Automorphism groups of Cayley digraphs

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The 3rd National Conference on Combinatorics and Graph Theory, July 20, 2008

Outline

Aim of this report

- In this report we shall give a brief survey of recent results on automorphism groups of Cayley digraphs concentrating on the normality of Cayley digraphs.
- The normality of Cayley digraphs is basic to investigate Cayley digraphs and it has been heavily applied to classify symmetric graphs, half-arc-transitive graphs, semisymmetric graph, etc, which are currently some of the main topics in algebraic graph theory. Also it has many other applications, specially in interconnection networks.

Permutation groups

- Permutation g on a finite set Ω: a bijection $g : \Omega \to \Omega$.
- Symmetric group S_Ω: the group of all permutations of Ω under permutation composition. For example, the composition of $g = (1\ 2)$ and $h = (2\ 3)$ yields $gh = (1\ 3\ 2)$, and $f = (1\ 3\ 2)$ has inverse $(1\ 2\ 3)$.
- Permutation group on Ω: a subgroup of S_Ω, that is, a subset of S_Ω closed under inverses and products (compositions).
- Example: $G = \langle (1\ 2\ 3\ 4\ 5) \rangle < S_\Omega$ on $\Omega = \{1, 2, 3, 4, 5\}$.

Regular permutation groups

- Let G be a permutation group on Ω, that is, $G \leq S_\Omega$.
- G is transitive on Ω: for any two points in Ω there is a permutation in G mapping one to the other.
- G is regular on Ω: for any two points in Ω there is one and only one permutation in G mapping one to the other, that is, only the identity element in the transitive subgroup fixes a point. A regular permutation group is ‘the smallest possible transitive group’.
- Example: $G = \langle (1\ 2\ 3\ 4\ 5) \rangle < S_\Omega$ on $\{1, 2, 3, 4, 5\}$.
  $H = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ on $\{1, 2, 3, 4\}$.

Notation for graphs

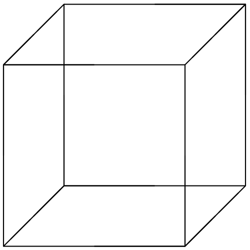
- X: a simple graph or digraph (directed graph).
- $V(X)$, $E(X)$: the vertex set and the edge set.
- The automorphism group $\mathrm{Aut}(X)$ of a graph X: the group of all permutations on $V(X)$ preserving the adjacency of X, that is, mapping an edge to an edge.
- X is vertex-transitive or edge-transitive: $\mathrm{Aut}(X)$ is transitive on $V(X)$ or $E(X)$, respectively.
- For a undirected graph X, each edge $\{u, v\}$ of X gives two ordered pairs (u, v) and (v, u), called arcs of X. If necessary, we can view a graph X as a directed graph in this way.

Notation for graphs

- s-arc: an $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices s.t. $(v_i, v_{i+1}) \in E(X)$, $v_{i-1} \neq v_{i+1}$.
- s-arc-transitive: $\mathrm{Aut}(X)$ acts transitively on the set of s-arcs in X
- 0-arc-transitive: vertex-transitive
- 1-arc-transitive: arc-transitive or symmetric
- For a subgroup G of $\mathrm{Aut}(X)$, G is s-regular or X is (G, s)-regular if G acts regularly on the set of s-arc of X.
- s-regular: X is $(\mathrm{Aut}(X), s)$-regular
- X half-arc-transitive: vertex-transitive, edge-transitive, but not arc-transitive

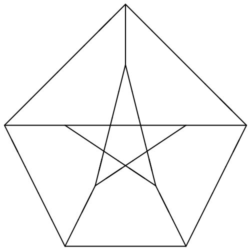
Automorphisms and symmetry

The symmetry of a graph can be measured by its automorphism group. For example, Petersen graph O_3 is more symmetry than the three dimensional hypercube Q_3.



$\mathrm{Aut}(Q_3) \cong S_4 \times \mathbb{Z}_2$

Q_3 is 2-arc-transitive



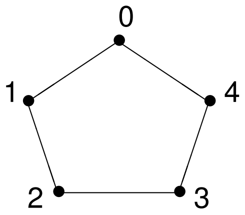
$\mathrm{Aut}(O_3) \cong S_5$

O_3 is 3-arc-transitive

Cayley digraphs

Let G be a finite group and $S \subset G$ with $1 \notin S$.

- Cayley digraph $\mathrm{Cay}(G, S)$: vertex set $V = G$, directed edge set $E = \{(g, sg) \mid g \in G, s \in S\}$
- Example: $G = \mathbb{Z}_5$, $S = \{1, 4\}$, $X = \mathrm{Cay}(G, S)$ is a cycle of length 5.



Basic properties of Cayley digraphs

- Cayley graph: if $S = S^{-1} = \{s^{-1} \mid \in S\}$, then $\mathrm{Cay}(G, S)$ can be viewed as an undirected graph, by identifying two directed edges (x, y) and (y, x) with one edge $\{x, y\}$.
- $\mathrm{Cay}(G, S)$ is connected $\Leftrightarrow G = \langle S \rangle$.
- Right regular representation $R(G)$ of G: the permutation group $\{R(g) \mid g \in G\}$ on G, where $R(g) : x \mapsto xg, \ \forall x \in G$ is a permutation on G. Clearly, $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$, which acts regularly on $V(X)$. Thus, any Cayley (di)graph is vertex-transitive.
- Characterization: A (di)graph X is isomorphic to a Cayley (di)graph on $G \Leftrightarrow \mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices.

Normal Cayley digraphs

Normal Cayley digraphs

- Normal Cayley digraph $\mathrm{Cay}(G, S)$: $R(G)$ is a normal subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$.
- Example: $G = \mathbb{Z}_5$, $S = \{1, 4\}$, $X = \mathrm{Cay}(G, S)$ is a normal Cayley graph on G. $\mathrm{Aut}(X) \cong D_{10}$.

- Set $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. Then, $\mathrm{Aut}(G, S)$ is a subgroup of the stabilizer $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$ of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$.
- $\mathrm{Cay}(G, S)$ normal: $\Leftrightarrow \mathrm{Aut}(\mathrm{Cay}(G, S)) = R(G) \rtimes \mathrm{Aut}(G, S)$ $\Leftrightarrow \mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$.
- Normal Cayley (di)graph has the ‘smallest possible automorphism group’.

Importance

Problem

Determine automorphism groups of Cayley digraphs.

- Up to now, it is no hope to solve the above problem completely.
- One of the main methods in algebraic graph theory is to use the automorphism group of a graph to study the graph.
- M.Y. Xu [40] conjectured that ‘most’ connected Cayley digraphs are normal.
- Studying normality, or determining automorphism groups of Cayley digraphs, has been becoming an very active topic in the algebraic graph theory, which also plays an important role in the investigation of various symmetry properties of digraphs.

Normality Depends on groups

- Being a normal Cayley digraph is not invariant under digraph isomorphisms, and so strictly depends upon which group the digraph is a Cayley digraph on. This means that it is possible a Cayley digraph on a group is normal and at the same time it is nonnormal as a Cayley graph on another group.

Example

The three-dimensional hypercube Q_3 is a Cayley graph on either the group $\mathbb{Z}_2^3$ or the group $\mathbb{Z}_4 \times \mathbb{Z}_2$, and the Cayley graph on the first group is normal, but the Cayley graph on the second group is nonnormal.

Equivalence

- Two subsets S and T of a group G with $1 \notin S$ and $1 \notin T$ are said to be equivalent if there is a group automorphism of G mapping S to T, denoted by $S \equiv T$.
- Let $S \equiv T$. Then $\mathrm{Cay}(G, S)$ is normal $\Leftrightarrow$ $\mathrm{Cay}(G, T)$ is normal.
- When a classification of normal (or nonnormal) Cayley digraphs on a given group is done, usually all non-equivalent subsets corresponding to normal (nonnormal) Cayley digraphs are given.

Remarks

Question

Let G be a finite group G. Are there two subsets S and T of G with $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ such that $\mathrm{Cay}(G, S)$ is normal and $\mathrm{Cay}(G, T)$ is nonnormal?

A basic result

All disconnected normal Cayley digraphs have been determined (see Wang, Wang and Xu [38], or [40, Proposition 2.4 and 2.5])).

- when the normality of Cayley digraphs on a given group is considered, it suffices to consider the connected ones.

Examples

Example 1 ([19])

Let G be a nonabelian group of order $2p^2$ for an odd prime p and S a two-element generating subset of G. Then $X = \mathrm{Cay}(G, S)$ is nonnormal if and only if

$$G = \langle a, b, c \mid a^p = b^p = c^2 = 1, [a, b] = [a, c] = 1, c^{-1}bc = b^{-1} \rangle$$

and $S \equiv \{ca, cba^{-1}\}$; further, $\mathrm{Aut}(X) \cong R(G) \cdot (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Examples

Example 2 ([46])

Let $X = \mathrm{Cay}(G, S)$ be a connected cubic Cayley graph of order $4p$ for a prime p. Then either $R(G) \trianglelefteq \mathrm{Aut}(X)$, or one of the following happens:

(1) $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, $S \equiv \{a, a^{-1}, b\}$, $\mathrm{Aut}(X) \cong \mathbb{Z}_2^3 \rtimes S_3$, and $X \cong Q_3$, the three-dimensional hypercube of order 8;

(2) $G = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$, $S \equiv \{b, a, a^{-1}\}$ or $\{b, ab, a^2b\}$, and $X \cong Q_3$;

(3) $G = \langle a, b \mid a^{2p} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ for $p \geq 3$, $S \equiv \{b, ab, a^pb\}$ and $\mathrm{Aut}(X) \cong \mathbb{Z}_2^p \rtimes D_{2p}$

Cayley (di)graphs of order p

- It is very difficult to determine the normality of Cayley (di)graphs for general valencies. In fact the only groups, for which the complete information about the normality of Cayley (di)graphs is available, are the groups of order p and pq, where p and q are primes.
- Burnside's Theorem: Every transitive permutation group of prime degree p is either 2-transitive or solvable with a regular normal Sylow p-subgroup. This implies the following result (also see Alspach [1]).

Normality of Cayley (di)graphs of order p

A Cayley (di)graph on the cyclic groups of order a prime p is normal except the Cayley (di)graph is isomorphic the complete (di)graph of order p.

Cayley digraphs of order *pq*

Xu 1998 [40] posed the following.

Problem

Determine the automorphism groups of Cayley digraphs of order a product of two primes?

Answer

Du, Wang and Xu 1998 [7]: the normality of Cayley digraphs of order twice a prime.

Dobson and Witte 2002 [6]: the normality of Cayley digraphs of order a prime square.

Lu and Xu 2003 [27]: the normality of Cayley digraphs of order a product of two distinct odd primes.

What is Minimal Cayley digraph?

A generating set S of a group G is called minimal if S generates G but $S \backslash \{s\}$ cannot generate G for any $s \in S$. A Cayley digraph $\mathrm{Cay}(G, S)$ is minimal if S is a minimal generating set of G.

Let S be a minimal generating subset of G. Xu [40, Problem 6] conjectured that $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, S \cup S^{-1})$ are normal for any minimal generating set S of G. The answer is negative.

Meng and Ying 2000 [32]: determined all finite abelian groups whose Cayley digraph with respect to any given minimal generating set is normal.

Feng, Liu and Xu 2002 [20]: determined all finite abelian groups whose Cayley graph with respect to any given minimal generating set and its inverse is normal.

Cayley (di)graphs on abelian groups

Baik et al.1998,2000 [4, 3]: determined all nonnormal Cayley graphs on abelian groups with valences less than or equal to 5.

Feng and Gao 1997 [15]: Let G be a finite abelian group such that the Sylow 2-subgroup of G is cyclic. A minimal Cayley (di)graph on G is normal.

Meng and Huang 1997 [31]: proved all strongly connected minimal Cayley digraphs on infinite cyclic group $\mathbb{Z}$ has automorphism group isomorphic to $R(\mathbb{Z})$.

Xu, Zhang and Zhou 2005 [42]: determined all nonnormal Cayley digraphs on abelian groups with valences less than or equal to 3.

Minimal Cayley (di)graphs on symmetric groups

Feng 2006 [13]: For any minimal generating set S of transpositions of S_n, the Cayley graph $\mathrm{Cay}(S_n, S)$ is normal. This generalizes a result of Godsil and Royle [24].

Many interconnection networks were constructed from those Cayley graphs $\mathrm{Cay}(S_n, S)$. Let $S_1 = \{(i\ i+1) \mid 1 \leq i \leq n-1\}$, $S_2 = \{(1\ i) \mid 2 \leq i \leq n\}$ and $S_3 = \{(i\ i+1) \mid 1 \leq i \leq n-1\} \cup (1\ n)$. The Cayley graphs $\mathrm{Cay}(S_n, S_1)$, $\mathrm{Cay}(S_n, S_2)$ and $\mathrm{Cay}(S_n, S_3)$ are called the bubble-sort network BS_n, the star network ST_n, and the modified bubble-sort network MB_n respectively (see [25]). Feng's result shows that the underlying graphs of those interconnection networks are normal Cayley graphs on S_n, so that their automorphism groups are known.

A general description

Fang, Praeger and Wang [12]: obtained a general description of the possibilities for the automorphism groups of connected Cayley graphs on a finite non-abelian simple group.

Concrete results

Praeger 1999 [33]: proved that all connected normal edge-transitive cubic Cayley graphs on non-abelian simple groups are normal.

Li 1996 [26]: showed that the only possibilities for connected nonnormal arc-transitive cubic Cayley graphs on nonabelian simple groups must arise from one of the groups A_5, $L_2(11)$, M_{11}, A_{11}, M_{23}, A_{23} and A_{47}.

S.J. Xu, X.G. Fang, J. Wang and M.Y. Xu 2005,2007 [44, 45]: Let G be a finite nonabelian simple group and let $X = \mathrm{Cay}(G, S)$ be a connected arc-transitive cubic Cayley graph on G. Then X is normal or $G = A_{47}$. Furthermore, There are exactly two nonnormal cubic Cayley graphs on A_{47} which are 5-regular with automorphism groups isomorphic to A_{48}.

Concrete results

Fang *et al.* 2002 [9]: proved that the vast majority of connected cubic Cayley graphs on non-abelian simple groups are normal.

Fang, Li and Xu 2004 [10]: gave a general description of the possibilities for the automorphism groups of tetravalent connected edge-transitive Cayley graphs on a finite non-abelian simple group.

Zhou and Feng 2007 [47]: gave two sufficient conditions for nonnormal Cayley graphs of valency 5. Using these, they determined all nonnormal Cayley graphs of A_5 with valency 5. Combining the result given by S.J. Xu and M.Y. Xu[43], all nonnormal Cayley graphs on A_5 with valency 4 or 5 are known.

Concrete results

Zhou and Feng 2007 [47]: constructed three infinite families of nonnormal Cayley graphs of valency 5 on $\mathrm{PSL}(2, p)$ and A_p. There are 6 known nonnormal Cayley graphs on nonabelian simple groups of valency 3 or 4.

Problem

Are there infinitely many connected nonnormal Cayley graphs of valency 3 or 4 on non-abelian simple groups?

Remark: There are some progress on this problem and infinite many nonnormal Cayley graphs of valency 4 on non-abelian simple groups are constructed recently.

Directed case

A Cayley digraph $\mathrm{Cay}(G, S)$ is a directed Cayley graph if $S^{-1} \neq S$.

Li 1996 [26]: Let X be a connected directed 2-valent Cayley graph on $G = \mathrm{PSL}_2(q)$. Then $R(G) \trianglelefteq \mathrm{Aut}(X)$.

Fang, Lu, Wang and Xu 2004 [11]: proved that the vast majority of connected directed 2-valent Cayley graphs on non-abelian simple groups are normal.

What is a regular p-group?

- A finite p-group P is called a regular p-group if for any two elements x and y in P, there exist $c_1, c_2, \cdots, c_r$ in the derived group $\langle x, y\rangle'$ such that $(xy)^p = x^p y^p c_1^p c_2^p \cdots c_r^p$.
- By Huppert [28, III. Theorem 10.2], a p-group of order p^n with $n \leq p$ is regular.

Cayley (di)graphs on p-groups

Feng, Wang and Xu 2002 [21]: determined the nonnormal Cayley digraphs of valency 2 on a regular p-group.

Feng and Xu 2005 [22]: determined the nonnormal Cayley graphs of valency 4 on a regular p-group.

A corollary: Let G be a p-group of order p^n with $n \leq p$. Then all connected tetravalent Cayley graph on G are normal except for $p = 5$.

Feng et al. 2001, 2005 [16, 23]: A tetravalent connected Cayley graph on a p-group G for an odd prime p is normal if G has nilpotent class 2 or has order p^3.

- If G has order p^4, there are nonnormal tetravalent connected Cayley graphs on G. This is also true if G has nilpotent class 3.

A nonnormal Cayley graph

Example

Let
$G = \langle a, b, c | a^9 = b^3 = c^3 = 1, [a, b] = a^3, [a, c] = b, [b, c] = 1 \rangle$ and $S = \{a, ac, a^{-1}, (ac)^{-1}\}$. Then $X = \mathrm{Cay}(G, S)$ is nonnormal. Moreover if we set $A = \mathrm{Aut}(X)$ and $A_1^* = \{\alpha \in A_1 | s^\alpha = s, \ \forall s \in S\}$, then $A_1^* \cong \mathrm{Aut}(G, S) \cong \mathbb{Z}_2$ and $A_1/A_1^* \cong D_8$. Note that $|G| = 3^4$ and G has nilpotent class 3.

Tetravalent half-arc-transitive case

Feng, Kwak, Xu and Zhou 2008 [18]: Let X be a connected tetravalent half-arc-transitive graph of order p^4 for a prime p. Then $p \geq 3$ and X is a normal Cayley graph on a non-abelian group G of order p^4.

- The above result is true if X has order p^3 (see Xu [41]).

A problem

- There exist connected tetravalent half-arc-transitive graphs whose stabilizers are not isomorphic to $\mathbb{Z}_2$ (see [5, 29, 30]).
- Every tetravalent vertex transitive graph with an odd prime power must be a Cayley graph (see [14]).

Problem

Does there exist a connected tetravalent half-arc-transitive Cayley graph with order an odd-prime power which is nonnormal?

Cayley graphs on dihedral groups

Du, Feng, Kwak and Xu 2004 [8, 17]: determined the normal cubic Cayley graphs on a dihedral group.

Wang and Xu 2007 [39]: Let $D_{2n} = \langle a, b \mid a^2 = b^n = 1, a^{-1}ba = b^{-1} \rangle$ be the dihedral group of order $2n$ ($n > 6$) and let $X = \mathrm{Cay}(D_{2n}, S)$ be a 1-regular Cayley graph of valency 4. Then X is normal, unless $n = 2m$, $m \geq 4$ is even, and $X \cong \mathrm{Cay}(G, \{a, a^{-1}, a^i b, a^{-i} b\})$, where $2 \leq i \leq m-2$, and $i^2 = \pm 1 \pmod{m}$. Moreover, if $i^2 = -1 \pmod{m}$ then $m = 2 \pmod{4}$, and the vertex-stabilizer is isomorphic to $\mathbb{Z}_4$; while if $i^2 = 1 \pmod{m}$ then $m = 0 \pmod{4}$, and the vertex-stabilizer is isomorphic to $\mathbb{Z}_2^2$.

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Thanks!