Adjacent Vertex Distinguishing Total Colorings of Graphs

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1 Definitions

¶ Let $G = (V, E)$ be a simple graph.

¶ Edge-k-coloring:

A mapping $f : E \rightarrow \{1, 2, \ldots, k\}$ such that $f(e) \neq f(e')$ for any adjacent edges $e, e' \in E$.

¶ Edge chromatic number:

$\chi'(G) = \min\{k | G \text{ is edge-}k\text{-colorable}\}$.

¶ **Total-k-coloring**:

A mapping $f : V \cup E \to \{1, 2, \ldots, k\}$ such that any two adjacent vertices, adjacent edges, and incident vertex and edge are assigned to different colors.

¶ **Total chromatic number:**

$\chi''(G) = \min\{k | G$ is total-k-colorable$\}$.

- For an edge coloring f of G and for a vertex $v \in V$, we define:

$$C_f(v) = \{f(e) | e \text{ is incident to } v\}.$$

- For a total coloring f of G and for a vertex $v \in V$, we define:

$$C_f[v] = \{f(e) | e \text{ is incident to } v\} \cup \{f(v)\}.$$

¶ **Adjacent-Vertex-Distinguishing edge coloring (AVD edge coloring):**

A proper edge coloring f such that

$$C_f(u) \neq C_f(v)$$

for any adjacent vertices $u, v \in V$.

¶ **Adjacent-Vertex-Distinguishing edge chromatic number (AVD edge chromatic number):**

$\chi'_a(G) = \min\{k | G$ is AVD edge-k-colorable$\}$.

¶ **Adjacent-Vertex-Distinguishing total coloring (AVD total coloring):**

A proper total coloring f such that

$$C_f[u] \neq C_f[v]$$

for any adjacent vertices $u, v \in V$.

¶ **Adjacent-Vertex-Distinguishing total chromatic number (AVD edge total chromatic number):**

$$\chi''_a(G) = \min\{k|G \text{ is AVD total-}k\text{-colorable}\}.$$

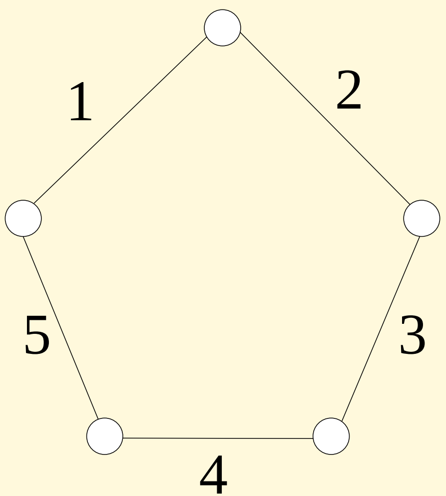
Examples

First Example: C_5

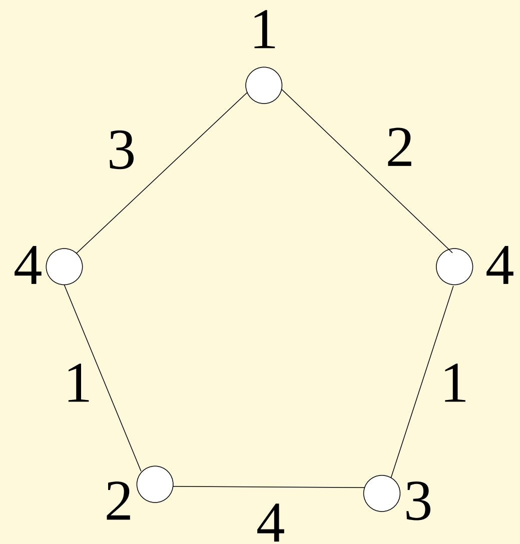
$$\chi'(C_5) = 3,$$

$$\chi'_a(C_5) = 5,$$

$$\chi''_a(C_5) = \chi''(C_5) = 4.$$



$$\chi_a'(C_5) = 5$$



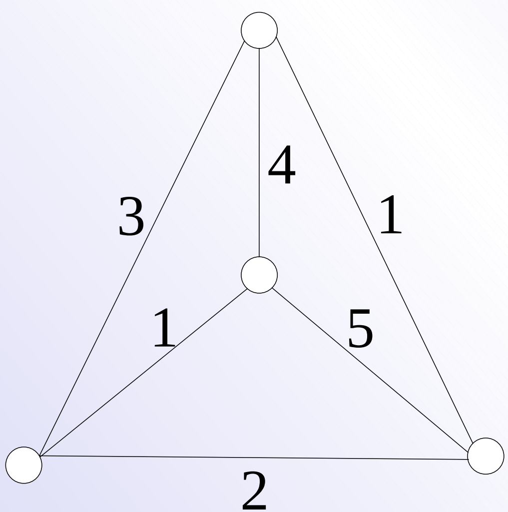
$$\chi_a''(C_5) = 4$$

Second Example: K_4

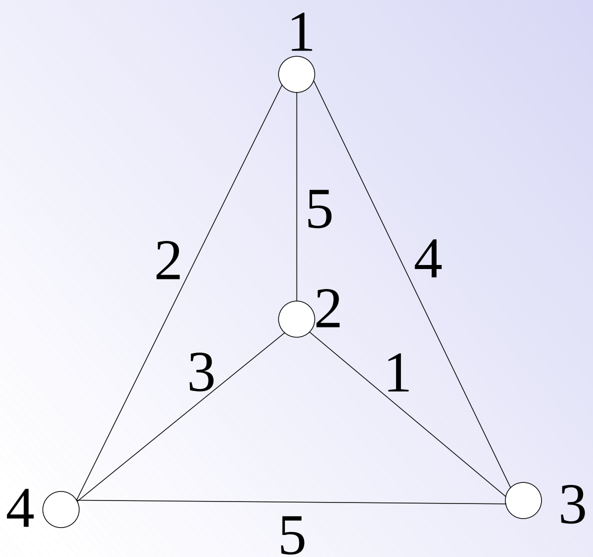
$$\chi'(K_4) = 3,$$

$$\chi'_a(K_4) = 5,$$

$$\chi''_a(K_4) = \chi''(K_4) = 5.$$



$$\chi'_a(K_4) = 5$$



$$\chi''_a(K_4) = 5$$

2 Some Known Results

Δ: the maximum degree of the graph G

δ: the minimum degree of the graph G

k-Vertex: a vertex of degree k

(2.1) AVD Edge Coloring

Normal graph: a graph without isolated edges

Vizing Theorem (Vizing, 1964)

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

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Conjecture 1 (Zhang et al. 2002)

For a normal graph $G(\neq C_5)$, $\chi'_a(G) \leq \Delta + 2$.

¶ $\Delta \leq \chi'(G) \leq \chi'_a(G)$.

¶ If G has two adjacent Δ-vertices, then $\chi'_a(G) \geq \Delta + 1$.

¶ For a normal tree T, $\chi'_a(T) \leq \Delta + 1$; $\chi'_a(T) = \Delta + 1 \Leftrightarrow T$ has adjacent Δ-vertices.

[Zhang Z, Liu L, Wang J, Appl. Math. Lett., 15(2002) 623-626]

¶ If G is normal and $\Delta = 3$, then $\chi'_a(G) \leq 5$.

¶ If G is normal and bipartite, then $\chi'_a(G) \leq \Delta + 2$.

¶ If G is normal, then $\chi'_a(G) \leq \Delta + O(\log \chi(G))$. ($\chi(G)$ is the vertex chromatic number of G).

[Balister, Győri, Lehel, Schelp, SIAM J. Discrete Math. 21(2007) 237-250]

¶ If G is normal and $\Delta > 10^{20}$, then $\chi'_a(G) \leq \Delta + 300$.

[Hatami, J. Combin. Theory Ser. B, 95(2005) 246-256]

(2.2) AVD Total Coloring

Conjecture 2 (Zhang et al. 2004)

For a graph G with $|G| \geq 2$, $\chi''_a(G) \leq \Delta + 3$.

¶ $\Delta + 1 \leq \chi''(G) \leq \chi''_a(G)$.

¶ If G has two adjacent Δ-vertices, then $\chi'_a(G) \geq \Delta + 2$.

¶ $\chi''_a(G) \le \chi(G) + \chi'(G)$.

If G is bipartite, then $\chi''_a(G) \le \Delta + 2$;

Equality holds $\Leftrightarrow$ G has adjacent Δ-vertices.

If G is planar, then $\chi''_a(G) \leq 4 + \Delta + 1 = \Delta + 5$.

(Using Four-Color Theorem and Vizing Theorem)

If $\chi'(G) = \Delta$ and $\chi(G) \leq 3$, then $\chi''_a(G) \leq \Delta + 3$.

¶ If $n \geq 4$, then $\chi''_a(C_n) = 4$.

¶ $\chi''_a(K_n) = n + 1$ if n is even, $\chi''_a(K_n) = n + 2$ otherwise.

¶ Let $n + m \geq 2$. Then $\chi''_a(K_{m,n}) = \Delta + 1$ if $m \neq n$, $\chi''_a(K_{m,n}) = \Delta + 2$ if $m = n$.

¶ For a tree T with $|T| \geq 2$, $\chi''_a(T) \leq \Delta + 2$; $\chi''_a(T) = \Delta + 2 \Leftrightarrow T$ has adjacent Δ-vertices.

[Zhang Z et al, Sci. China Ser. A 34(2004) 574-583]

¶ If $\Delta = 3$, then $\chi''_a(G) \leq 6$.

[Chen X, Discrete Math. 308(2008) 4003-4008]

[Wang H, J. Comb. Optim. 14(2007) 87-109]

3 Main Results

(3.1) χ''_a for outerplanar graphs.

A planar graph is called **outerplanar** if there is an embedding of G into the Euclidean plane such that all the vertices are incident to the unbounded face.

- If G is an outerplanar graph with $\Delta \geq 3$, then $\chi(G) \leq 3$, $\chi'(G) = \Delta$, thus

$$\chi''_a(G) \leq \chi(G) + \chi'(G) \leq \Delta + 3.$$

- If G is a 2-connected outerplanar graph with $\Delta = 3$, then $\chi''_a(G) = 5$.

  [Chen X, Zhang Z, J. Lanzhou Univ. Nat. Sci. 42(2006) 96-102]

¶ If G is a 2-connected outerplanar graph with $\Delta = 4$, then $5 \le \chi''_a(G) \le 6$; $\chi''_a(G) = 6 \Leftrightarrow G$ has adjacent Δ-vertices.

[Chen X, Zhang Z, J. Lanzhou Univ. Nat. Sci. 42(2006) 96-102]

¶ If G is a 2-connected outerplanar graph with $\Delta = 5$, then $6 \leq \chi''_a(G) \leq 7$; $\chi''_a(G) = 7 \Leftrightarrow G$ has adjacent Δ-vertices.

[Zhang S, Chen X, Liu X, Xibei Shifan Daxue Xuebao Ziran Kexue Ban 41(5)(2005) 8-13]

¶ If G is a 2-connected outerplanar graph with $\Delta = 6$, then $7 \le \chi''_a(G) \le 8$; $\chi''_a(G) = 8 \Leftrightarrow G$ has adjacent Δ-vertices.

[An Mingqiang, Hexi Xueyuan Xuebao 21(5)(2005) 25-29]

Theorem 1. Let G be an outerplane graph with $\Delta \geq 3$. Then

(1) $\Delta + 1 \leq \chi''_a(G) \leq \Delta + 2$;

(2) $\chi''_a(G) = \Delta + 2 \Leftrightarrow G$ has adjacent Δ-vertices.

(3.2) χ''_a for graphs with lower maximum average degree

The **maximum average degree** $\mathrm{mad}(G)$ of a graph G is defined by

$$\mathrm{mad}(G) = \max_{H \subseteq G} \{2|E(H)|/|V(H)|\}.$$

Theorem 2 . Let G be a graph with $\mathrm{mad}(G) = M$.

¶ If $M < 3$ and $\Delta \geq 5$, then

(1) $\Delta + 1 \leq \chi''_a(G) \leq \Delta + 2$;

(2) $\chi''_a(G) = \Delta + 2 \Leftrightarrow G$ has adjacent Δ-vertices.

¶ If $M < 3$ and $\Delta = 4$, then $\chi''_a(G) \leq 6$.

¶ If $M < \frac{8}{3}$ and $\Delta = 3$, then $\chi''_a(G) \leq 5$.

The **girth** g of a graph G is the length of a shortest cycle in G.

Let G be a planar graph. Then

$$\mathrm{mad}(G) < \frac{2g}{g-2}.$$

If G is planar and $g \geq 6$, $\mathrm{mad}(G) < 3$;

If G is planar and $g \geq 8$, $\mathrm{mad}(G) < \frac{8}{3}$.

Corollary 3. Let G be a planar graph.

- ¶ If $g \geq 6$ and $\Delta \geq 5$, then
  (1) $\Delta + 1 \leq \chi''_a(G) \leq \Delta + 2$;
  (2) $\chi''_a(G) = \Delta + 2 \Leftrightarrow G$ has adjacent Δ-vertices.
- ¶ If $g \geq 6$ and $\Delta = 4$, then $\chi''_a(G) \leq 6$.
- ¶ If $g \geq 8$ and $\Delta = 3$, then $\chi''_a(G) \leq 5$.

(3.3) χ'_a for graphs with lower maximum average degree

Theorem 4. Let G be a graph with $\text{mad}(G) = M$.

¶ If $M < 3$ and $\Delta \geq 3$, then $\chi'_a(G) \leq \Delta + 2$.

¶ If $M < \frac{5}{2}$ and $\Delta \geq 4$, then $\chi'_a(G) \leq \Delta + 1$.

¶ If $M < \frac{7}{3}$ and $\Delta = 3$, then $\chi'_a(G) \leq 4$.

¶ If $M < \frac{5}{2}$ and $\Delta \geq 5$, then

(1) $\Delta \leq \chi'_a(G) \leq \Delta + 1$;

(2) $\chi'_a(G) = \Delta + 1 \Leftrightarrow G$ has adjacent Δ-vertices.

Corollary 5. Let G be a planar graph. Then

¶ If $g \geq 6$ and $\Delta \geq 3$, then $\chi'_a(G) \leq \Delta + 2$.

¶ If $g \geq 10$ and $\Delta \geq 4$, then $\chi'_a(G) \leq \Delta + 1$.

¶ If $g \geq 14$ and $\Delta = 3$, then $\chi'_a(G) \leq 4$.

¶ If $g \geq 10$ and $\Delta \geq 5$, then

(1) $\Delta \leq \chi'_a(G) \leq \Delta + 1$;

(2) $\chi'_a(G) = \Delta + 1 \Leftrightarrow G$ has adjacent Δ-vertices.

4 Outline of Proofs

(4.1) Proof of Theorem 1

Lemma 1. Every outerplane graph G with $|G| \geq 2$ contains one of (C1)-(C5) as follows:

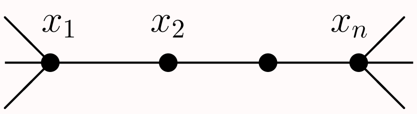
¶ (C1) a leaf is adjacent to a 3^--vertex.

¶ (C2) a path $x_1x_2\cdots x_n$, with $n \geq 4$, $d(x_1) \neq 2$, $d(x_n) \neq 2$, $d(x_2) = \cdots = d(x_{n-1}) = 2$.

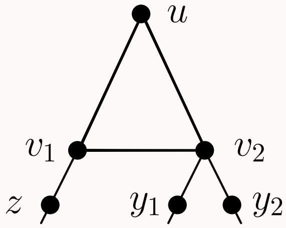
¶ (C3) a 4^+-vertex v is adjacent to a leaf and $d(v) - 3$ 2^--vertices.

¶ (C4) a 3-face $[uv_1v_2]$ with $d(u) = 2$, $d(v_1) = 3$.

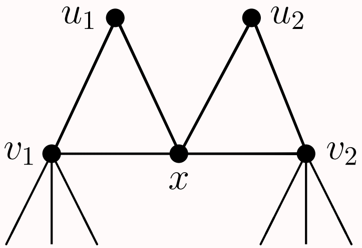
¶ (C5) two 3-faces $[u_1v_1x]$ and $[u_2v_2x]$ with $d(x) = 4$, $d(u_1) = d(u_2) = 2$.



(C2)



(C4)



(C5)

Lemma 2. Every outerplane graph G with $\Delta \leq 3$ contains one of (B1)-(B3):

- ¶ (B1) a vertex adjacent to at most one vertex that is not a leaf.
- ¶ (B2) a path $x_1x_2x_3x_4$ such that each of x_2 and x_3 is either a 2-vertex, or a 3-vertex that is adjacent to a leaf.
- ¶ (B3) a 3-face $[uxy]$ such that either $d(u) = 2$, or $d(u) = 3$ and u is adjacent to a leaf.

Lemma 3. Every outerplane graph G with $\Delta = 4$ contains one of (A1)-(A4):

¶ (A1) a vertex v with $d(v) \neq 3$ adjacent to a leaf.

¶ (A2) a 3-vertex adjacent to at least two leaves.

¶ (A3) a path $x_1x_2x_3x_4$ such that each of x_2 and x_3 is either a 2-vertex, or a 3-vertex that is adjacent to a leaf.

¶ (A4) a 3-face $[uxy]$ with $d(x) = 3$ such that either $d(u) = 2$, or $d(u) = 3$ and u is adjacent to a leaf.

Lemma 4. Every outerplane graph G with $\Delta = 3$ and without adjacent 3-vertices contains (D1) or (D2):

¶ (D1) a leaf.

¶ (D2) a cycle $x_1x_2\cdots x_nx_1$, with $n \geq 3$, $d(x_1) = 3$, $d(x_2) = \cdots = d(x_{n-1}) = 2$.

Theorem 1.1. If G is an outerplane graph with $\Delta \leq 3$, then $\chi''_a(G) \leq 5$.

Proof: The proof proceeds by induction on $\sigma(G)$ $(= |G| + ||G||)$. If $\sigma(G) \leq 5$, the theorem holds trivially. Suppose that G is an outerplane graph with $\Delta \leq 3$ and $\sigma(G) \geq 6$. By the induction assumption, any outerplane graph H with $\Delta(H) \leq 3$ and $\sigma(H) < \sigma(G)$ has a total-5-AVD-coloring f. By Lemma 2, G contains one of (B1)-(B3). We reduce each possible case to extend f to the whole graph G. □

Theorem 1.2. If G is an outerplane graph with $\Delta = 3$ and without adjacent 3-vertices, then $\chi''_a(G) = 4$.

Proof: By induction on $\sigma(G)$. By Lemma 4, we handle possible case (D1) or (D2). $\square$

Theorem 1.3. If G is an outerplane graph with $\Delta \geq 4$, then $\chi''_a(G) \leq \Delta + 2$.

Proof: By induction on $\sigma(G)$. By Lemma 1, we handle each possible case of (C1)-(C5). $\square$

Theorem 1.4. If G is an outerplane graph with $\Delta \geq 4$ and without adjacent Δ-vertices, then $\chi''_a(G) = \Delta + 1$.

Proof: By induction on $\sigma(G)$. By Lemma 3, we handle each possible case of (A1)-(A4). $\square$

(4.2) Proof of Theorem 2

Theorem 2.1. If G is a graph with $\text{mad}(G) < 3$ and $K(G) = \max\{\Delta + 2, 6\}$, then $\chi''_a(G) \leq K(G)$.

Proof: The proof proceeds by induction on $\sigma(G)$ ($= |G| + ||G||$). If $\sigma(G) \leq 5$, the theorem holds trivially. Suppose that G is a graph with $\text{mad}(G) < 3$ and $\sigma(G) \geq 6$. By the induction assumption, any proper subgraph H of G has a total-K-AVD-coloring f.

Claim 1. No 3^--vertex is adjacent to a leaf.

Claim 2. No path $x_1x_2\cdots x_n$ with $d(x_1), d(x_n) \geq 3$, $d(x_2) = \cdots = d(x_{n-1}) = 2$, where $n \geq 4$.

Claim 3. No k-vertex v, $k \geq 4$, with neighbors $v_1, v_2, \cdots, v_k$ such that $d(v_1) = 1$, $d(v_i) \leq 2$ for $2 \leq i \leq k-2$.

Claim 4. No 2-vertex adjacent to a 3-vertex.

Claim 5. No 4-vertex adjacent to three 2-vertices.

Claim 6. No 5-vertex v adjacent to five 2-vertices.

Let H be the graph obtained by removing all leaves of G. Then $\mathrm{mad}(H) \leq \mathrm{mad}(G) < 3$. H has the following properties:

Claim 7.

(1) $\delta(H) \geq 2$;

(2) If $2 \leq d_G(v) \leq 3$, then $d_H(v) = d_G(v)$;

(3) If $d_H(v) = 2$, then $d_G(v) = 2$;

(4) If $d_G(v) \geq 4$, then $d_H(v) \geq 3$.

We make use of discharging method. First, we define an initial charge function

$$w(v) = d_H(v) \text{ for every } v \in V(H).$$

Next, we design a discharging rule and redistribute weights accordingly. Once the discharging is finished, a new charge function w' is produced. However, the sum of all charges is kept fixed when the discharging is in progress. Nevertheless, we can show that $w'(v) \geq 3$ for all $v \in V(H)$. This leads to the following obvious contradiction:

$$
\begin{aligned}
3 &= \frac{3|V(H)|}{|V(H)|} \leq \frac{\sum_{v \in V(H)} w'(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w(v)}{|V(H)|} \\
&= \frac{2|E(H)|}{|V(H)|} \leq \mathrm{mad}(H) < 3.
\end{aligned}
$$

The discharging rule is defined as follows:

(R) Every 4^+-vertex gives $\frac{1}{2}$ to each adjacent 2-vertex.

Let $v \in V(H)$. So $d_H(v) \geq 2$ by Claim 7(1).

If $d_H(v) = 2$, then v is adjacent to two 4^+-vertices by Claim 4. By (R),

$$w'(v) \geq d_H(v) + 2 \times \frac{1}{2} = 2 + 1 = 3.$$

If $d_H(v) = 3$, then $w'(v) = w(v) = 3$.

If $d_H(v) = 4$, then v is adjacent to at most two 2-vertices by Claim 5. Thus,

$$w'(v) \geq 4 - 2 \times \frac{1}{2} = 3.$$

If $d_H(v) = 5$, then v is adjacent to at most four 2-vertices by Claim 6. Thus,

$$w'(v) \geq 5 - 4 \times \frac{1}{2} = 3.$$

If $d_H(v) \geq 6$, then v is adjacent to at most $d_H(v)$ 2-vertices and hence

$$w'(v) \geq d_H(v) - \frac{1}{2}d_H(v) = \frac{1}{2}d_H(v) \geq 3.$$

Thanks for your coming!