing to the cohomology, we have $H^*(\mathcal{H}_x^*) \cong H^*(G, kC_x)$.

□

On the other hand, we have shown that the complex $Hom_{kG}(Bar_*(kG)\otimes_{kG} k, kC_x)$ is isomorphic to the complex $Hom_{kC_G(x)}(Bar_*(kC_G(x))\otimes_{kC_G(x)} k, k)$, which computes the group cohomology $H^*(C_G(x), k)$ of the centralizer subgroup $C_G(x)$ with coefficients in the trivial module k. (cf. Section 3, from the fourth step to the six step.) Therefore we get another realization to the additive decomposition:

Theorem 6.2. *Let k be a field and G a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra kG:*

$$HH^*(kG) \cong \bigoplus_{x\in X} H^*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of G. We compute the Hochschild cohomology $HH^*(kG) = H^*(Hom_{k(G\times G)}(Bar_*(kG), kG))$ by the classical normalized bar resolution, and we compute the group cohomology $H^*(C_G(x), k)$ by $H^*(Hom_{kC_G(x)}(Bar_*(kC_G(x))\otimes_{kC_G(x)} k, k))$. Then, we can realize an isomorphism in additive decomposition as follows:

$$HH^*(kG) \xrightarrow{\sim} \bigoplus_{x\in X} H^*(C_G(x), k),$$

$$[\varphi_x : \overline{G}^{\times n} \longrightarrow kG], \varphi_x \in \mathcal{H}_x^n \longmapsto [\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k],$$

$$\widehat{\varphi}_x(h_1, \cdots, h_n) = a_{1,x}, \text{ where } \varphi_x(h_1, \cdots, h_n)h_n^{-1}\cdots h_1^{-1} = \sum_{i=1}^{n_x} a_{i,x}x_i \in kC_x.$$

In other word, $\widehat{\varphi}_x(h_1, \cdots, h_n)$ is just the coefficient of x in $\varphi_x(h_1, \cdots, h_n)h_n^{-1}\cdots h_1^{-1} \in kC_x$. The inverse of the above isomorphism is given as follows:

$$\bigoplus_{x\in X} H^*(C_G(x), k) \xrightarrow{\sim} HH^*(kG),$$

$$[\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k] \longmapsto [\varphi_x : \overline{G}^{\times n} \longrightarrow kG], \varphi_x \in \mathcal{H}_x^n,$$

$$\varphi_x(g_1, \cdots, g_n) = \sum_{i=1}^{n_x} \widehat{\varphi}_x(h_{i,1}, \cdots, h_{i,n}) x_i g_1 \cdots g_n,$$

where $h_{i,1}, \cdots, h_{i,n} \in \overline{C_G(x)}$ *are determined by the sequence* $\{g_1, \cdots, g_n\}$ *as follows:*

$$\gamma_{i,x} g_1 = h_{i,1} \gamma_{s_i^1,x}, \gamma_{s_i^1,x} g_2 = h_{i,2} \gamma_{s_i^2,x}, \cdots, \gamma_{s_i^{n-1},x} g_n = h_{i,n} \gamma_{s_i^n,x}.$$

Proof This is a combination of Lemma 6.1 and the correspondence from the fourth step to the six step in Section 3.

□

Remark 6.3. Comparing Theorem 5.1 with Theorem 6.2 we see that the two realizations of the additive decomposition are very close to each other. In the sequel, we prefer to the second realization since it is simpler.

7. The cup product formula

We keep the notations of the previous sections: k is a field, and G is a finite group, and so on. We describe the cup product formula for the Hochschild cohomology ring $HH^*(kG)$ in terms of the additive decomposition.

Theorem 7.1. *With the notations in Theorem 6.2, the cup product in the Hochschild cohomology ring* $HH^*(kG) \cong H^*(Hom_{k(G\times G)}(Bar_*(kG), kG))$ *is given as follows. Let* $[\widehat{\varphi}_x](\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k)$ *and* $[\widehat{\varphi}_y](\widehat{\varphi}_y : \overline{C_G(y)}^{\times m} \longrightarrow k)$ *be two elements in* $H^n(C_G(x), k)$ *and in* $H^m(C_G(y), k)$*, respectively. Denote by* $[\varphi_x : \overline{G}^{\times n} \longrightarrow kG](\varphi_x \in \mathcal{H}_x^n)$ *and* $[\varphi_y : \overline{G}^{\times m} \longrightarrow kG](\varphi_y \in \mathcal{H}_y^m)$ *be the corresponding elements in* $HH^*(kG)$*, and denote by* $\varphi_x \cup \varphi_y : \overline{G}^{\times(n+m)} \longrightarrow kG$ *the cup product. Then, for any* $z \in X$*, we have the following cup product formula:*

$$(\varphi_x \cup \varphi_y)_z : \overline{G}^{\times(n+m)} \longrightarrow kG, \quad (\varphi_x \cup \varphi_y)_z \in \mathcal{H}_z^{n+m},$$

$$(g_1, \cdots, g_n, \cdots, g_{n+m}) \longmapsto \sum_{k=1}^{n_z} \sum_{(i,j)} \widehat{\varphi}_x(h_{i,1}, \cdots, h_{i,n}) \widehat{\varphi}_y(h_{j,1}, \cdots, h_{j,m}) z_k,$$

where the second sum takes over all pairs (i,j) *such that* $x_i g_1 \cdots g_n y_j g_{n+1} \cdots g_{n+m} = z_k$*. In particular, we have the following formula:*

$$(\widehat{\varphi_x \cup \varphi_y})_z : \overline{C_G(z)}^{\times(n+m)} \longrightarrow k,$$

$$(\widehat{\varphi_x \cup \varphi_y})_z(h_1, \cdots, h_n, \cdots, h_{n+m}) = \sum_{(i,j)} a_{i,x} a_{j,y},$$

where $\varphi_x(h_1, \cdots, h_n) = \sum_{i=1}^{n_x} a_{i,x} x_i$*,* $\varphi_y(h_{n+1}, \cdots, h_{n+m}) = \sum_{j=1}^{n_y} a_{j,y} y_j$*, and where the sum takes over all pairs* (i,j) *such that* $x_i(h_1 \cdots h_n) y_j (h_1 \cdots h_n)^{-1} = z_1 = z$*.*

Proof This is obvious from the correspondence in Theorem 6.2. Notice that $h_{i,1}, \cdots, h_{i,n} \in \overline{C_G(x)}$ depend on the sequence $\{g_1, \cdots, g_n\}$ by the following steps:

$$\gamma_{i,x} g_1 = h_{i,1} \gamma_{s_i^1,x}, \gamma_{s_i^1,x} g_2 = h_{i,2} \gamma_{s_i^2,x}, \cdots, \gamma_{s_i^{n-1},x} g_n = h_{i,n} \gamma_{s_i^n,x},$$

and that $h_{j,1}, \cdots, h_{j,m} \in \overline{C_G(y)}$ depend on the sequence $\{g_{n+1}, \cdots, g_{n+m}\}$ by the following steps:

$$\gamma_{j,y} g_{n+1} = h_{j,1} \gamma_{s_j^1,y}, \gamma_{s_j^1,y} g_{n+2} = h_{j,2} \gamma_{s_j^2,y}, \cdots, \gamma_{s_j^{m-1},y} g_{n+m} = h_{j,m} \gamma_{s_j^m,y}.$$

□

Theorem 7.2. *With the notations in Theorem 5.1, the cup product in the Hochschild cohomology ring $HH^*(kG) \cong H^*(Hom_{k(G\times G)}(Bar_*(kG), kG))$ is given as follows. Let $[\widehat{\varphi}_x](\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k)$ and $[\widehat{\varphi}_y](\widehat{\varphi}_y : \overline{C_G(y)}^{\times m} \longrightarrow k)$ be two elements in $H^n(C_G(x), k)$ and in $H^m(C_G(y), k)$, respectively. Denote by $[\varphi_x : \overline{G}^{\times n} \longrightarrow kC_x]$ and $[\varphi_y : \overline{G}^{\times m} \longrightarrow kC_y]$ be the corresponding elements in $HH^*(kG)$, and denote by $\varphi_x \cup \varphi_y : \overline{G}^{\times(n+m)} \longrightarrow kG$ the cup product. Then, for any $z \in X$, we have the following cup product formula:*

$$(\varphi_x \cup \varphi_y)_z : \overline{G}^{\times(n+m)} \longrightarrow kC_z,$$

$$(g_1, \cdots, g_n, \cdots, g_{n+m}) \longmapsto (-1)^{\frac{n^2+m^2+n+m}{2}} \sum_{k=1}^{n_z} \sum_{(i,j)} \widehat{\varphi}_x(h'_{i,1}, \cdots, h'_{i,n})\widehat{\varphi}_y(h'_{j,1}, \cdots, h'_{j,m}) z_k,$$

where the second sum takes over all pairs (i, j) such that $g_1 \cdots g_n x_i g_{n+1} \cdots g_{n+m} y_j = z_k$. In particular, we have the following formula:

$$\widehat{(\varphi_x \cup \varphi_y)}_z : \overline{C_G(z)}^{\times(n+m)} \longrightarrow k,$$

$$\widehat{(\varphi_x \cup \varphi_y)}_z(h_1, \cdots, h_n, \cdots, h_{n+m}) = (-1)^{\frac{(n+m+1)(n+m)}{2}} \sum_{(i,j)} a_{i,x} a_{j,y},$$

where $\varphi_x(h^{-1}_{n+m}, \cdots, h^{-1}_{1+m}) = \sum_{i=1}^{n_x} a_{i,x} x_i$, $\varphi_y(h^{-1}_m, \cdots, h^{-1}_1) = \sum_{j=1}^{n_y} a_{j,y} y_j$, and where the sum takes over all pairs (i, j) such that $h_1 \cdots h_n \cdots h_{n+m} x_i y_j = z_1 = z$.

Proof This is obvious from the correspondence in Theorem 5.1. Notice that $h'_{i,1}, \cdots, h'_{i,n} \in \overline{C_G(x)}$ depend on the sequence $\{g_n^{-1}, \cdots, g_1^{-1}\}$ by the following steps:

$$\gamma_{i,x} g_n^{-1} = h'_{i,1}\gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x} g_{n-1}^{-1} = h'_{i,2}\gamma_{s_i^2,x}, \quad \cdots, \quad \gamma_{s_i^{n-1},x} g_1^{-1} = h'_{i,n}\gamma_{s_i^n,x},$$

and that $h'_{j,1}, \cdots, h'_{j,m} \in \overline{C_G(y)}$ depend on the sequence $\{g^{-1}_{n+m}, \cdots, g^{-1}_{n+1}\}$ by the following steps:

$$\gamma_{j,y} g^{-1}_{n+m} = h'_{j,1}\gamma_{s_j^1,y}, \quad \gamma_{s_j^1,y} g^{-1}_{n+m-1} = h'_{j,2}\gamma_{s_j^2,y}, \quad \cdots, \quad \gamma_{s_j^{m-1},y} g^{-1}_{n+1} = h'_{j,m}\gamma_{s_j^m,y}.$$

□

Remark 7.3. (1) We prefer to the cup product formula in Theorem 7.1 since it is simpler.

(2) By Remark 5.2 (a), our cup product formula in Theorem 7.2 is consistent with Siegel and Witherspoon's formula in [9, Theorem 5.1].

8. The △ operator

Let k be a field and G a finite group. Recall that the group algebra kG is a symmetric algebra with the bilinear form

$$\langle\ ,\ \rangle : kG \times kG \longrightarrow k,$$

$$\langle g, h\rangle = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for $g, h \in G$. For $n \geq 1$, the operator $\triangle : HH^n(kG) \longrightarrow HH^{n-1}(kG)$ on the Hochschild cohomology is induced by the following equation:

$$\langle \triangle(\varphi)(g_1, \cdots, g_{n-1}), g_n\rangle = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \cdots, g_{n-1}, g_n, g_1, \cdots, g_{i-1}), 1\rangle$$

for $\varphi \in C^n(kG) \cong Map(\overline{G}^{\times n}, kG)$, $\triangle(\varphi) \in C^{n-1}(kG) \cong Map(\overline{G}^{\times n-1}, kG)$. This operator together with the cup product $\cup$ and the Lie bracket $[\ ,\ ]$ define a BV algebra structure on $HH^*(kG)$.

We know from Section 4 that, for a conjugacy class C_x of G, $\mathcal{H}^*_x = \bigoplus_{n\geq 0} \mathcal{H}^n_x$ is a subcomplex of the Hochschild cohomology complex $\mathcal{H}^*$, where

$$\mathcal{H}^n_x = \{\varphi : \overline{G}^{\times n} \longrightarrow kG | \varphi(g_1, \cdots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \cdots, g_n \in \overline{G}\}.$$

Lemma 8.1. *The operator $\triangle : \mathcal{H}^n \longrightarrow \mathcal{H}^{n-1}$ restricts to $\triangle_x : \mathcal{H}^n_x \longrightarrow \mathcal{H}^{n-1}_x$ for each conjugacy class C_x.*

Proof We need to show that $\triangle(\varphi) \in \mathcal{H}^{n-1}_x$ for each $\varphi \in \mathcal{H}^n_x$. Suppose that $\triangle(\varphi)(g_1, \cdots, g_{n-1}) \neq 0$ for some $g_1, \cdots, g_{n-1} \in \overline{G}$. Notice that $g_n \in \overline{G}$ with $\langle \triangle(\varphi)(g_1, \cdots, g_{n-1}), g_n\rangle \neq 0$ if and only if the coefficient of g_n^{-1} in $\triangle(\varphi)(g_1, \cdots, g_{n-1})$ is nonzero. For each such g_n, there exists some i such that $\langle \varphi(g_i, \cdots, g_{n-1}, g_n, g_1, \cdots, g_{i-1}), 1\rangle \neq 0$. This implies that $1 \in g_i \cdots g_{n-1} g_n g_1 \cdots g_{i-1} C_x$, or equivalently, $g_n^{-1} \in g_i \cdots g_i C_x g_i \cdots g_{n-1} = g_1 \cdots g_{n-1} C_x$. It follows that $\triangle(\varphi)(g_1, \cdots, g_{n-1}) \in k[g_1 \cdots g_{n-1} C_x]$.

□

Now we can determine the behavior of the operator $\triangle$ under the additive decomposition.

Theorem 8.2. *Let k be a field and G a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra kG:*

$$HH^*(kG) \cong \bigoplus_{x \in X} H^*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of G. Let $\widehat{\triangle}_x : H^n(C_G(x), k) \longrightarrow H^{n-1}(C_G(x), k)$ be the map induced by the operator $\triangle_x : HH^n(kG) \longrightarrow HH^{n-1}(kG)$. Then $\widehat{\triangle}_x$ is defined as follows:

$$\widehat{\triangle}_x(\psi)(h_1, \cdots, h_{n-1}) = \sum_{i=1}^{n} (-1)^{i(n-1)} \psi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_1, \cdots, h_{i-1})$$

for $\psi : \overline{C_G(x)}^{\times n} \longrightarrow k$ and for $h_1, \cdots, h_{n-1} \in \overline{C_G(x)}$.

Proof We have the following commutative diagram

$$\begin{tikzcd} H^n(\mathcal{H}^*_x) \arrow[r, "\triangle_x"] \arrow[d, "\wr"'] & H^{n-1}(\mathcal{H}^*_x) \arrow[d, "\wr"'] \\ H^n(C_G(x), k) \arrow[r, "\widehat{\triangle}_x"] & H^{n-1}(C_G(x), k). \end{tikzcd}$$

Take an element $\psi : \overline{C_G(x)}^{\times n} \longrightarrow k$ in $Hom_{kC_G(x)}(Bar_n(kC_G(x)) \otimes_{kC_G(x)} k, k)$ and denote by $\varphi : \overline{G}^{\times n} \longrightarrow kG$ the corresponding element in $\mathcal{H}^n_x$. By Theorem 6.2, for any $h_1, \cdots, h_n \in \overline{C_G(x)}$, $\psi(h_1, \cdots, h_n)$ is equal to the coefficient of x in $\varphi(h_1, \cdots, h_n) h_n^{-1} \cdots h_1^{-1} \in kC_x$. On the other hand, $\triangle_x(\varphi)(g_1, \cdots, g_{n-1}) \in k[g_1 \cdots g_{n-1} C_x]$ is defined by the following equation:

$$\langle \triangle_x(\varphi)(g_1, \cdots, g_{n-1}), g_n\rangle = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \cdots, g_{n-1}, g_n, g_1, \cdots, g_{i-1}), 1\rangle,$$

where $g_n \in G$. Under the vertical isomorphism, $\triangle_x(\varphi)$ corresponds to $\widehat{\triangle}_x(\psi)$. For any $h_1, \cdots, h_{n-1} \in \overline{C_G(x)}$, $\widehat{\triangle}_x(\psi)(h_1, \cdots, h_{n-1})$ is the coefficient of x in $\triangle_x(\varphi)(h_1, \cdots, h_{n-1}) h_{n-1}^{-1} \cdots h_1^{-1} \in kC_x$, or equivalently, the coefficient of $x h_1 \cdots h_{n-1}$ in $\triangle_x(\varphi)(h_1, \cdots, h_{n-1}) \in k[h_1 \cdots h_{n-1} C_x]$. This coefficient is equal to

$$\begin{aligned} &\langle \triangle_x(\varphi)(h_1, \cdots, h_{n-1}), h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}\rangle \\ &= \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_1, \cdots, h_{i-1}), 1\rangle. \end{aligned}$$

We also know that $\psi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_1, \cdots, h_{i-1})$ is equal to the coefficient of x in

$$\begin{aligned} &\varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_1, \cdots, h_{i-1}) h_{i-1}^{-1} \cdots h_1^{-1} x h_1 \cdots h_{n-1} h_{n-1}^{-1} \cdots h_i^{-1} \\ &= \varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_1, \cdots, h_{i-1}) x \in kC_x, \end{aligned}$$

which is again equal to $\langle \varphi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_1, \cdots, h_{i-1}), 1\rangle$. It follows that

$$\widehat{\triangle}_x(\psi)(h_1, \cdots, h_{n-1}) = \sum_{i=1}^{n} (-1)^{i(n-1)} \psi(h_i, \cdots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_1, \cdots, h_{i-1}).$$

□

Remark 8.3. By [5, Corollary 2.2], we know that $H^n(G,k)$ is a Gerstenhaber subalgebra of $HH^*(kG)$ under the inclusion map (cf. Remark **??**):

$$Hom_{kG}(Bar_*(kG)\otimes_{kG}k,k)\hookrightarrow Hom_{k(G\times G)}(Bar_*(kG),kG),$$
$$(\varphi:\overline{G}^{\times n}\longrightarrow k)\longmapsto(\psi:\overline{G}^{\times n}\longrightarrow kG),\psi(g_1,\cdots,g_n)=\varphi(g_1,\cdots,g_n)g_1\cdots g_n.$$

Notice that by notations in Section 4, $\psi\in\mathcal{H}_1^n$. So motivated by Theorem 8.2, we can similarly define an operator $\triangle_1:H^n(G,k)\longrightarrow H^{n-1}(G,k)$ in the group cohomology $H^*(G,k)$ as follows:

$$\triangle_1(\varphi)(g_1,\cdots,g_{n-1})=\sum_{i=1}^{n}(-1)^{i(n-1)}\varphi(g_i,\cdots,g_{n-1},g_{n-1}^{-1}\cdots g_1^{-1},g_1,\cdots,g_{i-1})$$

for $\varphi:\overline{G}^{\times n}\longrightarrow k$ and for $g_1,\cdots,g_{n-1}\in\overline{G}$.

We prove that $H^n(G,k)$ is in fact a BV subalgebra of $HH^*(kG)$.

Proposition 8.4. *Let k be a field and G a finite group. Then $H^n(G,k)\hookrightarrow HH^*(kG)$ is a BV subalgebra.*

Proof Suppose that under the above inclusion map $H^n(G,k)\hookrightarrow HH^*(kG)$, $\varphi:\overline{G}^{\times n}\longrightarrow k$ corresponds to $\widehat{\varphi}:\overline{G}^{\times n}\longrightarrow kG$, where $\widehat{\varphi}(g_1,\cdots,g_n)=\varphi(g_1,\cdots,g_n)g_1\cdots g_n$. We need to show that $\widehat{\triangle_1(\varphi)}=\triangle(\widehat{\varphi})$. For $g_1,\cdots,g_{n-1}\in\overline{G}$, $\triangle(\widehat{\varphi})(g_1,\cdots,g_{n-1})$

$$\begin{aligned}&=\sum_{i=1}^{n}(-1)^{i(n-1)}\widehat{\varphi}(g_i,\cdots,g_{n-1},g_{n-1}^{-1}\cdots g_1^{-1},g_1,\cdots,g_{i-1})\\&=\sum_{i=1}^{n}(-1)^{i(n-1)}\varphi(g_i,\cdots,g_{n-1},g_{n-1}^{-1}\cdots g_1^{-1},g_1,\cdots,g_{i-1})\\&=\widehat{\triangle_1(\varphi)}(g_1,\cdots,g_{n-1}).\end{aligned}$$

□

Now let G be an abelian group. In this case, the Hochschild cohomology ring $HH^*(kG)$ of the group algebra kG is isomorphic to the tensor product algebra of kG and the group cohomology ring $H^*(G,k)$: $HH^*(kG)\cong kG\otimes_k H^*(G,k)$. According to [3], this isomorphism is given as follows. For G an abelian group, conjugacy classes are elements of G, hence a cochain φ_x of $\mathcal{H}_x^n$ for $x\in G$ attributes a scalar multiple of $g_1\cdots g_nx$ for each $(g_1,\cdots,g_n)\in\overline{G}^{\times n}$; we denote by $\overline{\varphi_x}(g_1,\cdots,g_n)$ the corresponding scalar and we obtain in this way a map $\overline{\varphi_x}:\overline{G}^{\times n}\longrightarrow k$. It is not difficult to verify that the map $\varphi\longmapsto\Sigma_{x\in G}(x\otimes\overline{\varphi_x})$ defines a ring isomorphism $C^*(kG)\longrightarrow kG\otimes C^*(kG,k)$ compatible with the differentials, and therefore it induces the above isomorphism.

Proposition 8.5. *Let k be a field and G a finite abelian group. Under the above isomorphism $HH^*(kG)\cong kG\otimes_k H^*(G,k)$, the operator $\triangle:HH^n(kG)\longrightarrow HH^{n-1}(kG)$ corresponds to the sum of operators $x\otimes\overline{\triangle_x}:x\otimes H^n(G,k)\longrightarrow x\otimes H^{n-1}(G,k)$, where $x\in G$ and $\overline{\triangle_x}:H^n(G,k)\longrightarrow H^{n-1}(G,k)$ is defined as follows:*

$$\overline{\triangle_x}(\overline{\varphi})(g_1,\cdots,g_{n-1})=\sum_{i=1}^{n}(-1)^{i(n-1)}\overline{\varphi}(g_i,\cdots,g_{n-1},g_{n-1}^{-1}\cdots g_1^{-1}x^{-1},g_1,\cdots,g_{i-1})$$

for $\overline{\varphi}:\overline{G}^{\times n}\longrightarrow k$ and for $g_1,\cdots,g_{n-1}\in\overline{G}$.

Proof The proof is similar to Theorem 8.2. According to Lemma 8.1, the operator $\triangle:\mathcal{H}^n\longrightarrow\mathcal{H}^{n-1}$ restricts to $\triangle_x:\mathcal{H}_x^n\longrightarrow\mathcal{H}_x^{n-1}$ for each $x\in G$. Let $\overline{\varphi}\in H^n(G,k)$ corresponds to $\varphi\in\mathcal{H}_x^n$. It suffices to show that $\overline{\triangle_x(\varphi)}=\overline{\triangle_x}(\overline{\varphi})$. For $g_1,\cdots,g_{n-1}\in\overline{G}$, $\overline{\triangle_x(\varphi)}(g_1,\cdots,g_{n-1})$ is the coefficient of $g_1\cdots g_{n-1}x$ in $\triangle_x(\varphi)(g_1,\cdots,g_{n-1})$, and it is equal to

$$\langle\triangle_x(\varphi)(g_1,\cdots,g_{n-1}),x^{-1}g_{n-1}^{-1}\cdots g_1^{-1}\rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \cdots, g_{n-1}, x^{-1} g_{n-1}^{-1} \cdots g_1^{-1}, g_1, \cdots, g_{i-1}), 1 \rangle \\
&= \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \cdots, g_{n-1}, x^{-1} g_{n-1}^{-1} \cdots g_1^{-1}, g_1, \cdots, g_{i-1}) x, x^{-1} \rangle \\
&= \sum_{i=1}^{n} (-1)^{i(n-1)} \overline{\varphi}(g_i, \cdots, g_{n-1}, x^{-1} g_{n-1}^{-1} \cdots g_1^{-1}, g_1, \cdots, g_{i-1}) \\
&= \sum_{i=1}^{n} (-1)^{i(n-1)} \overline{\varphi}(g_i, \cdots, g_{n-1}, g_{n-1}^{-1} \cdots g_1^{-1} x^{-1}, g_1, \cdots, g_{i-1}) \\
&= \overline{\triangle}_x(\overline{\varphi})(g_1, \cdots, g_{n-1}).
\end{aligned}
$$

□

9. The Lie bracket

Then the Gerstenhaber algebra $(HH^*(A), \cup, [\ ,\ ])$ together with the operator $\triangle$ is a *Batalin-Vilkovsky algebra* (BV-algebra), that is, $(HH^*(A), \triangle)$ is a complex and

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)|\beta|}(\triangle(\alpha \cup \beta) - \triangle(\alpha) \cup \beta - (-1)^{|\alpha|}\alpha \cup \triangle(\beta))$$

for all homogeneous elements $\alpha, \beta \in HH^*(A)$.

10. Some examples

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Yuming Liu
School of Mathematical Sciences
Beijing Normal University
Beijing 100875
P.R.China
E-mail address: `ymliu@math.bun.edu.cn`

Guodong Zhou
Ecole Polytechnique Fédérale de Lausanne
SB/IGAT/CTG,
Batîment MA B3 424,
CH-1015, Lausanne,
Switzerland
E-mail address: `guodong.zhou@epfl.ch`