山东理工大学·数学与信息科学学院
2008年

# 图的因子与韧度

## Toughness and factors of graph

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## 1 Definition

- **factor of graph**

  **Let $g$ and $f$ be two integer-valued functions defined on $V(G)$. A spanning subgraph $F$ of $G$ is called a $(g, f)$-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for all $x \in V(F)$. A $(g, f)$-factor is called a $f$-factor,$k$-factor if $g(x) = f(x)$,$g(x) = f(x) = k$. In particular, 1-factor is also called perfect matching.**

- **fractional factor of graph** **Let** $h : E(G) \to [0, 1]$ **be a function. A function** $h$ **is called a fractional** $(g, f)$**-factor if and only if** $g(x) \leq h(E_x) \leq f(x)$ **holds for any vertex** $x \in V(G)$**, where** $h(E_x) = \sum_{e \in E_x} h(e)$**,** $E_x = \{e \in E(G) | e$ **is incident with** $x$ **in** $E(G)\}$**. If** $g(x) = f(x)$ **or** $g(x) = f(x) = k$**, then a fractional** $(g, f)$**-factor is called a fractional** $f$**-factor or a fractional** $k$**-factor. In particular, a fractional [0,1]-factor is also called a fractional matching and a fractional 1-factor is also called a fractional perfect matching**

- **toughness**

  **A graph is $t$-tough if for any $S \subseteq V(G)$ and $\omega(G-S) > 1$, we have**

$$|S| \geq t\omega(G-S)$$

**holds where $\omega(G-S)$ denotes the number of components of $G-S$.**

**A complete graph is $t$-tough for any positive real number $t$. If $G$ is not complete, there exists a largest $t$ such that $G$ is $t$-tough. This number is called the toughness of $G$ and denoted by $t(G)$.**

**We define $t(K_n) = \infty$. If $G$ is not complete,**

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} | S \subseteq V(G), \omega(G-S) \geq 2\}.$$

**The toughness of a graph was first introduced by Chvátal.**

# 2 Some results

- **Since then a lot of research has been done. mainly relating**
- **toughness conditions to the existence of cycle structure. Most of these are still open.**

- **Another stream involved finding toughness conditions for the existence of certain factors of graphs.**

- **Research on toughness has also focused on computational complexity issues. We now know that it is NP-hard to compute the toughness of a graph.**

⋆ **main results**

- **Theorem 1(H.Enomoto)** ***Let $G$ be a graph. If $G$ is $k$-tough, $|V(G)| \geq k+1$ and $k|V(G)|$ is even, then G has a $k$-factor.***

  **The result was originally a conjecture by Chvátal and is sharp since for any positive real number $\epsilon$, there exists a graph $G$ that has no $k$-factor with $t(G) \geq k - \epsilon$.**

⋆ **Another toughness**

Another toughness was Defined

$$\tau'(G) = min\{\frac{|S|}{\omega(G-S)-1}|\omega(G-S)\geq 2\}$$

if $G$ is not complete. Otherwise $\tau'(G)=\infty$.

Obviously $\tau'(G) > t(G)$

H.Enomoto improved Th for $k = 1, 2$ and sufficiently large number of vertices for k.

⋆ Theorem 2 (Katerinis) Let $a \leq b$ and G be a graph .

1) Suppose that $t(G) \geq \frac{(b-a)^2+2(b-a)}{4a}$ when $b \equiv a(mod2)$ and $t(G) \geq \frac{(b+a)^2+2(b-a)+1}{4a}$ otherwise. If $f$ is an integer-valued function such that $a \leq f(x) \leq b$ and $f(V(G)) \equiv 0(mod2)$. then $G$ has an $f$-factor.

2) If $a < b$ or $bn$ is even, hen $G$ has an $[a, b]$-factor if $t(G) \geq a + \frac{a}{b} - 1$.

Progress has also been made on the relationship between toughness and $(k, n)$-factor critical graphs for $k = 1, 2, 3$.
Also t-edge toughness was introduced by Katona.

⋆ toughness and fractional factors Recently many authors are studying fractional factors. G.Liu and L. Zhang discussed the sufficient condition of fractional $k$-factors with $k \geq 1$ related to toughness.

**Theorem 3(Liu and Zhang)** *Let $G$ be a graph with $|V(G)| \geq 2$. Then $G$ has a fractional 1-factor if $t(G) \geq 1$.*

**Theorem 4(Liu and Zhang)** *Let $k \geq 2$ be an integer. A graph $G$ has a fractional $k$-factor if $t(G) \geq k - \frac{1}{k}$.*

They also give two important lemmas.

**Lemma 1** *Let $G$ be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of $H$ is isomorphic to $K_k$ where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set $I$ and the covering set $C = V(H) \backslash I$ of $H$ satisfying*

$$|V(H)| \leq (k - \frac{1}{k+1})|I|, \quad |C| \leq (k - 1 - \frac{1}{k+1})|I|.$$

**Lemma 2** *Let $G$ be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. let $T_1, \cdots, T_{k-1}$ be a partition of the vertices of $H$ satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some $T_j$ to be empty. If each component of $H$ has a vertex of degree at most $k-2$ in $G$, then $G$ has a maximal independent set $I$ and a covering set $C = V(H) \setminus I$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

*where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for every $j = 1, \cdots, k-1$.*

we consider the relationship between the toughness and the existence of fractional $f$-factors.

**Theorem 5** *Let $G$ be a graph and $f$ an integer-valued function on $V(G)$ satisfying $a \leq f(x) \leq b$ with $1 \leq a \leq b$ and $b \geq 2$ for all $x \in V(G)$. If $t(G) \geq \frac{b^2+b}{a} - \frac{b+1}{b}$, then $G$ has a fractional $f$-factor.*

Obviously, we can obtain the above results with $a = f(x) = b$ for all $x \in V(G)$. Our result is also sharp in the sense of $f(x) = k$ for all $x \in V(G)$.

⋆ **Liu and Ma's isolated toughness** G. Liu and Y. Ma defined a parameter isolated toughness of a graph

$$I(G) = \min\{\frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2\},$$

if $G$ is not complete, where $i(G-S)$ denotes the number of isolated vertices of $G-S$; $I(G) = \infty$, if $G$ is complete. Obviously, $I(G) \geq t(G)$.

⋆ **An isolated toughness in Bipartite graph**

**we defined a new isolated toughness in Bipartite graph**

$$I'(G) = \min\{\frac{|S|}{i(G-S)} : S \subseteq X \textbf{ or } Y, i(G-S) \geq 2\}$$

**where $G$ is not a complete bipartite graph. Otherwise $I'(G) = \infty$.**

**We apply this parameter to characterize $(g, f)$-factors in bipartite graph and obtain some sufficient conditions related to the new toughness for a bipartite graph to have $(g, f)$-factors.**

Our main results are as follows.

**Theorem 6** *Let* $G = (X, Y, E(G))$ *be a bipartite graph and* $g, f$ *be two positive integer-valued functions defined on* $V(G)$ *such that* $a \le g(x) \le f(x) \le b$ *for all* $x \in V(G)$*, where* $a, b$ *are integers and* $2 \le a \le b$*. Let* $|X| = |Y| = n \ge \frac{(a+b-1)(a+b-2)-1}{a}$*,* $\delta(G) \ge b$ *and* $f(X) = f(Y)$*,* $g(X) = g(Y)$*. If* $I'(G) > \frac{(a+b-1)(n-1)}{an}$*, then* $G$ *has a* $(g, f)$*-factor.*

**Theorem 7** *Let* $G = (X, Y, E(G))$ *be a bipartite graph and* $g, f$ *be two positive integer-valued functions defined on* $V(G)$ *such that* $a \le g(x) \le f(x) \le b$ *for all* $x \in V(G)$*, where* $a, b$ *are integers and* $1 \le a \le b$*. Let* $|X| = |Y| = n > \frac{(a+b+2)(a+b-2)}{4a}$*,* $f(X) = f(Y)$ *and* $g(X) = g(Y)$*. If* $I'(G) > \frac{n-1}{2\sqrt{an+1}-a-b}$*, then* $G$ *has a* $(g, f)$*-factor.*

Furthermore, these results are proved to be sharp in some sense.

# 谢 谢 大 家 ！