Schur Positivity and q-Log-convexity of the Narayana Polynomials

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Definitions

$(a_n)_{n\geq 0}$: a sequence of real numbers.

- unimodal:

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq a_{m+2} \geq \cdots$$

- log-concave:

$$a_m^2 \geq a_{m+1}a_{m-1}, \quad m \geq 1$$

- log-convex:

$$a_m^2 \leq a_{m+1}a_{m-1}, \quad m \geq 1$$

q-Log-concavity

$f(q), g(q)$: Two polynomials over $\mathbb{R}$

$$f(q) - g(q) \geq_q 0 \quad \text{if } f(q) - g(q) \in \mathbb{R}^+[q]$$

$(f_n)_{n \geq 0}$: a sequence of polynomials over $\mathbb{R}$

- q-log-concave:—[Stanley]

$$f_m(q)^2 - f_{m+1}(q) f_{m-1}(q) \geq_q 0, m \geq 1$$

- strongly q-log-concave:

$$f_m(q) f_n(q) - f_{m+1}(q) f_{n-1}(q) \geq_q 0, m \geq n \geq 1$$

q-Log-convexity

- q-log-convex:—[Liu and Wang, 2006]

$$f_m(q)^2 - f_{m+1}(q)f_{m-1}(q) \leq_q 0, m \geq 1$$

- strongly q-log-convex:

$$f_m(q)f_n(q) - f_{m+1}(q)f_{n-1}(q) \leq_q 0, m \geq n \geq 1$$

Examples on q-Log-concavity and q-Log-convexity:

- q-binomial coefficient: Butler(1990), Krattenthaler(1989)
- q-Stirling number: Sagan(1992), Leroux(1990)
- Bell polynomial, Eulerian polynomial: Liu and Wang(2006)

Narayana numbers

- Catalan number: $C_n = \frac{1}{n+1}\binom{2n}{n}$

Narayana number: $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$

Narayana polynomial: $N_n(q) = \sum_{k=0}^{n} N(n,k)q^k$

q-Catalan number: $\frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}$

q-Narayana number: $N_q(n,k) = \frac{1}{[n]}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ k+1 \end{bmatrix} q^{k^2+k}$

$$[k] := \frac{(1-q^k)}{(1-q)}, [k]! = [1][2]\cdots[k], \begin{bmatrix} n \\ j \end{bmatrix} := \frac{[n]!}{[j]![n-j]!}.$$

Two conjectures

- Liu and Wang (2006) have shown that for a given positive real number q the sequence $(N_n(q))_{n\geq 0}$ is log-convex.

Liu and Wang's Conjecture I: The Narayana polynomials $N_n(q)$ form a q-log-convex sequence.

Liu and Wang's Conjecture II: If the positive sequence $(a_n)_{n\geq 0}$ is log-convex, then the sequence $(b_n)_{n\geq 0}$ defined by $b_n = \sum_{k=0}^{n} N(n,k)a_k$ is log-convex.

Main Results

- The Narayana polynomials $N_n(q)$ are strongly q-log-convex.
- The Narayana transformation preservers log-convexity.
- Given k, the polynomial sequence $(N_q(n,k))_{n\geq 0}$ is strongly q-log-concave.
- Given n, the polynomial sequence $(N_q(n,k))_{k\geq 0}$ is strongly q-log-concave.

Main Tools

Symmetric Functions

- Young tableaux
- Schur functions
- Littlewood-Richardson’s rule
- Hook-content formula
- Brändén’s formula

Partition and Young diagram

- λ: a partition with components $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$

$\lambda \vdash n$: if $\sum_{i=1}^{k} \lambda_i = n$

$\ell(\lambda)$: the number of nonzero components λ_i

$\mathrm{Par}(n)$: the set of all partitions of n

$\mu \subseteq \lambda$: if $\lambda_i \geq \mu_i$ holds for each i.

The Young diagram of λ is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i.

Partition and Young diagram

- When $\mu \subseteq \lambda$, a skew partition λ/μ is the diagram obtained from the diagram of λ by removing the diagram of μ at the top-left corner.

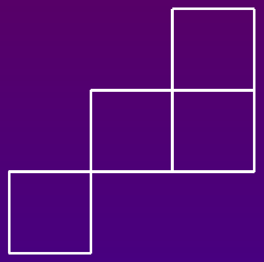


Fig 1: The diagram $(3,3,1)/(2,1)$

Semistandard Young Tableau

- A semistandard Young tableau (SSYT) of shape λ/μ is an array $T = (T_{ij})$ of positive integers of shape λ/μ that is weakly increasing in every row and strictly increasing in every column.

$\mathrm{type}(T) = \alpha$: α_i is the number of i's in T.

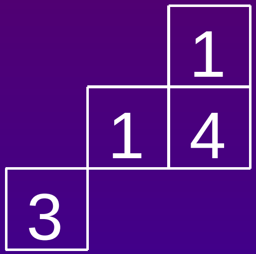


Fig 2: SSYT of shape $(3, 3, 1)/(2, 1)$

Schur Function

If T has type $\mathrm{type}(T) = \alpha$, then we write

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

The skew Schur function $s_{\lambda/\mu}(x)$ of shape λ/μ is defined as the generating function

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

summed over all semistandard Young tableaux T of shape λ/μ. We set $s_\emptyset(x) = 1$.

Hook-content Formula

- For a symmetric function $f(x)$, define

$$\begin{aligned} \mathrm{ps}_n(f) &= f(1, q, \ldots, q^{n-1}), \\ \mathrm{ps}_n^1(f) &= \mathrm{ps}_n(f)|_{q=1} = f(1^n). \end{aligned}$$

Given a partition λ and a square (i, j) in row i and column j, let

$$h(i,j) = \lambda_i + \lambda'_j - i - j + 1, \text{(hook length)}$$

$$c(i,j) = j - i, \text{(content)}$$

Hook-content Formula

- Theorem: (Stanley, Enumerative Combinatorics)
  For any partition λ and $n \geq 1$, we have

$$\begin{aligned}
\mathrm{ps}_n(s_\lambda) &= q^{\sum_{k\geq 1}(k-1)\lambda_k} \prod_{(i,j)\in\lambda} \frac{[n+c(i,j)]}{[h(i,j)]} \\
\mathrm{ps}_n^1(s_\lambda) &= \prod_{(i,j)\in\lambda} \frac{n+c(i,j)}{h(i,j)}.
\end{aligned}$$

Littlewood-Richardson Tableau

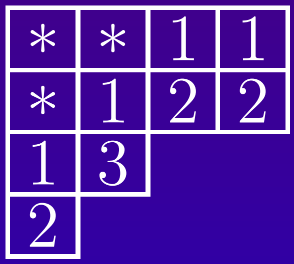
- Lattice permutation $w_1 w_2 \cdots w_n$:

$$\#i's \geq \#i+1's \text{ in } w_1 w_2 \cdots w_j, \text{ for any } i, j.$$

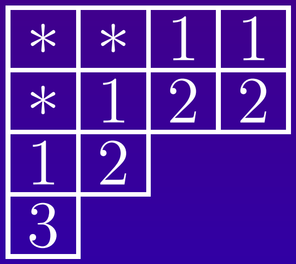
Littlewood-Richardson tableau T:

reverse reading word T^{rev} is a lattice permutation,

$$T_1^{\text{rev}} = 112213312, \quad T_2^{\text{rev}} = 11221213$$



*	*	1	1
*	1	2	2
1	3		
2			



*	*	1	1
*	1	2	2
1	2		
3			

Fig 3: Skew Littlewood-Richardson tableaux

Littlewood-Richardson's Rule

- Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ can be defined by the following relation

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda .$$

Theorem: (Stanley, Enumerative Combinatorics)
The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of Littlewood-Richardson tableaux of shape λ/μ and type ν.

Pieri's Rule

Horizontal strip: no two squares in the same column
Vertical strip: no two squares in the same row

Theorem: (Stanley, Enumerative Combinatorics)

$$s_\mu s_{(n)} = \sum_\lambda s_\lambda$$

summed over all partitions λ such that λ/μ is a horizontal strip of size n, and

$$s_\mu s_{(1^n)} = \sum_\lambda s_\lambda$$

summed over all partitions λ such that λ/μ is a vertical strip of size n.

Brändén's formula

Theorem: (Brändén, 2004)

For all $n, k \in \mathbb{N}$, we have

$$N_q(n,k) = s_{(2^k)}(q, q^2, \ldots, q^{n-1}). \tag{1}$$

In particular,

$$N(n,k) = \mathrm{ps}^1_{n-1}(s_{(2^k)}). \tag{2}$$

$$N_n(q) = \sum_{k=0}^{n} N(n,k)q^k = \sum_{k=0}^{n} \mathrm{ps}^1_{n-1}(s_{(2^k)})q^k$$

The q-Log-convexity of $N_n(q)$

- To prove:

$$N_{m+1}(q)N_{n-1}(q) - N_m(q)N_n(q) \geq_q 0, m \geq n \geq 1.$$

For any $r \geq 0$, let

$$C_1 = [q^r]N_{m+1}(q)N_{n-1}(q) = \sum_{k=0}^{r} \mathrm{ps}_m^1\left(s_{(2^k)}\right)\mathrm{ps}_{n-2}^1\left(s_{(2^{r-k})}\right),$$

$$C_2 = [q^r]N_m(q)N_n(q) = \sum_{k=0}^{r} \mathrm{ps}_{m-1}^1\left(s_{(2^k)}\right)\mathrm{ps}_{n-1}^1\left(s_{(2^{r-k})}\right).$$

It suffices to prove that $C_1 - C_2 \geq 0$.

The q-Log-convexity of $N_n(q)$

Note that

$$\begin{aligned}
\mathrm{ps}^1_m\left(s_{(2^k)}\right) &= \sum_{0\le a\le b\le m-n+2} \mathrm{ps}^1_{n-2}\big(s_{(2^{k-b},1^{b-a})}\big)\mathrm{ps}^1_{m-n+2}\big(s_{(2^a,1^{b-a})}\big),\\
\mathrm{ps}^1_{m-1}\left(s_{(2^k)}\right) &= \sum_{0\le a\le b\le m-n+1} \mathrm{ps}^1_{n-2}\big(s_{(2^{k-b},1^{b-a})}\big)\mathrm{ps}^1_{m-n+1}\big(s_{(2^a,1^{b-a})}\big),\\
\mathrm{ps}^1_{n-1}\left(s_{(2^{r-k})}\right) &= \mathrm{ps}^1_{n-2}\left(s_{(2^{r-k})} + s_{(2^{r-k-1},1)} + s_{(2^{r-k-1})}\right).
\end{aligned}$$

Theorem [Stanley, Enumerative Combinatorics]:

$$s_{\lambda/\mu}(x,y) = \sum_{\nu} s_{\lambda/\nu}(x)s_{\nu/\mu}(y),$$

ranging over all partitions ν satisfying $\mu \subseteq \nu \subseteq \lambda$.

The q-Log-convexity of $N_n(q)$

Thus

$$C_1 - C_2 =$$

$$\sum_{k=0}^{r} \sum_{0\le a\le b\le m-n+2} \mathrm{ps}^1_{m-n+2}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})})$$

$$-\sum_{k=0}^{r} \sum_{0\le a\le b\le m-n+1} \mathrm{ps}^1_{m-n+1}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})})$$

$$-\sum_{k=0}^{r} \sum_{0\le a\le b\le m-n+1} \mathrm{ps}^1_{m-n+1}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1},1)})$$

$$-\sum_{k=0}^{r} \sum_{0\le a\le b\le m-n+1} \mathrm{ps}^1_{m-n+1}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1})}).$$

Let $d = m - n + 1$. By simplifying, we obtain

$$C_1 - C_2 \quad = \quad \mathrm{ps}^1_{n-2}\left(\sum_{0\le a\le b\le d-1} \mathrm{ps}^1_d(s_{(2^a,1^{b+1-a})}) \sum_{k=0}^{r} D(a,b,k,r)\right)$$

$D(a,b,k,r)$

- Given $a, b, r \in \mathbb{N}$ and $0 \le k \le r$, let

$$\begin{aligned} D_1(a,b,k,r) &= s_{(2^{k-b-1},1^{b+2-a})} s_{(2^{r-k-1})}, \\ D_2(a,b,k,r) &= s_{(2^{k-b},1^{b-a})} s_{(2^{r-k-1})}, \\ D_3(a,b,k,r) &= s_{(2^{k-b-1},1^{b+1-a})} s_{(2^{r-k-1},1)}. \end{aligned}$$

and let

$$D(a,b,k,r) = D_1(a,b,k,r) + D_2(a,b,k,r) - D_3(a,b,k,r),$$

where $s_{(2^i,1^j)} = 0$ for $i < 0$ or $j < 0$. It is easy to see that $D(a,b,r,r) \equiv 0$.

The q-Log-convexity of $N_n(q)$

- Note that

$$C_1 - C_2 = \mathrm{ps}^1_{n-2}\left(\sum_{0\le a\le b\le d-1} \mathrm{ps}^1_d(s_{(2^a,1^{b+1-a})})\sum_{k=0}^{r} D(a,b,k,r)\right)$$

It suffices to prove:

$$\sum_{0\le a\le b\le d-1} \mathrm{ps}^1_d(s_{(2^a,1^{b+1-a})})\sum_{k=0}^{r} D(a,b,k,r) \text{ is } s\text{-positive.}$$

Given a symmetric function $f = \sum_\lambda a_\lambda s_\lambda$,

f is s-positive iff $a_\lambda \ge 0$ for any λ

Schur Positivity

Theorem I:
For any $b \geq a \geq 0$ and $r \geq 0$, the symmetric function $\sum_{k=0}^{r} D(a,b,k,r)$ is s-positive.

Theorem II: [for $a = 0, b = 0, r = m$]

For any $m \geq 0$, we have

$$\sum_{i=0}^{m} \left(s_{(2^{i-1})} s_{(2^{m-i})} + s_{(2^{i-2},1^2)} s_{(2^{m-i})} - s_{(2^{i-1},1)} s_{(2^{m-i-1},1)} \right) = \sum_{\lambda \in \mathrm{Par}_{\{2,4\}}(2m-2)} s_\lambda,$$

where $\mathrm{Par}_{\{2,4\}}(2m-2)$ denotes the set of partitions of $2m-2$ whose parts belong to $\{2,4\}$.

Schur Positivity

Taking $m = 3, 4, 5$ and using the Maple packages:

SF (Stembridge,1995); ACE (Veigneau, 1998)

we observe that

$$\sum_{k=0}^{3} \left(s_{(2^{k-1})} s_{(2^{3-k})} + s_{(2^{k-2},1^2)} s_{(2^{3-k})} - s_{(2^{k-1},1)} s_{(2^{3-k-1},1)} \right)$$

$$= s_{(4)} + s_{(2,2)}.$$

$$\sum_{k=0}^{4} \left(s_{(2^{k-1})} s_{(2^{4-k})} + s_{(2^{k-2},1^2)} s_{(2^{4-k})} - s_{(2^{k-1},1)} s_{(2^{4-k-1},1)} \right)$$

$$= s_{(4,2)} + s_{(2,2,2)}.$$

$$\sum_{k=0}^{5} \left(s_{(2^{k-1})} s_{(2^{5-k})} + s_{(2^{k-2},1^2)} s_{(2^{5-k})} - s_{(2^{k-1},1)} s_{(2^{5-k-1},1)} \right)$$

$$= s_{(4,4)} + s_{(4,2,2)} + s_{(2,2,2,2)}.$$

Proof of Theorem II

- For a symmetric function f, suppose that f has the expansion $f = \sum_\lambda a_\lambda s_\lambda$, and then the action of Δ^μ on f is given by

$$\Delta^\mu(f) = \sum_\lambda a_\lambda s_{\lambda\cup\mu}.$$

Exam:

$$\begin{aligned} f &= s_{(4,3,2)} + 3s_{(2,2,1)} + 2s_{(5)} \\ \Delta^{(3,1)} f &= s_{(4,3,3,2,1)} + 3s_{(3,2,2,1,1)} + 2s_{(5,3,1)}. \end{aligned}$$

Proof of Theorem II

Given $m \in \mathbb{N}$ and $0 \leq i \leq m$, let

$$\begin{aligned} D^{(1)}_{m,i} &= s_{(2^i)} s_{(2^{m-i-1})}, \\ D^{(2)}_{m,i} &= s_{(2^{i-1},1^2)} s_{(2^{m-i-1})}, \\ D^{(3)}_{m,i} &= s_{(2^{i-1},1)} s_{(2^{m-i-1},1)}, \end{aligned}$$

and let

$$D_{m,i} = D^{(1)}_{m,i} + D^{(2)}_{m,i} - D^{(3)}_{m,i},$$

where $s_{(2^i,1)} = s_{(2^i,1^2)} = 0$ for $i < 0$ by convention. It is clear that $D_{m,m} \equiv 0$.

Proof of Theorem II

- For any $n \geq k \geq 1$, we have

$$\begin{aligned}
s_{(2^k)} s_{(2^{n+1})} &= \Delta^{(2)}\big(s_{(2^k)} s_{(2^n)}\big), \\
s_{(2^{k-1},1^2)} s_{(2^{n+1})} &= \Delta^{(2)}\big(s_{(2^{k-1},1^2)} s_{(2^n)}\big), \\
s_{(2^k)} s_{(2^{n+1},1^2)} &= \Delta^{(2)}\big(s_{(2^k)} s_{(2^n,1^2)}\big), \\
s_{(2^{k-1},1)} s_{(2^{n+1},1)} &= \Delta^{(2)}\big(s_{(2^{k-1},1)} s_{(2^n,1)}\big).
\end{aligned}$$

$$\begin{matrix} * & * & 1 & 1 \\ * & * & 2 & \\ * & * & 3 & \\ * & * & & \\ * & * & & \\ 2 & & & \\ 3 & & & \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} * & * & 1 & 1 \\ * & * & 2 & \\ * & * & 3 & \\ * & * & & \\ * & * & & \\ & & & \\ 2 & & & \\ 3 & & & \end{matrix}$$

Proof of Theorem II

- For any $n \geq k \geq 1$, we have

$$\begin{aligned} s_{(2^k)} s_{(2^{n+1})} &= \Delta^{(2)}\big(s_{(2^k)} s_{(2^n)}\big), \\ s_{(2^{k-1},1^2)} s_{(2^{n+1})} &= \Delta^{(2)}\big(s_{(2^{k-1},1^2)} s_{(2^n)}\big), \\ s_{(2^k)} s_{(2^{n+1},1^2)} &= \Delta^{(2)}\big(s_{(2^k)} s_{(2^n,1^2)}\big), \\ s_{(2^{k-1},1)} s_{(2^{n+1},1)} &= \Delta^{(2)}\big(s_{(2^{k-1},1)} s_{(2^n,1)}\big). \end{aligned}$$

Proof of Theorem II

- Let $m = 2k+1$ for some $k \in \mathbb{N}$.

(i) We have

$$\begin{aligned} D_{m,k} &= s_{(3^k)} s_{(1^k)}, \\ D_{m,k+1} &= s_{(4^k)} - s_{(3^k)} s_{(1^k)} - \Delta^{(2)}\big(s_{(3^k)} s_{1^{(k-2)}}\big). \end{aligned}$$

(ii) For any $0 \le i \le k-1$, we have

$$\begin{aligned} D_{m,i} &= \Delta^{(2)}(D_{m-1,i}), \\ D_{m,m-i} &= \Delta^{(2)}(D_{m-1,m-1-i}). \end{aligned}$$

Proof of Theorem II

- Let $m = 2k$ for some $k \in \mathbb{N}$.

  (i) We have

$$\begin{aligned} D_{m,k-1} &= s_{(3^k)} s_{(1^{k-2})} + \Delta^{(2)}\left(s_{(3^{k-1})} s_{(1^{k-1})}\right), \\ D_{m,k} &= -s_{(3^k)} s_{(1^{k-2})}. \end{aligned}$$

  (ii) For any $0 \leq i \leq k-2$, we have

$$\begin{aligned} D_{m,i} &= \Delta^{(2)}(D_{m-1,i}), \\ D_{m,m-i} &= \Delta^{(2)}(D_{m-1,m-1-i}), \\ D_{m,m-k+1} &= \Delta^{(2)}(D_{m-1,m-k}). \end{aligned}$$

$D_{m,k}$ for $m = 7$

	$m = 7$
$D_{7,0}$	$s_{(2^6)}$
$D_{7,1}$	$s_{(4,2^4)} + s_{(3^2,2^3)} + s_{(3,2^4,1)}$
$D_{7,2}$	$s_{(3^2,2^2,1^2)} + s_{(4,3^2,2)} + s_{(4^2,2^2)} + s_{(3^3,2,1)} + s_{(4,3,2^2,1)}$
$D_{7,3}$	$s_{(4,3^2,1^2)} + s_{(3^3,1^3)} + s_{(4^2,3,1)} + s_{(4^3)}$
$D_{7,4}$	$-s_{(4,3^2,2)} - s_{(4,3^2,1^2)} - s_{(3^3,2,1)} - s_{(3^3,1^3)} - s_{(4^2,3,1)}$
$D_{7,5}$	$-s_{(3^2,2^3)} - s_{(3^2,2^2,1^2)} - s_{(4,3,2^2,1)}$
$D_{7,6}$	$-s_{(3,2^4,1)}$
$D_{7,7}$	0

$D_{m,k}$ **for** $m = 8$

	$m = 8$
$D_{8,0}$	$s_{(2^7)}$
$D_{8,1}$	$s_{(4,2^5)} + s_{(3^2,2^4)} + s_{(3,2^5,1)}$
$D_{8,2}$	$s_{(3^2,2^3,1^2)} + s_{(4,3^2,2^2)} + s_{(4^2,2^3)} + s_{(3^3,2^2,1)} + s_{(4,3,2^3,1)}$
$D_{8,3}$	$s_{(4,3^2,2,1^2)} + s_{(3^3,2,1^3)} + s_{(4^2,3,2,1)} + s_{(4^3,2)}$ $+s_{(3^4,1^2)} + s_{(4^2,3^2)} + s_{(4,3^3,1)}$
$D_{8,4}$	$-s_{(3^4,1^2)} - s_{(4^2,3^2)} - s_{(4,3^3,1)}$
$D_{8,5}$	$-s_{(4^2,3,2,1)} - s_{(3^3,2^2,1)} - s_{(3^3,2,1^3)} - s_{(4,3^2,2,1^2)} - s_{(4,3^2,2^2)}$
$D_{8,6}$	$-s_{(3^2,2^4)} - s_{(3^2,2^3,1^2)} - s_{(4,3,2^3,1)}$
$D_{8,7}$	$-s_{(3,2^5,1)}$
$D_{8,8}$	0

$D_{m,k}$ **for** $m = 9$

	$m = 9$
$D_{9,0}$	$s_{(2^8)}$
$D_{9,1}$	$s_{(4,2^6)} + s_{(3^2,2^5)} + s_{(3,2^6,1)}$
$D_{9,2}$	$s_{(3^2,2^4,1^2)} + s_{(4,3^2,2^3)} + s_{(4^2,2^4)} + s_{(3^3,2^3,1)} + s_{(4,3,2^4,1)}$
$D_{9,3}$	$s_{(4,3^2,2^2,1^2)} + s_{(3^3,2^2,1^3)} + s_{(4^2,3,2^2,1)} + s_{(4^3,2^2)}$ $+s_{(3^4,2,1^2)} + s_{(4^2,3^2,2)} + s_{(4,3^3,2,1)}$
$D_{9,4}$	$s_{(4,3^3,1^3)} + s_{(4^2,3^2,1^2)} + s_{(4^4)} + s_{(4^3,3,1)} + s_{(3^4,1^4)}$
$D_{9,5}$	$-s_{(4,3^3,1^3)} - s_{(4^2,3^2,1^2)} - s_{(4^4)} - s_{(4^3,3,1)} - s_{(3^4,1^4)}$ $-s_{(3^4,2,1^2)} - s_{(4^2,3^2,2)} - s_{(4,3^3,2,1)}$
$D_{9,6}$	$-s_{(4^2,3,2^2,1)} - s_{(3^3,2^3,1)} - s_{(3^3,2^2,1^3)} - s_{(4,3^2,2^2,1^2)} - s_{(4,3^2,2^3)}$
$D_{9,7}$	$-s_{(3^2,2^5)} - s_{(3^2,2^4,1^2)} - s_{(4,3,2^4,1)}$
$D_{9,8}$	$-s_{(3,2^6,1)}$
$D_{9,9}$	0

Proof of Theorem II

Recurrence relation:

$$\sum_{i=0}^{m+1} D_{m+1,i} = \begin{cases} \Delta^{(2)}\left(\sum_{i=0}^{m} D_{m,i}\right), & \text{if } m = 2k-1 \\ s_{(4^k)} + \Delta^{(2)}\left(\sum_{i=0}^{m} D_{m,i}\right), & \text{if } m = 2k \end{cases}$$

The end of the proof for Theorem II.

Proof of Theorem I

- Recall that

$$\begin{aligned} D_1(a,b,k,r) &= s_{(2^{k-b-1},1^{b+2-a})}s_{(2^{r-k-1})}, \\ D_2(a,b,k,r) &= s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1})}, \\ D_3(a,b,k,r) &= s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k-1},1)}, \\ D(a,b,k,r) &= D_1(a,b,k,r)+D_2(a,b,k,r)-D_3(a,b,k,r). \end{aligned}$$

For $i = 1, 2, 3$, it is clear that

$$D_i(a,b,k,r) = D_i(a-1,b-1,k-1,r-1),$$

hence

$$D(a,b,k,r) = D(a-1,b-1,k-1,r-1).$$

Proof of Theorem I

- Given a pair (λ, μ) of partitions and a pair (f_1, f_2) of symmetric functions, suppose that

$$\begin{aligned} \Delta^{\lambda}(f_1) &= \sum_{\nu} a_{\nu} s_{\nu}, \\ \Delta^{\mu}(f_2) &= \sum_{\nu} b_{\nu} s_{\nu}. \end{aligned}$$

Define

$$\tilde{\Delta}^{\lambda,\mu}(f_1, f_2) = \sum_{\nu} \max(a_{\nu}, b_{\nu}) s_{\nu}.$$

Proof of Theorem I

- Lemma:

For any $r \geq k \geq b \geq a \geq 0$ and $i = 1, 2, 3$, we have the following recurrence relations

$$D_i(a, b, k, r) = \tilde{\Delta}^{(1),(3)}(D_i(a, b-1, k-1, r-1), D_i(a, b-1, k-1, r-2)).$$

In particular, for $i = 1$, we have

$$s_{(2^{k-b-1},1^{b+2-a})} s_{(2^{r-k-1})} = \tilde{\Delta}^{(1),(3)}\big(s_{(2^{k-b-1},1^{b+1-a})} s_{(2^{r-k-1})}, s_{(2^{k-b-1},1^{b+1-a})} s_{(2^{r-k-2})}\big).$$

Proof of Theorem I

shape $\mu/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+1-a})$

$$\Downarrow$$

shape $\mu \cup (1)/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+2-a})$

$$\begin{array}{|c|c|c|c|}
\hline
* & * & 1 & 1 \\
\hline
* & * & 2 & 2 \\
\hline
* & * & 3 \\
\cline{1-3}
* & * & 4 \\
\cline{1-3}
* & * & 5 \\
\cline{1-3}
* & * & 6 \\
\cline{1-3}
* & * \\
\cline{1-2}
3 & 7 \\
\cline{1-2}
4 & 8 \\
\cline{1-2}
\end{array}
\quad \Rightarrow \quad
\begin{array}{|c|c|c|c|}
\hline
* & * & 1 & 1 \\
\hline
* & * & 2 & 2 \\
\hline
* & * & 3 \\
\cline{1-3}
* & * & 4 \\
\cline{1-3}
* & * & 5 \\
\cline{1-3}
* & * & 6 \\
\cline{1-3}
* & * \\
\cline{1-2}
3 & 7 \\
\cline{1-2}
4 & 8 \\
\cline{1-2}
9 \\
\cline{1-1}
\end{array}$$

Proof of Theorem I

shape $\mu/(2^{r-k-2})$ and type $(2^{k-b-1}, 1^{b+1-a})$

$$\Downarrow$$

shape $\mu \cup (3)/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+2-a})$

$$\begin{array}{cccc} * & * & 1 & 1 \\ * & * & 2 & 2 \\ * & * & 3 & \\ * & * & 4 & \\ * & * & 5 & \\ * & * & & \\ 3 & 6 & & \\ 4 & 7 & & \\ 8 & & & \end{array} \Rightarrow \begin{array}{cccc} * & * & 1 & 1' \\ * & * & 2 & 2' \\ * & * & 3' & \\ * & * & 4' & \\ * & * & 5' & \\ * & * & & \\ 3 & 6' & & \\ 4 & 7' & & \\ 8' & & & \end{array} \Rightarrow \begin{array}{cccc} * & * & 1 & 1' \\ * & * & 2 & 2' \\ & & 3' & \\ * & * & 4 & \\ * & * & 5 & \\ * & * & 6 & \\ * & * & & \\ 3' & 7 & & \\ 4' & 8 & & \\ 9 & & & \end{array} \Rightarrow \begin{array}{cccc} * & * & 1 & 1 \\ * & * & 2 & 2 \\ * & * & 3 & \\ * & * & 4 & \\ * & * & 5 & \\ * & * & 6 & \\ * & * & & \\ 3 & 7 & & \\ 4 & 8 & & \\ 9 & & & \end{array}$$

Proof of Theorem I

- Induct on the difference $b - a$. When $a = b$, note that

$$\sum_{k=0}^{r} D(a,b,k,r) = \sum_{k=a}^{r} D(0,0,k-a,r-a) = \sum_{i=0}^{r-a} D_{r-a,i}.$$

The negative terms of $D(a,b,k,r)$ come from either

$$\Delta^{(1)}(D(a,b-1,k-1,r-1))$$

or

$$\Delta^{(3)}(D(a,b-1,k-1,r-2)).$$

∎

The Narayana Transformation

Theorem:

If the sequence $(a_k)_{k\geq 0}$ of positive real numbers is log-convex, then the sequence

$$b_n = \sum_{k=0}^{n} N(n,k)a_k, \quad n \geq 0$$

is log-convex.

The Narayana Transformation

- For any $n \geq 1$, $0 \leq r \leq 2n$ and $0 \leq k \leq \lfloor \frac{r}{2} \rfloor$, let

$$\begin{aligned} \alpha(n,r,k) &= N(n+1,k)N(n-1,r-k) + N(n+1,r-k) \\ &\quad N(n-1,k) - 2N(n,r-k)N(n,k). \end{aligned}$$

For any $n, r, k \geq 0$, let

$$\alpha'(n,r,k) = \begin{cases} \alpha(n,r,k)/2, & \text{if } r \text{ is even and } k = r/2, \\ \alpha(n,r,k), & \text{otherwise.} \end{cases}$$

The Narayana Transformation

Note that for $n \geq 1$

$$
\begin{aligned}
b_{n-1}b_{n+1} - b_n^2 &= \sum_{r=0}^{2n} \left(\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) a_k a_{r-k} \right) \\
N_{n-1}(q)N_{n+1}(q) - N_n(q)^2 &= \sum_{r=0}^{2n} \left(\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) \right) q^r.
\end{aligned}
$$

The q-log-convexity of $N_n(q)$ implies that for any $r \geq 0$

$$\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) \geq 0.$$

The Narayana Transformation

Since the sequence $(a_k)_{k\geq 0}$ is log-convex , we obtain that

$$a_0 a_r \geq a_1 a_{r-1} \geq a_2 a_{r-2} \geq \cdots .$$

There exists an integer $k' = k'(n,r)$ such that

$$\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) a_k a_{r-k} \geq \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) a_{k'} a_{r-k'} \geq 0.$$

Therefore, $(b_n)_{n\geq 0}$ is log-convex. ∎

Lemma:

For given n and r, there always exists an integer $k' = k'(n,r)$ such that $\alpha(n,r,k) \geq 0$ for $k \leq k'$ and $\alpha(n,r,k) \leq 0$ for $k > k'$.

The Narayana Transformation

- Clearly, if $k \leq r-n-1$, then $n \leq (r-k)-1$ and $\alpha(n,r,k)=0$. We only need to determine the sign of $\alpha(n,r,k)$ for $r-n-1 < k \leq \lfloor \frac{r}{2} \rfloor$.

$$\begin{aligned} \alpha(n,r,k) &= N(n+1,k)N(n-1,r-k)+N(n+1,r-k) \\ & N(n-1,k)-2N(n,r-k)N(n,k). \end{aligned}$$

Note that $N(m,k)=\mathrm{ps}^1_{m-1}(s_{(2^k)})$ for any $m \in \mathbb{N}$.
By Hook-content formula we find that

$$N(m,k)=\frac{((n-1)(n-2)\cdots(n-r+k))\cdot(n(n-1)\cdots(n-r+k+1))}{k!(k+1)!}.$$

The Narayana Transformation

Let

$$\begin{aligned}
f_1(x) &= (n+1)(n-x+1)(n-x)^2(n-x-1), \\
f_2(x) &= (n+1)(n-(r-x)+1)(n-(r-x))^2(n-(r-x)-1), \\
f_3(x) &= (n-1)(n-x)(n-x+1)(n-(r-x))(n-(r-x)+1), \\
f(x) &= f_1(x)+f_2(x)-2f_3(x). \\
C &= (n-1)(n-2)^2(n-3)^2\cdots(n-k+2)^2(n-k+1), \\
C' &= (n-1)(n-2)^2(n-3)^2\cdots(n-(r-k)+2)^2(n-(r-k)+1).
\end{aligned}$$

Then we have

$$\alpha(n,r,k) = \frac{C}{k!(k+1)!}\cdot\frac{C'}{(r-k)!(r-k+1)!}\cdot f(k).$$

q-Log-concavity of $(N_q(n,k))_{k\geq 0}$

Theorem: Given a positive integer n, the sequence $(N_q(n,k))_{k\geq 0}$ is strongly q-log-concave.

Proof: For any $k \geq l \geq 1$, we get

$$N_q(n,k)N_q(n,l) - N_q(n,k+1)N_q(n,l-1) = s_{(2^k)}s_{(2^l)} - s_{(2^{k+1})}s_{(2^{l-1})},$$

where each Schur function on the righthand side is over the variable set $\{q, q^2, \ldots, q^{n-1}\}$. Note that

$$s_{(2^k)}s_{(2^l)} - s_{(2^{k+1})}s_{(2^{l-1})} = \Delta^{(2)}(s_{(2^{k-1})}s_{(2^l)} - s_{(2^k)}s_{(2^{l-1})}),$$

By induction on the difference $k-l$, $s_{(2^k)}s_{(2^l)} - s_{(2^{k+1})}s_{(2^{l-1})}$ is s-positive.

Bergeron and McNamara (2004); Kleber (2001)

q-Log-concavity of $(N_q(n,k))_{n\geq 0}$

Theorem:

Given a positive integer k, the sequence $(N_q(n,k))_{n\geq 0}$ is strongly q-log-concave.

Lemma: [Lam-Postnikov-Pylyavaskyy (2007)]

Let k be a positive integer. If I, J are partitions with $I \subseteq (2^{k-1})$ and $J \subseteq (2^{k-1},1)$, then both

$$s_{(2^{k-1})} s_{(2^k)/I} - s_{(2^{k-1})/I} s_{(2^k)} \qquad (3)$$

and

$$s_{(2^{k-1},1)} s_{(2^k)/J} - s_{(2^{k-1},1)/J} s_{(2^k)} \qquad (4)$$

are s-positive.

q-Log-concavity of $(N_q(n,k))_{n\geq 0}$

- Lemma (using hook-content formula):

For any $r \geq 1$, let

$$X_r = \{q, q^2, \ldots, q^{r-1}\}, \quad X_r^{-1} = \{q^{-1}, q^{-2}, \ldots, q^{-(r-1)}\}.$$

For any $m \geq n \geq 1$ and $k \geq 1$, we have

$$\begin{aligned} & q^{n-1} s_{(2^{k-1},1)}(X_{n-1}) s_{(2^k)}(X_m) - q^m s_{(2^{k-1},1)}(X_m) s_{(2^k)}(X_{n-1}) \\ = \quad & q^{k-1} \left(s_{(2^{k-1},1)}(X_{n-1}) s_{(2^k)}(X_m) - s_{(2^{k-1},1)}(X_m) s_{(2^k)}(X_{n-1}) \right) \end{aligned} \tag{5}$$

and

$$\begin{aligned} & q^{2(n-1)} s_{(2^{k-1})}(X_{n-1}) s_{(2^k)}(X_m) - q^{2m} s_{(2^{k-1})}(X_m) s_{(2^k)}(X_{n-1}) \\ = \quad & q^{2k(m+n-1)} \left(s_{(2^{k-1})}(X_{n-1}^{-1}) s_{(2^k)}(X_m^{-1}) - s_{(2^{k-1})}(X_m^{-1}) s_{(2^k)}(X_{n-1}^{-1}) \right). \end{aligned} \tag{6}$$

q-Log-concavity of $(N_q(n,k))_{n\geq 0}$

- For any $m \geq n \geq 1$, let

$$A_{m,n}(q) = N_q(m,k)N_q(n,k) - N_q(m+1,k)N_q(n-1,k).$$

By Brändén's formula, we have

$$A_{m,n}(q) \quad = \quad s_{(2^k)}(X_m)s_{(2^k)}(X_n) - s_{(2^k)}(X_{m+1})s_{(2^k)}(X_{n-1}),$$

which equals

$$\begin{aligned} & s_{(2^k)}(X_m)\left(s_{(2^k)}(X_{n-1}) + q^{n-1}s_{(2^{k-1},1)}(X_{n-1}) + q^{2(n-1)}s_{(2^{k-1})}(X_{n-1})\right) \\ & \qquad - \left(s_{(2^k)}(X_m) + q^m s_{(2^{k-1},1)}(X_m) + q^{2m}s_{(2^{k-1})}(X_m)\right) s_{(2^k)}(X_{n-1}) \\ = & \quad \left(q^{n-1}s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - q^m s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})\right) \\ & \qquad + \left(q^{2(n-1)}s_{(2^{k-1})}(X_{n-1})s_{(2^k)}(X_m) - q^{2m}s_{(2^{k-1})}(X_m)s_{(2^k)}(X_{n-1})\right). \end{aligned}$$

q-Log-concavity of $(N_q(n,k))_{n\geq 0}$

- Therefore, $A_{m,n}(q)$ equals

$$\begin{aligned}
&q^{k-1}\left(s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})\right)\\
&\qquad +q^{2k(m+n-1)}\left(s_{(2^{k-1})}(X_{n-1}^{-1})s_{(2^k)}(X_m^{-1}) - s_{(2^{k-1})}(X_m^{-1})s_{(2^k)}(X_{n-1}^{-1})\right)\\
&\quad = q^{k-1}s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(Z)\\
&\qquad +q^{k-1}\textstyle\sum_{J\subseteq(2^{k-1},1)} s_J(Z)\left(s_{(2^{k-1},1)}s_{(2^k)/J} - s_{(2^{k-1},1)/J}s_{(2^k)}\right)(X_{n-1})\\
&\qquad +q^{2k(m+n-1)}s_{(2^{k-1})}(X_{n-1}^{-1})s_{(2^k)}(Z^{-1})\\
&\qquad +q^{2k(m+n-1)}s_{(2^{k-1})}(X_{n-1}^{-1})s_{(2^{k-1},1)}(Z^{-1})s_{(1)}(X_{n-1}^{-1})\\
&\qquad +q^{2k(m+n-1)}\textstyle\sum_{I\subseteq(2^{k-1})} s_I(Z)\left(s_{(2^{k-1})}s_{(2^k)/I} - s_{(2^{k-1})/I}s_{(2^k)}\right)(X_{n-1}^{-1}),
\end{aligned}$$

where $Z=\{q^{n-1},\dots,q^{m-1}\}$ and $Z^{-1}=\{q^{1-n},\dots,q^{1-m}\}$.

Thank You!