Polynomials with only Real Zeros

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Unimodal and log-concave sequences

Def. Let $\{a_0, a_1, \ldots\}$ be a sequence of nonnegative numbers.

It is *unimodal* (**UM**) if $a_0 \le \cdots \le a_{m-1} \le a_m \ge a_{m+1} \ge \cdots$.

- m is called a *mode* of the sequence.

It is *log-concave* (**LC**) if $a_{i-1}a_{i+1} \le a_i^2$ for all $i > 0$.

- LC $\iff \left\{\frac{a_{i+1}}{a_i}\right\} \downarrow \Longrightarrow$ UM.

Ex. $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ has a mode $n/2$ or two modes $(n \pm 1)/2$.

Combinatorialists love to prove that counting sequences are unimodal.

— D.Zeilberger

Generating functions

Newton Inequality Suppose that $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathrm{RZ}$. Then

$$a_i^2 \geq a_{i-1}a_{i+1}\frac{(i+1)(n-i+1)}{i(n-i)}. \qquad (\star)$$

If $a_i \geq 0$, then $a_0, \dots, a_n$ is LC and UM with at most two modes.

- **Euler version:** $f(x) = \sum_{i=0}^{n} \binom{n}{i} a_i x^i \in \mathrm{RZ} \Longrightarrow a_i^2 \geq a_{i-1}a_{i+1}$.

Proof. $(\star) \iff \frac{a_i^2}{\binom{n}{i}^2} \geq \frac{a_{i-1}}{\binom{n}{i-1}}\frac{a_{i+1}}{\binom{n}{i+1}}$.

- Denote by **RZ** the set of real polynomials with only **R**eal **Z**eros. (Such polynomials are called *hyperbolic*.)

Darroch Thm. If $f(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{j=1}^{n}(x + r_j) \in \mathrm{RZ}$, where $r_j \geq 0$, then

$$|\text{mode } (a_i) - M| < 1,$$

where

$$M := \frac{f'(1)}{f(1)} = \frac{\sum_{i=1}^{n} i a_i}{\sum_{i=0}^{n} a_i} = \sum_{j=1}^{n} \frac{1}{r_j + 1}.$$

- **Benoumhani version:** $\lfloor M \rfloor \leq \text{mode } (a_i) \leq \lceil M \rceil$.

Ex. $\sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n$. $M = \frac{n}{2}$.

Zeros of the derivative

Prop. If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathrm{RZ}$, then

$$Df := f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in \mathrm{RZ}.$$

Proof. (1) $(x-r)^m \| f(x) \Longrightarrow (x-r)^{m-1} \| f'(x)$.

(2) (**Rolle**) $f(a) = f(b) = 0 \Longrightarrow f'(c) = 0$ for some $c \in (a, b)$. $\square$

- **Gauss-Lucas Thm.** The convex hull of the zeros of a polynomial $f(x)$ contains all the zeros of its derivative $f'(x)$.

Ex. Let $f(x) = (x+1)^3(x+5)$, $r_1 = r_2 = r_3 = -1, r_4 = -5$.

Then $f'(x) = 4(x+1)^2(x+4)$, $s_1 = s_2 = -1, s_3 = -4$.

$$r_4 \le s_3 \le r_3 \le s_2 \le r_2 \le s_1 \le r_1.$$

Proof of Newton Inequality.

$$f(x) = a_n x^n + \cdots + a_{i+1} x^{i+1} + a_i x^i + a_{i-1} x^{i-1} + \cdots + a_1 x + a_0 \in \mathrm{RZ}$$

$$\overset{D^{(i-1)}}{\Longrightarrow} \tfrac{n!}{(n-i+1)!} a_n x^{n-i+1} + \cdots + \tfrac{(i+1)!}{2!} a_{i+1} x^2 + \tfrac{i!}{1!} a_i x + (i-1)! a_{i-1} \in \mathrm{RZ}$$

$$\overset{*}{\Longrightarrow} \quad (i-1)! a_{i-1} x^{n-i+1} + i! a_i x^{n-i} + \tfrac{(i+1)!}{2} a_{i+1} x^{n-i-1} + \cdots \in \mathrm{RZ}$$

$$\overset{D^{(n-i-1)}}{\Longrightarrow} \tfrac{(i-1)!(n-i+1)!}{2} a_{i-1} x^2 + i!(n-1)! a_i x + \tfrac{(i+1)!(n-i-1)!}{2} a_{i+1} \in \mathrm{RZ}$$

$$\Longrightarrow \quad a_i^2 \geq a_{i-1} a_{i+1} \tfrac{(i+1)(n-i+1)}{i(n-i)}. \qquad \text{(discriminant } \geq 0) \qquad \square$$

- If $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathrm{RZ}$, then its reciprocal polynomial

$$f^*(x) := x^n f(1/x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in \mathrm{RZ}.$$

Prop. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathrm{RZ}$, where $a_0, a_n \neq 0$. Then for $0 < i < n$,

$$i a_i^2 > (i+1) a_{i-1} a_{i+1}.$$

So no two consecutive coefficients of f can be equal to zero.

Proof. Let $r_1, \ldots, r_n$ be all zeros of f. Then

$$a_1^2 - 2a_0 a_2 = a_0^2 \sum_{k=1}^{n} r_k^{-2} > 0.$$

The statement follows by applying this inequality to

$$D^{i-1} f = (i-1)! a_{i-1} + \frac{i!}{1!} a_i x + \frac{(i+1)!}{2!} a_{i+1} x^2 + \cdots . \quad \square$$

Lemma If $g(x) \in \mathrm{RZ}$, then for any $r \in \mathbb{R}$,

$$(D - r)g := g'(x) - rg(x) \in \mathrm{RZ}.$$

Proof. Apply Rolle theorem to $g'(x) - rg(x) = e^{rx}\left[e^{-rx}g(x)\right]'$. $\square$

Hermite-Poulain Thm. If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in \mathrm{RZ}$, then for arbitrary $g(x) \in \mathrm{RZ}$,

$$f(D)g := a_0 g(x) + a_1 g'(x) + a_2 g''(x) + \cdots + a_n g^{(n)}(x) \in \mathrm{RZ}.$$

Proof. Let $f(x) = a_n \prod_{i=1}^{n}(x - r_i)$. Then $f(D) = a_n \prod_{i=1}^{n}(D - r_i)$. $\square$

Laguerre Thm. If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in \mathrm{RZ}$, then

$$F(x) := a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \cdots + \frac{a_n}{n!}x^n \in \mathrm{RZ}.$$

Proof. Let $f^*(x) := x^n f\left(\frac{1}{x}\right) = a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n$ and $g(x) := \frac{x^n}{n!}$. Then by Hermite-Poulain Thm.,

$$f(x) \in \mathrm{RZ} \Longrightarrow f^*(x) \in \mathrm{RZ} \Longrightarrow F(x) = f^*(D)g \in \mathrm{RZ}. \quad \square$$

Composition of polynomials

Schur Thm. Let $f(x) = \sum a_i x^i \in \mathrm{RZ}, g(x) = \sum b_j x^j \in \mathrm{RZ}$ and zeros of $g(x)$ have the same sign. Then

$$(f \odot g)(x) := \sum k! a_k b_k x^k \in \mathrm{RZ}.$$

Maló Thm. $(f \cdot g)(x) := \sum a_k b_k x^k \in \mathrm{RZ}.$

Schur-Szegö Thm. Let $f = \sum_{i=0}^{n} \binom{n}{i} a_i x^i \in \mathrm{RZ}, g = \sum_{i=0}^{n} \binom{n}{i} b_i x^i \in \mathrm{RZ}$ and zeros of $g(x)$ have the same sign. Then

$$(f * g)(x) := \sum_{i=0}^{n} \binom{n}{i} a_i b_i x^i \in \mathrm{RZ}.$$

Multiplier sequences

Def. Let $\{\gamma_i\}_{i=0}^{+\infty}$ be a sequence of real numbers. Define a linear operator Γ on $\mathbb{R}[x]$ by

$$\Gamma(a_0 + a_1 x + \cdots + a_n x^n) = \gamma_0 a_0 + \gamma_1 a_1 x + \cdots + \gamma_n a_n x^n.$$

We say that $\{\gamma_i\}_{i=0}^{+\infty}$ is a *multiplier sequence* (**MS**) (of the first kind) if $f(x) \in \mathrm{RZ}$ implies $\Gamma(f) \in \mathrm{RZ}$.

Ex. $\{0, 1, 2, 3, \ldots\}$: $f = \sum a_i x^i \in \mathrm{RZ} \Longrightarrow xf' = \sum i a_i x^i \in \mathrm{RZ}$.

Ex. $\{1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots\}$: $\sum a_i x^i \in \mathrm{RZ} \Longrightarrow \sum \dfrac{a_i}{i!} x^i \in \mathrm{RZ}$.

Prop. Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ be a multiplier sequence. Then

(1) $\gamma_k, \gamma_{k+1}, \ldots, \gamma_{k+n}, \ldots$ is also a multiplier sequence;

(2) $\gamma_i^2 \geq \gamma_{i-1}\gamma_{i+1}$;

(3) $\gamma_{i-1}\gamma_{i+1} > 0$;

(4) $\gamma_i = 0 \Longrightarrow \gamma_{i+1} = \gamma_{i+2} = \cdots = 0$.

Proof. (1) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathrm{RZ}$. Then

$$a_0\gamma_k + a_1\gamma_{k+1}x + \cdots + a_n\gamma_{k+n}x^n = x^{-k}\Gamma\left[x^k f(x)\right] \in \mathrm{RZ}.$$

(2) $\Gamma\left[x^{i-1}(x+1)^2\right] = (\gamma_{i+1}x^2 + 2\gamma_i x + \gamma_{i-1})x^{i-1} \in \mathrm{RZ} \Longrightarrow \gamma_i^2 \geq \gamma_{i-1}\gamma_{i+1}$.

(3) $\Gamma\left(x^{i+1} - x^{i-1}\right) = (\gamma_{i+1}x^2 - \gamma_{i-1})x^{i-1} \in \mathrm{RZ} \Longrightarrow \gamma_{i-1}\gamma_{i+1} > 0$.

(4) (2)+(3) $\Longrightarrow$ (4).

Pólya-Schur Thm. (transcendental characterization)

Let $\{\gamma_k\}_{k=0}^{+\infty}$ be a sequence of non-negative real numbers and $\gamma_0 \neq 0$. Then it is a multiplier sequence if and only if

$$\phi(z) := \sum_{k=0}^{+\infty} \frac{\gamma_k}{k!} z^k$$

is a real entire function which can be written as

$$\phi(z) = ce^{az} \prod_{k=1}^{+\infty} (1 + b_k z),$$

where $a \geq 0, b_k > 0$ and $\sum_{k=1}^{+\infty} b_k < +\infty$.

Pólya-Schur Thm.(algebraic characterization) A sequence $\{\gamma_k\}_{k=0}^{+\infty}$ of real numbers is a multiplier sequence if and only if the zeros of

$$\Gamma\left[(1+x)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k, \quad n = 1, 2, 3, \ldots$$

are all real and of the same sign.

Proof. $\Longleftarrow$ Let $\sum_{k=0}^{n} a_k x^k \in \mathrm{RZ}$. Then by Schur-Szegö Thm.,

$$\sum_{k=0}^{n} \gamma_k a_k x^k = \sum_{k=0}^{n} \binom{n}{k} \gamma_k \binom{n}{k}^{-1} a_k x^k \in \mathrm{RZ}. \quad \square$$

A key result on the boundary between algebra and analysis. — Boas

Totally positive matrices and Pólya frequency sequencess

Def. An infinite matrix is **TP** if its minors are all nonnegative. Given an infinite sequence $\{a_0, a_1, a_2, \ldots\}$, define its Toeplitz matrix

$$A = (a_{j-i})_{i,j\geq 0} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ & a_0 & a_1 & a_2 & \cdot \\ \mathrm{O} & & a_0 & a_1 & \cdot \\ & & & a_0 & \cdot \\ & & & & \cdot \end{pmatrix}.$$

A finite sequence $\{a_0, a_1, \ldots, a_n\} \cong \{a_0, a_1, \ldots, a_n, 0, 0, \ldots\}$.

Def. The sequence $\{a_i\}$ is **PF** if the matrix A is TP.

- PF$\Longrightarrow$LC$\Longrightarrow$UM.

Schoenberg-Edrei Thm. Let $a_i > 0$. Then

$$a_0 = 1, a_1, a_2, \ldots \text{ is PF} \iff \sum_{i \geq 0} a_i x^i = \frac{\prod_{j \geq 1}(1 + \alpha_j x)}{\prod_{j \geq 1}(1 - \beta_j x)}\, e^{\gamma x},$$

where $\gamma, \alpha_j, \beta_j \geq 0$ and $\sum(\alpha_j + \beta_j) < \infty$.

- Basic PF sequences:

(1) $1 + \alpha x$;

(2) $\frac{1}{1-\beta x} = 1 + \beta x + \beta^2 x^2 + \cdots + \beta^n x^n + \cdots$;

(3) $e^{\gamma x} = 1 + \gamma x + \frac{\gamma^2}{2!} x^2 + \cdots + \frac{\gamma^n}{n!} x^n + \cdots$.

Aissen-Schoenberg-Whitney Thm. Let $a_i > 0$. Then

$$a_0, \ldots, a_n \text{ is PF } \iff \sum_{i=0}^{n} a_i x^i \in \mathrm{RZ}.$$

Ex. a_0, a_1, a_2 is PF $\iff a_1^2 \geq 4a_0a_2$.

- **Kurtz Thm.** Let $a_i > 0$. If $a_i^2 > 4a_{i-1}a_{i+1}$, then

$$\sum_{i=0}^{n} a_i x^i \in \mathrm{RZ}.$$

Interlacing and alternating of polynomials

Suppose that $f(x) = \alpha \prod_{i=1}^{n}(x - r_i), \quad r_n \le \cdots \le r_1$

and $g(x) = \beta \prod_{j=1}^{m}(x - s_j), \quad s_m \le \cdots \le s_1.$

We say that g *interlaces* f if $m = n - 1$ and

$$r_n \le s_{n-1} \le r_{n-1} \le \cdots \le r_2 \le s_1 \le r_1.$$

We say that g *alternates* f if $m = n$ and

$$s_n \le r_n \le s_{n-1} \le \cdots \le r_2 \le s_1 \le r_1.$$

Ex. $f \in \mathrm{RZ} \Longrightarrow f' \preccurlyeq_{\mathrm{int}} f; \quad f \in \mathrm{PF} \Longrightarrow f \preccurlyeq_{\mathrm{alt}} xf'.$

Linear combinations of polynomials

Prop. $g(x) \preccurlyeq f(x) \iff \alpha f(x) + \beta g(x) \in \mathrm{RZ}$ for any $\alpha, \beta \in \mathbb{R}$.

Def. A polynomial is *standard* if its leading coefficient is positive.

Thm. (Wang & Yeh, J. Combin. Theory Ser. A, 2005)

Suppose that f, g are standard and $g \preccurlyeq f$. If $ad \leq bc$, then

$$(ax + b)f(x) + (cx + d)g(x) \in \mathrm{RZ}.$$

Coro. Suppose that $f, g \in \mathrm{PF}$ and $g \preccurlyeq_{\mathrm{int}} f$. If $ad \geq bc$, then

$$(ax + b)f(x) + x(cx + d)g(x) \in \mathrm{RZ}.$$

Polynomial sequences with only real zeros

Ex. The Stirling number of the second kind $S(n,k)$ is the number of partitions of the set $[n]$ into k blocks. Clearly,

$$S(n,k) = kS(n-1,k) + S(n-1,k-1), \quad S(0,0) = 1.$$

Define the Stirling polynomial $S_n(x) = \sum_{k=0}^{n} S(n,k)x^k$. Then

$$S_n(x) = xS_{n-1}(x) + xS'_{n-1}(x)$$

with $S_0(x) = 1$. Harper showed that $S_n(x) \in \mathrm{RZ}$ by induction on n.

Ex. Let π be a permutation in the symmetric group S_n on $[n]$. An element $i \in [n]$ is called an excedance of π if $\pi(i) > i$. Denote by $\operatorname{exc}(\pi)$ the number of excedances of π. For example, $\operatorname{exc}(\dot{2}\dot{4}3\dot{5}1) = 3$. Define the Eulerian polynomial $A_0(q) = 1$ and

$$A_n(q) = \sum_{\pi \in S_n} q^{\operatorname{exc}(\pi)+1}.$$

Then $A_1(q) = q, A_2(q) = q + q^2$ and

$$A_n(q) = nqA_{n-1}(q) + q(1-q)A'_{n-1}(q).$$

It is well known that $A_n(q) \in \mathrm{RZ}$ for $n \geq 1$.

Sturm sequences of polynomials

Def. Let $\{P_i(x)\}$ be a sequence of standard polynomials. It is called a *Sturm sequence* if $P_i \in \mathrm{RZ}$ and $P_i \preccurlyeq_{\mathrm{int}} P_{i+1}$ for all n.

Prop. $\{P_i(x)\}$ is a Sturm sequence $\iff$ $P_{i-1}(r)P_{i+1}(r) < 0$ whenever $P_i(r) = 0$ and $i \geq 1$.

Ex. The classical orthogonal polynomials sequence $\{p_n(x)\}$ satisfies

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

with $p_{-1}(x) = 0$ and $p_0(x) = 1$, where $a_n, c_n > 0$ and $b_n \in \mathbb{R}$.

A standard result in the theory of orthogonal polynomials is that for each $n \geq 1$, the zeros of $p_n(x)$ are real, simple, and strictly interlace those of $p_{n+1}(x)$. In other words, $\{p_n(x)\}$ forms a Sturm sequence.

Sturm chains and Sturm theorem

Sturm Thm. Let P be standard. Apply Euclidean algorithm to $P_0 = P$ and $P_1 = P'$ to obtain $\{P_i\}$ (**Sturm chain** of P):

$$\begin{aligned} P_0(x) &= q_1(x)P_1(x) - P_2(x) \\ P_1(x) &= q_2(x)P_2(x) - P_3(x) \\ &\cdots \\ P_{k-2}(x) &= q_{k-1}(x)P_{k-1}(x) - P_k(x) \\ P_{k-1}(x) &= q_k(x)P_k(x). \end{aligned}$$

Then $P \in \mathrm{RZ} \iff P_i$ are standard and $\deg P_i = \deg P_{i-1} - 1$.

Ex. Given $P(x) = x^4 + 7x^3 + 17x^2 + 17x + 6$, let $P_0(x) = P(x)$ and $P_1(x) = P'(x) = 4x^3 + 21x^2 + 34x + 17$. Then

$$\begin{aligned}
P_0(x) &= \frac{1}{16}(4x+7)P_1(x) \underbrace{- \frac{1}{16}(11x^2+34x+23)}_{P_2(x)} \\
P_1(x) &= \frac{16}{121}(44x+95)P_2(x) \underbrace{- \frac{128}{121}(x+1)}_{P_3(x)} \\
P_2(x) &= \frac{121}{16 \cdot 128}(11x+23)P_3(x)
\end{aligned}$$

It follows that $P(x) \in \mathrm{RZ}$. Actually, $P(x) = (x+1)^2(x+2)(x+3)$.

$$F(x) = a(x)f(x) + b(x)g(x)$$

Thm. (Liu & Wang, Adv. in Appl. Math., 2007a)

Let $F(x) = a(x)f(x) + b(x)g(x)$. Suppose that

(1) $f, g \in \mathrm{RZ}$ and $g \preccurlyeq f$;

(2) F and g are standard;

(3) $\deg F = \deg f$ or $\deg f + 1$.

If $b(r) \le 0$ whenever $f(r) = 0$, then $F \in \mathrm{RZ}$ and $f \preccurlyeq F$.

Ex. (1) Orthogonal polynomials: $p_{-1}(x) = 0, p_0(x) = 1$ and

$$p_n(x) = (a_n x + b_n)p_{n-1}(x) - c_n p_{n-2}(x), \quad a_n, c_n > 0.$$

(2) Stirling polynomials: $S_0(x) = 1$ and

$$S_n(x) = xS_{n-1}(x) + xS'_{n-1}(x).$$

(3) Eulerian polynomials: $A_0(x) = 1$ and

$$A_n(x) = nxA_{n-1}(x) + x(1-x)A'_{n-1}(x).$$

(4) Narayana polynomials: $N_1(x) = x, N_2(x) = x + x^2$ and

$$(n+1)N_n(x) = (2n-1)(1+x)N_{n-1}(x) - (n-2)(1-x)^2 N_{n-2}(x).$$

Thm. (Ma & Wang, 2008)

Let f, F be standard and $F(x) = a(x)f(x) + b(x)f'(x)$. Suppose that $f \in \mathrm{RZ}$ and $b(r) \le 0$ whenever $f(r) = 0$. Then $F \in \mathrm{RZ}$ and $f \preccurlyeq F$. Furthermore, if $(x-r)^m \| f$, then $(x-r)^\ell \| F$, where

$$\ell = \begin{cases} m-1, & \text{if } b(r) \neq 0. \\ m, & \text{if } b(r) = 0 \text{ but } a(r) + mb'(r) \neq 0. \\ m+1, & \text{if } b(r) = 0 \text{ and } a(r) + mb'(r) = 0. \end{cases}$$

Thm. (Ma & Wang, 2008)

Let f, F be standard and $F(x) = a(x)f(x) + b(x)f'(x)$. Suppose that $f \in \mathrm{RZ}$ and $b(r) \leq 0$ whenever $f(r) = 0$.

(1) Assume that $f \in \mathrm{RZ}(-\infty, t]$, where t is the largest zero of f, with the multiplicity m. Then $F \in \mathrm{RZ}(-\infty, t]$ if and only if $b(t) = 0$ and $a(t) + mb'(t) \geq 0$.

(2) Assume that $f \in \mathrm{RZ}[s, +\infty)$, where s is the smallest zero of f, with the multiplicity m. Then $F \in \mathrm{RZ}[s, +\infty)$ if and only if $\deg F = \deg f$, or $b(s) = 0$ and $a(s) + mb'(s) \leq 0$.

Ex. (Wagner's T-linear transformation)

Let $F = (x - \alpha)f + xf'$ and $f \in \mathrm{RZ}(-\infty, 0]$. Then

(1) $F \in \mathrm{RZ}$.

(2) Let $x^m \| f$. Then $F \in \mathrm{RZ}(-\infty, 0]$ if and only if $\alpha \leq m_0$.

(3) Furthermore, $x^m \| F$ if $\alpha \neq m$, and $x^{m+1} | F$ if $\alpha = m$.

Ex. (Brändén's E-linear transformation)

Let $F = (x - \alpha)f + x(x+1)f'$. Then $f \in \mathrm{RZ}[-1, 0]$ implies that $F \in \mathrm{RZ}[-1, 0]$ and $f \preccurlyeq F$.

Ex. (Alternating Runs)

$$R_{n+2}(x) = x(nx+2)R_{n+1}(x) + x\left(1-x^2\right)R'_{n+1}(x), \quad R_1(x) = 1.$$

Knuth: $R_n(x) = \left(\frac{1+x}{2}\right)^{n-1}(1+w)^{n+1}A_n\left(\frac{1-w}{1+w}\right), \quad w = \sqrt{\frac{1-x}{1+x}}.$

Wilf: $R_n(x) \in \mathrm{RZ}[-1,0].$

Bóna: The multiplicity of the zero $x=-1$ is $\left\lfloor \frac{n}{2} \right\rfloor - 1.$

Ex. $A_{n+1}(x;q) = (nx+q)A_n(x;q) + x(1-x)A'_n(x;q), \quad A_0 = 1.$

Brenti: $A_n(x,q) \in \mathrm{RZ}$ for $q > 0!$ $\quad A_n(x,q) \in \mathrm{RZ}$ for $q \in \mathbb{Z}^-$?

Brändén: Let $E_n(x;q) = (1+x)^n A_n\left(\frac{x}{1+x};q\right)$. Then

$$E_{n+1}(x;q) = q(1+x)E_n(x;q) + x(1+x)E'_n(x;q).$$

(1) If $q > 0$, then $A_n(x;q)$ have nonpositive and simple zeros.

(2) If $n+q \le 0$, then $A_{n+1}(x;q) \in \mathrm{RZ}[1,+\infty)$.

(3) If $q \in \mathbb{Z}^-$, then $A_n(x;q) \in \mathrm{RZ}[1,+\infty)$ and $(x-1)^m \| A_n(x;q)$ where $m = \max\{n+q, 0\}$.

$$F(x) = a(x)f(x) + b_1(x)g_1(x) + \cdots + b_k(x)g_k(x)$$

Thm. (Liu & Wang, Adv. in Appl. Math., 2007a)

Let $F(x) = a(x)f(x) + b_1(x)g_1(x) + \cdots + b_k(x)g_k(x)$. Suppose that

(1) $f, g_j \in \mathrm{RZ}$ and $g_j \preccurlyeq f$ for each j.

(2) F and $g_1, \ldots, g_k$ are standard.

(3) $\deg F = \deg f$ or $\deg f + 1$.

If $b_j(r) \leq 0$ for each j whenever $f(r) = 0$, then $F \in \mathrm{RZ}$ and $f \preccurlyeq F$.

Ex. 1. Matching polynomials:

$$M(G,x) = xM(G-\{v\},x) - \sum_{u\sim v} M(G-\{v,u\},x).$$

2. Heilmann-Lieb partition functions: $W \geq 0$ and

$$Q(G,x) = xQ(G-\{v\},x) - \sum_{u\sim v} W(u,v)Q(G-\{v,u\},x).$$

3. Brenti derangement polynomials: $d_0(q)=1, d_1(q)=0$ and

$$d_n(q) = (n-1)qd_{n-1}(q) + q(1-q)d'_{n-1}(q) + (n-1)qd_{n-2}(q).$$

$$F(x) = a(x)f(x) + b(x)g(x), G(x) = c(x)f(x) + d(x)g(x)$$

Thm. (Liu & Wang, Adv. in Appl. Math., 2007)

Let $\begin{pmatrix} F(x) \\ G(x) \end{pmatrix} = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$ and

(1) $f, g \in \mathrm{RZ}$ and $g \preccurlyeq f$.

(2) F, G, f, g are standard.

(3) $\deg F = \deg G$ or $\deg G + 1$.

(4) $a(x)d(x) - b(x)c(x) \geq 0$ whenever $G(x) = 0$.

Suppose either that $c(x) \equiv c > 0$ and $\deg G \leq \deg g + 1$ or that $d(x) \equiv d > 0$ and $\deg G \leq \deg f$. Then $F, G \in \mathrm{RZ}$ and $G \preccurlyeq F$.

Coro. (Wang & Yeh, JCTA, 2005)

Let $F(x) = (ax + b)f(x) + (cx + d)g(x)$. Suppose that

(1) $f, g \in \mathrm{RZ}$ and $g \preccurlyeq f$.

(2) f and g are standard.

If $ad \leq bc$, then $F(x) \in \mathrm{RZ}$.

Proof. Let $G(x) = af(x) + cg(x)$. Then $F, G \in \mathrm{RZ}$ and $G \preccurlyeq F$. □

Ex. Let $a, b, c, d \in \mathbb{R}^+$. Suppose that f, g are standard and $g \preccurlyeq f$.

1. If $ad \geq bc$, then $cf + dg \preccurlyeq af + bg$. In particular, $g \preccurlyeq af + bg \preccurlyeq f$.

2. If $af - bg$ is standard, then $cf + dg \preccurlyeq af - bg$. In particular, $f, g \preccurlyeq af - bg$.

3. If $-af + bg$ is standard, then $-af + bg \preccurlyeq cf + dg$, and in particular, $-af + bg \preccurlyeq f, g$.

4. If $af - bg$ and $cf - dg$ are standard, then $ad \leq bc$ implies that $cf - dg \preccurlyeq af - bg$.

Ex. Given a finite graph G and a nonnegative integer k, let $\gamma(G,k)$ denote the number of distinct embeddings of the graph G into an oriented surface of genus k. Define the genus polynomial

$$GP(G,x)=\sum_{k\geq 0}\gamma(G,k)x^k.$$

Gross-Robbins-Tucker Conjecture. $\gamma(G,k)$ is log-concave in k.

Stahl Conjecture. $GP(G,x)\in \mathrm{RZ}$.

Stahl considered the H-linear family of graphs obtained by consistently amalgamating additional copies of a graph H. For such a family $\{G_n\}$, there is a square matrix M and vector v with entries in $\mathbb{Z}[x]$ such that the genus polynomial of G_n is the first entry of $M^n v$.

Stahl Example.

1. $M_1 = \begin{pmatrix} 4 & 2 \\ 6x & 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ x \end{pmatrix}.$ $\square$

2. $M_2 = \begin{pmatrix} 0 & 4 \\ 2x & 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$ $\square$

3. $M_3 = 4\begin{pmatrix} 2+3x & 1 \\ 4x & 2x \end{pmatrix}, \quad v_3 = 2\begin{pmatrix} 1+x \\ 2x \end{pmatrix}.$ $\square$

4. $M_4 = 6\begin{pmatrix} 3x & 3 \\ 2x & 1+3x \end{pmatrix}, \quad v_4 = 2\begin{pmatrix} 2 \\ 1+x \end{pmatrix}.$

5. $M_5 = \begin{pmatrix} 192x & 96+288x \\ 72+192x^2 & 24+288x \end{pmatrix}, \quad v_5 = \begin{pmatrix} 18+18x \\ 6+30x \end{pmatrix}.$

6. $M_6 = \begin{pmatrix} 8+68x & 4+16x \\ 32x+48x^2 & 16x \end{pmatrix}, \quad v_6 = \begin{pmatrix} 2+14x \\ 8x+8x^2 \end{pmatrix}.$

Stahl Question. Let $M(x) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{R}[x]$.

1. Under what conditions can it be guaranteed that if $(f(x), g(x))$ is a pair of polynomials whose zeros interlace, then so do the zeros of the two components of the vector $(f(x), g(x))M(x)$ interlace?

2. Under what conditions can it be guaranteed that the zeros of each of the entries of $M^k(x)$ are all real for $k = 1, 2, \ldots$.

- Let $M^k = \begin{pmatrix} a_k & c_k \\ b_k & d_k \end{pmatrix}$. Then $(a_{k+1}, c_{k+1}) = (a_k, c_k)M$.

Coro. Let $M = \begin{pmatrix} a(x) & c(x) \\ b(x) & d(x) \end{pmatrix}$ be a **nice** matrix of polynomials:

(1) $\deg a \le 1, \deg b \le 2, \deg d \le 1$ and c is a positive constant.

(2) $\det(M) \ge 0$ for $x \le 0$.

Suppose that $f, g \in \mathrm{PF}$ and $g \preccurlyeq f$. Then

1. If $(F, G) = (f, g)M$, then $F, G \in \mathrm{PF}$ and $G \preccurlyeq F$.
2. If $(G_1, F_1)^T = M(g, f)^T$, then $F_1, G_1 \in \mathrm{PF}$ and $G_1 \preccurlyeq F_1$.
3. Each entry of M^k has only real zeros for $k = 1, 2, \ldots$.

Proof. $F = af + bg$ and $G = cf + dg$. $\square$

Stahl Example. The matrices M_1, M_2, M_3, M_4 are nice and M_5, M_6 can decompose into the product of two nice matrices

$$\begin{aligned} M_5 &= 24 \begin{pmatrix} 4+12x & 8 \\ 1+12x & 3+8x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \\ M_6 &= 4 \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} 8+12x & 4 \\ 2+17x & 1+4x \end{pmatrix}., \end{aligned}$$

For $1 \le i \le 6, v_i^{(1)} \preccurlyeq v_i^{(2)}$, so every entry of $M_i^k v_i$ is in RZ.

Open problems

Characterize all real polynomial matrices that can be decomposed into the product of finite nice matrices and find an algorithm of decompositions for such matrices.

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Thank you for your attention!