Triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules of rank 2

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Abstract

The theory of (φ_q, Γ)-modules is a generalization of Fontaine's theory of (φ, Γ)-modules, which classifies G_F-representations on $\mathcal{O}_F$-modules and F-vector spaces for any finite extension F of $\mathbb{Q}_p$. In this paper following Colmez's method we classify triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules of rank 2. In this process we establish two kinds of cohomology theories for $\mathcal{O}_F$-analytic (φ_q, Γ)-modules. Using them we show that, if D is an $\mathcal{O}_F$-analytic (φ_q, Γ)-module such that $D^{\varphi_q=1,\Gamma=1} = 0$ i.e. $V^{G_F} = 0$ where V is the Galois representation attached to D, then any overconvergent extension of the trivial representation of G_F by V is $\mathcal{O}_F$-analytic. In particular, contrarily to the case of $F = \mathbb{Q}_p$, there are representations of G_F that are not overconvergent.

Contents

Introduction

The present paper heavily depends on the theory of (φ, Γ)-modules for Lubin-Tate extensions, a generalization of Fontaine's theory of (φ, Γ)-modules. The existence of this generalization was more or less implicit in [14, 8]. See also [15] and [25, Remark 2.3.1]. In [17], Kisin and Ren provided details, where (φ, Γ)-modules for Lubin-Tate extensions are called (φ_q, Γ)-modules.

To recall this theory, let F be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ the ring of integers in F and π a unifomizer of $\mathcal{O}_F$. Fix an algebraic closure of F denoted by $\bar{F}$, and put $G_F = \mathrm{Gal}(\bar{F}/F)$. Let k_F be the residue field of F, $q = \# k_F$. Let $\mathrm{W} = \mathrm{W}(k_F)$ be the ring of Witt vectors over k_F, $F_0 = \mathrm{W}[1/p]$. Then F_0 is the maximal absolutely unramified subfield of F. Let $\mathcal{F}$ be a Lubin-Tate group over F corresponding to the uniformizer π. Then $\mathcal{F}$ is a formal $\mathcal{O}_F$-module. Let X be a local coordinate on $\mathcal{F}$. Then the formal Hopf algebra $\mathcal{O}_{\mathcal{F}}$ may be identified with $\mathcal{O}_F[[X]]$. For any $a \in \mathcal{O}_F$, let $[a]_{\mathcal{F}} \in \mathcal{O}_F[[X]]$ be the power series giving the endomorphism a of $\mathcal{F}$. If $n \geq 1$, let $F_n \subset \bar{F}$ be the subfield generated by the π^n-torsion points of $\mathcal{F}$. Write $F_\infty = \cup_n F_n$, $\Gamma = \mathrm{Gal}(F_\infty/F)$ and $G_{F_\infty} = \mathrm{Gal}(\bar{F}/F_\infty)$. For any integer $n \geq 0$, let $\Gamma_n \subset \Gamma$ be the subgroup $\mathrm{Gal}(F_\infty/F_n)$. Let $T\mathcal{F}$ be the Tate module of $\mathcal{F}$. It is a free $\mathcal{O}_F$-module of rank 1. The action of G_F on $T\mathcal{F}$ factors through Γ and induces an isomorphism $\chi_{\mathcal{F}} : \Gamma \to \mathcal{O}_F^\times$. For any $a \in \mathcal{O}_F^\times$ we write $\sigma_a := \chi_{\mathcal{F}}^{-1}(a)$. Using the periods of $T\mathcal{F}$, one can construct a ring $\mathcal{O}_{\mathscr{E}}$ with actions of $\varphi_q = \varphi^{\log_p q}$ and Γ. We will recall the construction in Section 1. Kisin and Ren [17] defined étale (φ_q, Γ)-modules over $\mathcal{O}_{\mathscr{E}}$ and classified G_F-representations on $\mathcal{O}_F$-modules in terms of these modules.

In this paper we are interested in triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules over a Robba ring $\mathscr{R}_L$, where L is a finite extension of F. A *triangulable* (φ_q, Γ)*-module* over $\mathscr{R}_L$ means a (φ_q, Γ)-module D that has a filtration consisting of (φ_q, Γ)-submodules $0 = D_0 \subset D_1 \subset \cdots \subset D_d = D$ such that D_i/D_{i-1} is free of rank 1 over $\mathscr{R}_L$.

In the spirit of Colmez's work [9] on the classification of triangulable (φ, Γ)-modules of rank 2, in the present paper we will classify triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules over $\mathscr{R}_L$ of rank 2. One motivation for doing this, is that the authors believe that under the hypothetical p-adic local Langlands correspondence these (φ_q, Γ)-modules should correspond to certain unitary principal series of $\mathrm{GL}_2(F)$. Colmez [13] and Liu–Xie–Zhang [21] respectively determined the spaces of locally analytic vectors of the unitary principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ based on this kind of (φ, Γ)-modules. Our computations of dimensions of $\mathrm{Ext}^1_{\mathrm{an}}$ match those of Kohlhaase on extensions of locally analytic representations [19]. Nakamura [22] gave a generalization of Colmez's work in another direction. But we think that Nakamura's point of view is probably not the best one for applications to the p-adic local Langlands correspondence.

For our purpose we consider two kinds of cohomology theories for $\mathcal{O}_F$-analytic (φ_q, Γ)-modules.

For a (φ_q, Γ)-module D over $\mathscr{R}_L$, we define $H^\bullet(D)$ by the cohomology of the semigroup $\varphi_q^{\mathbb{N}} \times \Gamma$ as in [13]. Then the first cohomology group $H^1(D)$ is isomorphic to $\mathrm{Ext}(\mathscr{R}_L, D)$, the L-vector space of extensions of $\mathscr{R}_L$ by D in the category of (φ_q, Γ)-modules.

If D is $\mathcal{O}_F$-analytic, we consider the following complex

$$C^\bullet_{\varphi_q, \nabla}(D) : \quad 0 \longrightarrow D \xrightarrow{f_1} D \oplus D \xrightarrow{f_2} D \longrightarrow 0 ,$$

where $f_1 : D \to D \oplus D$ is the map defined as $m \mapsto ((\varphi_q - 1)m, \nabla m)$ and $f_2 : D \oplus D \to D$ is $(m, n) \mapsto \nabla m - (\varphi_q - 1)n$. The operator ∇ is defined in Section 1.3. Put $H^i_{\varphi_q,\nabla}(D) := H^i(C^\bullet_{\varphi_q,\nabla}(D))$, $i = 0, 1, 2$. Each of these modules admits a Γ-action. We set $H^i_{\mathrm{an}}(D) = H^i_{\varphi_q,\nabla}(D)^\Gamma$.

Theorem 0.1. *Let D be an $\mathcal{O}_F$-analytic (φ_q, Γ)-module over $\mathscr{R}_L$. Then there is a natural isomorphism $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D) \to H^1_{\mathrm{an}}(D)$, where $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D)$ is the L-vector space that consists of extensions of $\mathscr{R}_L$ by D*

in the category of $\mathcal{O}_F$-analytic (φ_q, Γ)-modules.

The proof is given in Section 4, which is due to the referee and much simpler than that in our original version.

Theorem 0.2. *Let D be an $\mathcal{O}_F$-analytic (φ_q, Γ)-module over $\mathscr{R}_L$. Then $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D)$ is of codimension $([F : \mathbb{Q}_p] - 1) \dim_L D^{\varphi_q=1,\Gamma=1}$ in $\mathrm{Ext}(\mathscr{R}_L, D)$. In particular, if $D^{\varphi_q=1,\Gamma=1} = 0$, then $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D) = \mathrm{Ext}(\mathscr{R}_L, D)$.*

To prove Theorem 0.2, we will construct a (non canonical) projection from $\mathrm{Ext}(\mathscr{R}_L, D)$ onto $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D)$ whose kernel is of dimension $([F : \mathbb{Q}_p] - 1) \dim_L D^{\varphi_q=1,\Gamma=1}$.

If V is an overconvergent L-representation of G_F (in the sense of Definition 1.4), Δ is the (φ_q, Γ)-module over $\mathscr{E}_L^\dagger$ attached to V, and $D = \mathscr{R}_L \otimes_{\mathscr{E}_L^\dagger} \Delta$, then $\mathrm{Ext}(\mathscr{R}_L, D)$ measures the set of extensions of the trivial representation by V that are overconvergent (cf. Proposition 1.5 and Proposition 1.6). Theorem 0.2 tells us that, if $V^{G_F} = D^{\varphi_q=1,\Gamma=1} = 0$, then any such extension is $\mathcal{O}_F$-analytic.

Let $\mathscr{I}(L)$ (resp. $\mathscr{I}_{\mathrm{an}}(L)$) be the set of continuous (resp. locally F-analytic) characters $\delta : F^\times \to L^\times$. Let δ_{unr} denote the character of $F^\times$ such that $\delta_{\mathrm{unr}}(\pi) = q^{-1}$ and $\delta_{\mathrm{unr}}|_{\mathcal{O}_F^\times} = 1$. Then δ_{unr} is a locally F-analytic character. If $\delta \in \mathscr{I}(L)$, let $\mathscr{R}_L(\delta)$ be the (φ_q, Γ)-module over $\mathscr{R}_L$ of rank 1 that has a basis e_δ such that $\varphi_q(e_\delta) = \delta(\pi)e_\delta$ and $\sigma_a(e_\delta) = \delta(a)e_\delta$. If $\delta \in \mathscr{I}_{\mathrm{an}}(L)$, then $\mathscr{R}_L(\delta)$ is $\mathcal{O}_F$-analytic.

For locally F-analytic characters we have the following

Theorem 0.3. *For any $\delta \in \mathscr{I}_{\mathrm{an}}(L)$, we have*

$$\dim_L H^1_{\mathrm{an}}(\mathscr{R}_L(\delta)) = \begin{cases} 2 & \text{if } \delta = x^{-i}, i \in \mathbb{N} \text{ or } x^i\delta_{\mathrm{unr}}, i \in \mathbb{Z}_+ \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\dim_L H^1(\mathscr{R}_L(\delta)) = \begin{cases} [F : \mathbb{Q}_p] + 1 & \text{if } \delta = x^{-i}, i \in \mathbb{N} \\ 2 & \text{if } \delta = x^i\delta_{\mathrm{unr}}, i \in \mathbb{Z}_+ \\ 1 & \text{otherwise.} \end{cases}$$

For the proof of Theorem 0.3 we follow Colmez's method. In his paper [9] Colmez used the theory of p-adic Fourier transform for $\mathbb{Z}_p$. For our case we use the p-adic Fourier transform for $\mathcal{O}_F$ developed by Schneider and Teitelbaum [24] instead. But this transform can not be applied to our situation directly because, except for the case of $F = \mathbb{Q}_p$, it is defined over $\mathbb{C}_p$ and can not be defined over any finite extension L of F. We overcome this difficulty by applying it to $\mathscr{R}_{\mathbb{C}_p}$ and then descending certain results to $\mathscr{R}_L$. As a result, we obtain that, if δ_1 and δ_2 are in $\mathscr{I}_{\mathrm{an}}(L)$, then $\mathscr{R}_L(\delta_1)^{\psi=0}$ and $\mathscr{R}_L(\delta_2)^{\psi=0}$ are isomorphic to each other as $L[\Gamma]$-modules. This is exactly what we need. In fact, we will show that $S_\delta := (\mathscr{R}_L e_\delta / \mathscr{R}_L^+ e_\delta)^{\psi=0,\Gamma=1}$ is 1-dimensional over L for any $\delta \in \mathscr{I}_{\mathrm{an}}(L)$, and that $H^1_{\mathrm{an}}(\mathscr{R}_L(\delta))$ is isomorphic to S_δ when $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$ and δ is not of the form x^i.

For characters that are not locally F-analytic we have the following

Theorem 0.4. *For any $\delta \in \mathscr{I}(L) \backslash \mathscr{I}_{\mathrm{an}}(I)$ we have $H^1(\mathscr{R}_L(\delta)) = 0$. Consequently every extension of $\mathscr{R}_L$ by $\mathscr{R}_L(\delta)$ splits.*

To state our result on the classification, we need some parameter spaces. These parameter spaces are analogues of Colmez's parameter spaces [9]. Let $\mathscr{S}$ be the analytic variety over $\mathscr{I}_{\mathrm{an}}(L) \times \mathscr{I}_{\mathrm{an}}(L)$ whose fiber over (δ_1, δ_2) is isomorphic to $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$, $\mathscr{S}_{\mathrm{an}}$ the analytic variety over $\mathscr{I}_{\mathrm{an}}(L) \times \mathscr{I}_{\mathrm{an}}(L)$ whose fiber over (δ_1, δ_2) is isomorphic to $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$. There is a natural inclusion $\mathscr{S}_{\mathrm{an}} \hookrightarrow \mathscr{S}$. Let $\mathscr{S}_+, \mathscr{S}_+^{\mathrm{an}}, \mathscr{S}_+^{\mathrm{ng}}, \mathscr{S}_+^{\mathrm{cris}}, \mathscr{S}_+^{\mathrm{st}}, \mathscr{S}_+^{\mathrm{ord}}$ and $\mathscr{S}_+^{\mathrm{ncl}}$ be the subsets of $\mathscr{S}$ defined in Section 6. We can assign to any $s \in \mathscr{S}$ (resp. $s \in \mathscr{S}_{\mathrm{an}}$) a triangulable (resp. triangulable and $\mathcal{O}_F$-analytic) (φ_q, Γ)-module $D(s)$.

Theorem 0.5. (a) *For $s \in \mathscr{S}$, $D(s)$ is of slope zero if and only if s is in $\mathscr{S}_+ - \mathscr{S}_+^{\mathrm{ncl}}$; $D(s)$ is of slope zero and the Galois representation attached to $D(s)$ is irreducible if and only if s is in $\mathscr{S}_* - (\mathscr{S}_*^{\mathrm{ord}} \cup \mathscr{S}_*^{\mathrm{ncl}})$; $D(s)$ is of slope zero and $\mathcal{O}_F$-analytic if and only if s is in $\mathscr{S}_+^{\mathrm{an}} - \mathscr{S}_+^{\mathrm{ncl}}$.*

(b) *Let $s = (\delta_1, \delta_2, \mathscr{L})$ and $s' = (\delta_1', \delta_2', \mathscr{L}')$ be in $\mathscr{S}_+ - \mathscr{S}_+^{\mathrm{ncl}}$. If $\delta_1 = \delta_1'$, then $D(s) \cong D(s')$ if and only if $s = s'$. If $\delta_1 \neq \delta_1'$, then $D(s) \cong D(s')$ if and only if $s, s' \in \mathscr{S}_+^{\mathrm{cris}} \cup \mathscr{S}_+^{\mathrm{ord}}$ with $\delta_1' = x^{w(s)}\delta_2$, $\delta_2' = x^{-w(s)}\delta_1$.*

In the case when $F=\mathbb{Q}_p$, this becomes Colmez's result [9]. The proof of Theorem 0.5 will be given at the end of Section 6.

We give another application of Theorem 0.3. In the case of $F=\mathbb{Q}_p$, i.e. the cyclotomic extension case, Cherbonnier and Colmez [6] showed that all representations of $G_{\mathbb{Q}_p}$ are overconvergent. But our following result shows that this is not the case when $[F:\mathbb{Q}_p]\geq 2$.

Theorem 0.6. *Suppose that $[F:\mathbb{Q}_p]\geq 2$. Then there exist 2-dimensional L-representations of G_F that are not overconvergent (in the sense of Definition 1.4).*

By Kedlaya's Theorem [16], any (φ_q,Γ)-module of slope zero $D(s)$ in Theorem 0.5 (a) comes from a 2-dimensional L-representation of G_F that is overconvergent.

We outline the structure of this paper. We recall Fontaine's rings, the theory of (φ_q,Γ)-modules and the relation between (φ_q,Γ)-modules and Galois representations in Section 1.1 and Section 1.2, and then define $\mathcal{O}_F$-analytic (φ_q,Γ)-modules over the Robba ring $\mathscr{R}_L$ in Section 1.3. We define ψ in Section 2.1, and study the properties of ∂ and Res in Section 2.2. In Section 3.1 we extend ψ to $\mathscr{R}_{\mathbb{C}_p}$, in Section 3.2 we define operators m_α on $\mathscr{R}_{\mathbb{C}_p}$, and then in Section 3.3 we study the Γ-action on $\mathscr{R}_L(\delta)^{\psi=0}$ for all $\delta\in\mathscr{I}_{\mathrm{an}}(L)$. The cohomology theories for $\mathcal{O}_F$-analytic (φ_q,Γ)-modules are given in Section 4. In Section 5 we compute $H^1_{\mathrm{an}}(\mathscr{R}_L(\delta))$ and $H^1(\mathscr{R}_L(\delta))$ for all $\delta\in\mathscr{I}_{\mathrm{an}}(L)$. After providing preliminary lemmas in Section 5.1, we compute $H^0(\delta)$ for all $\delta\in\mathscr{I}(L)$ and $H^1_{\mathrm{an}}(\delta)$ for $\delta\in\mathscr{I}_{\mathrm{an}}(L)$ satisfying $v_\pi(\delta(\pi))<1-v_\pi(q)$ respectively in Section 5.2 and Section 5.3. For the purpose of computing $H^1_{\mathrm{an}}(\delta)$ for all $\delta\in\mathscr{I}_{\mathrm{an}}(L)$, we construct a transition map $\partial: H^1_{\mathrm{an}}(x^{-1}\delta)\rightarrow H^1_{\mathrm{an}}(\delta)$, which is done in Section 5.4. The computation of $H^1_{\mathrm{an}}(\delta)$ is given in Section 5.5. In section 5.6 we define two maps ι_k and $\iota_{k,\mathrm{an}}$. Applying results in Section 5 we classify triangulable $\mathcal{O}_F$-analytic (φ_q,Γ)-modules in Section 6.

1 (φ_q,Γ)-modules and $\mathcal{O}_F$-analytic (φ_q,Γ)-modules

In this section we recall the theory of (φ_q,Γ)-modules built in [8, 15, 17]. We keep using notations in the introduction.

1.1 The rings of formal series

Put $\widetilde{\mathrm{E}}^+=\varprojlim \mathcal{O}_{\bar{F}}/p$ with the transition maps given by Frobenius, and let $\widetilde{\mathrm{E}}$ be the fractional field of $\widetilde{\mathrm{E}}^+$. We may also identify $\widetilde{\mathrm{E}}^+$ with $\varprojlim \mathcal{O}_{\bar{F}}/\pi$ with the transition maps given by the q-Frobenius $\varphi_q=\varphi^{\log_p q}$. Evaluation of X at π-torsion points induces a map $\iota: T\mathcal{F}\rightarrow\widetilde{\mathrm{E}}^+$. Precisely, if $v=(v_n)_{n\geq 0}\in T\mathcal{F}$ with $v_n\in\mathcal{F}[\pi^n](\mathcal{O}_{\bar{F}})$ and $\pi\cdot v_{n+1}=v_n$, then $\iota(v)=(v_n^*(X)+\pi\mathcal{O}_{\bar{F}})_{n\geq 0}$.

Let $\{\cdot\}$ be the unique lifting map $\widetilde{\mathrm{E}}^+\rightarrow \mathrm{W}(\widetilde{\mathrm{E}}^+)_F:=\mathrm{W}(\widetilde{\mathrm{E}}^+)\otimes_{\mathcal{O}_{F_0}}\mathcal{O}_F$ such that $\varphi_q\{x\}=[\pi]_{\mathcal{F}}(\{x\})$ (see [8, Lemma 9.3]). When $\mathcal{F}$ is the cyclotomic Lubin-Tate group $\mathbb{G}_{\mathrm{m}}$, we have $\{x\}=[1+x]-1$, where $[1+x]$ is the Teichmüller lifting of $1+x$. This map respects the action of G_F. If $v\in T\mathcal{F}$ is an $\mathcal{O}_F$-generator, there is an embedding $\mathcal{O}_F[[u_{\mathcal{F}}]]\hookrightarrow \mathrm{W}(\widetilde{\mathrm{E}}^+)_F$ sending $u_{\mathcal{F}}$ to $\{\iota(v)\}$ which identifies $\mathcal{O}_F[[u_{\mathcal{F}}]]$ with a G_F-stable and φ_q-stable subring of $\mathrm{W}(\widetilde{\mathrm{E}}^+)_F$. The G_F-action on $\mathcal{O}_F[[u_{\mathcal{F}}]]$ factors through Γ. By [8, Lemma 9.3] we have

$$\varphi_q(u_{\mathcal{F}})=[\pi]_{\mathcal{F}}(u_{\mathcal{F}}),\quad \sigma_a(u_{\mathcal{F}})=[a]_{\mathcal{F}}(u_{\mathcal{F}}).$$

In the case of $\mathcal{F}=\mathbb{G}_{\mathrm{m}}$, $u_{\mathcal{F}}$ is denoted by T in [9]. Here T is used to denote the Tate module of a Lubin-Tate group.

Let $\mathcal{O}_{\mathscr{E}}$ be the π-adic completion of $\mathcal{O}_F[[u_{\mathcal{F}}]][1/u_{\mathcal{F}}]$. Then $\mathcal{O}_{\mathscr{E}}$ is a complete discrete valuation ring with uniformizer π and residue field $k_F((u_{\mathcal{F}}))$. The topology induced by this valuation is called the *strong topology*. Usually we consider the *weak topology* on $\mathcal{O}_{\mathscr{E}}$, i.e. the topology with $\{\pi^i\mathcal{O}_{\mathscr{E}}+u_{\mathcal{F}}^j\mathcal{O}_F[[u_{\mathcal{F}}]]:\ i,j\in\mathbb{N}\}$ as a fundamental system of open neighborhoods of 0. Let $\mathscr{E}$ be the field of fractions of $\mathcal{O}_{\mathscr{E}}$. Let $\mathscr{E}^+$ be the subring $F\otimes_{\mathcal{O}_F}\mathcal{O}_F[[u_{\mathcal{F}}]]$ of $\mathscr{E}$.

For any $r \in \mathbb{R}_+ \cup \{+\infty\}$, let $\mathscr{E}^{]0,r]}$ be the ring of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i$ with coefficients in F that are convergent on the annulus $0 < v_p(u_{\mathcal{F}}) \leq r$. For any $0 < s \leq r$ we define the valuation $v^{\{s\}}$ on $\mathscr{E}^{]0,r]}$ by

$$v^{\{s\}}(f) = \inf_{i\in\mathbb{Z}}(v_p(a_i) + is) \in \mathbb{R} \cup \{\pm\infty\}.$$

We equip $\mathscr{E}^{]0,r]}$ with the Fréchet topology defined by the family of valuations $\{v^{\{s\}} : 0 < s \leq r\}$. Then $\mathscr{E}^{]0,r]}$ is complete. We equip the Robba ring $\mathscr{R} := \cup_{r>0}\mathscr{E}^{]0,r]}$ with the inductive limit topology. The subring of $\mathscr{R}$ consisting of Laurent series of the form $\sum_{i\geq 0} a_i u_{\mathcal{F}}^i$ is denoted by $\mathscr{R}^+$.

Put $\mathscr{E}^\dagger := \{\sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i \in \mathscr{R} \mid a_i$ are bounded when $i \to +\infty\}$. This is a field contained in both $\mathscr{E}$ and $\mathscr{R}$. Put $\mathscr{E}^{(0,r]} = \mathscr{E}^\dagger \cap \mathscr{E}^{]0,r]}$. Let $v^{[0,r]}$ be the valuation defined by $v^{[0,r]}(f) = \min_{0\leq s\leq r} v^{\{s\}}(f)$. Let $\mathcal{O}_{\mathscr{E}^{(0,r]}}$ be the ring of integers in $\mathscr{E}^{(0,r]}$ for the valuation $v^{[0,r]}$. We equip $\mathcal{O}_{\mathscr{E}^{(0,r]}}[1/u_{\mathcal{F}}]$ with the topology induced by the valuation $v^{\{r\}}$ and then equip $\mathscr{E}^{(0,r]} = \cup_{m\in\mathbb{N}}\pi^{-m}\mathscr{E}^{(0,r]}[1/u_{\mathcal{F}}]$ with the inductive limit topology. The resulting topology on $\mathscr{E}^{(0,r]}$ is called the *weak topology* [11]. Note that the restriction of the weak topology to the subset $\{f(u_{\mathcal{F}}) = \sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i \in \mathscr{E}^{(0,r]} : a_i = 0 \text{ if } i \geq 0\}$ coincides with the topology defined by the valuation $v^{\{r\}}$ and its restriction to $\mathscr{E}^+$ coincides with the weak topology on $\mathscr{E}^+$. Then we equip $\mathscr{E}^\dagger = \cup_{r>0}\mathscr{E}^{(0,r]}$ with the inductive limit topology.

We extend the actions of φ_q and Γ on $\mathcal{O}_F[[u_{\mathcal{F}}]]$ to $\mathscr{E}^+$, $\mathcal{O}_{\mathscr{E}}$, $\mathscr{E}$, $\mathscr{E}^\dagger$ and $\mathscr{R}$ continuously.

Put $t_{\mathcal{F}} = \log_{\mathcal{F}}(u_{\mathcal{F}})$, where $\log_{\mathcal{F}}$ is the logarithmic of $\mathcal{F}$. Then $t_{\mathcal{F}}$ is in $\mathscr{R}$ but not in $\mathscr{E}^\dagger$. When $\mathcal{F} = \mathbb{G}_{\mathrm{m}}$, $t_{\mathcal{F}}$ coincides with the usual t in [9]. Note that $\varphi_q(t_{\mathcal{F}}) = \pi t_{\mathcal{F}}$ and $\sigma_a(t_{\mathcal{F}}) = a t_{\mathcal{F}}$ for any $a \in \mathcal{O}_F^\times$. Put $Q = Q(u_{\mathcal{F}}) = [\pi]_{\mathcal{F}}(u_{\mathcal{F}})/u_{\mathcal{F}}$.

We have the following analogue of [3, Lemma I.3.2].

Lemma 1.1. *If I is a Γ-stable principal ideal of $\mathscr{R}^+$, then I is generated by an element of the form $u_{\mathcal{F}}^{j_0}\prod_{n=0}^{+\infty}\Big(\varphi_q^n(Q(u_{\mathcal{F}})/Q(0))\Big)^{j_{n+1}}$. Furthermore the following hold:*

(a) *If $\mathscr{R}^+ \cdot \varphi_q(I) \subseteq I$, then the sequence $\{j_n\}_{n\geq 0}$ is decreasing.*

(b) *If $\mathscr{R}^+ \cdot \varphi_q(I) \supseteq I$, then the sequence $\{j_n\}_{n\geq 0}$ is increasing.*

Proof. The argument is similar to the proof of [3, Lemma I.3.2]. Let $f(u_{\mathcal{F}})$ be a generator of I. For any $\rho \in (0,1)$ put $V_\rho(I) = \{z \in \mathbb{C}_p : f(z) = 0, 0 \leq |z| \leq \rho\}$. If I is stable by Γ, then $V_\rho(I)$ is stable by $[a]_{\mathcal{F}}$ for any $a \in \mathcal{O}_F^\times$. As $V_\rho(I)$ is finite, for any $z \in V_\rho(I)$, there must be some element $a \in \mathcal{O}_F^\times$, $a \neq 1$ such that $[a]_{\mathcal{F}}(z) = z$. Note that $[\pi]_{\mathcal{F}}(z)$ satisfies $[a]_{\mathcal{F}}([\pi]_{\mathcal{F}}(z)) = [\pi]_{\mathcal{F}}(z)$ if $[a]_{\mathcal{F}}(z) = z$. But the cardinal number of the set $\{z \in \mathbb{C}_p : [a]_{\mathcal{F}}(z) = z, |z| \leq \rho\}$ is finite. Thus for any $z \in V_I(\rho)$ there exists a positive integer $m = m(\rho)$ such that $[\pi^m]_{\mathcal{F}}(z) = 0$. Therefore I is generated by an element of the form $u_{\mathcal{F}}^{j_0}\prod_{n=0}^{+\infty}(\varphi_q^n(Q(u_{\mathcal{F}})/Q(0)))^{j_{n+1}}$. The other two assertions are easy to prove. □

Corollary 1.2. *We have*

$$(t_{\mathcal{F}}) = \Big(u_{\mathcal{F}}\prod_{n\geq 0}\varphi_q^n(Q(u_{\mathcal{F}})/Q(0))\Big) \tag{1.1}$$

in the ring $\mathscr{R}^+$.

Proof. Because the ideal $(t_{\mathcal{F}})$ is Γ-invariant and $\mathscr{R}^+ \cdot \varphi_q(t_{\mathcal{F}}) = (t_{\mathcal{F}})$, by Lemma 1.1 there exists $j \in \mathbb{N}$ such that $(t_{\mathcal{F}}) = \Big(u_{\mathcal{F}}^j\prod_{n\geq 0}\varphi_q^n\big(Q(u_{\mathcal{F}})/Q(0)\big)^j\Big)$. From the fact $(t_{\mathcal{F}}/u_{\mathcal{F}}) \equiv 1 \mod u_{\mathcal{F}}\mathscr{R}^+$ we obtain $j = 1$. □

If $\mathcal{F}'$ is another Lubin-Tate group over F corresponding to π, by the theory of Lubin-Tate groups there exists a unique continuous ring isomorphism $\eta_{\mathcal{F},\mathcal{F}'} : \mathcal{O}^+_{\mathscr{E}_{\mathcal{F}}} \to \mathcal{O}^+_{\mathscr{E}_{\mathcal{F}'}}$ with

$$\eta_{\mathcal{F},\mathcal{F}'}(u_{\mathcal{F}}) = u_{\mathcal{F}'} + \text{ higher degree terms in } \mathcal{O}_F[[u_{\mathcal{F}'}]]$$

such that $\eta_{\mathcal{F},\mathcal{F}'}\circ[a]_{\mathcal{F}}=[a]_{\mathcal{F}'}\circ\eta_{\mathcal{F},\mathcal{F}'}$ for all $a\in\mathcal{O}_F$. We extend $\eta_{\mathcal{F},\mathcal{F}'}$ to isomorphisms

$$\mathcal{O}_{\mathscr{E}_{\mathcal{F}}}\xrightarrow{\sim}\mathcal{O}_{\mathscr{E}_{\mathcal{F}'}},\quad \mathscr{E}^+_{\mathcal{F}}\xrightarrow{\sim}\mathscr{E}^+_{\mathcal{F}'},\quad \mathscr{E}_{\mathcal{F}}\xrightarrow{\sim}\mathscr{E}_{\mathcal{F}'},\quad \mathscr{E}^\dagger_{\mathcal{F}}\to\mathscr{E}^\dagger_{\mathcal{F}'},\quad \mathscr{R}_{\mathcal{F}}\to\mathscr{R}_{\mathcal{F}'}.$$

By abuse of notations these isomorphisms are again denoted by $\eta_{\mathcal{F},\mathcal{F}'}$.

Let $\ell_u=\log u_{\mathcal{F}}$ be a variable over $\mathscr{R}[1/t_{\mathcal{F}}]$. We extend the φ_q,Γ-actions to $\mathscr{R}[1/t_{\mathcal{F}},\ell_u]$ by

$$\varphi_q(\ell_u)=q\ell_u+\log\frac{[\pi]_{\mathcal{F}}(u_{\mathcal{F}})}{u_{\mathcal{F}}^q},\quad \sigma_a(\ell_u)=\ell_u+\log\frac{[a]_{\mathcal{F}}(u_{\mathcal{F}})}{u_{\mathcal{F}}}.$$

1.2 Galois representations and (φ_q,Γ)-modules

Let L be a finite extension of F. Let $\mathrm{Rep}_L G_F$ be the category of finite dimensional L-vector spaces V equipped with a linear action of G_F.

If A is any of $\mathscr{E}^+$, $\mathscr{E}$, $\mathscr{E}^\dagger$, $\mathscr{R}$, we put $A_L=A\otimes_F L$. Then we extend the φ_q, Γ-actions on A to A_L by L-linearity. Let R denote any of $\mathscr{E}_L$, $\mathscr{E}^\dagger_L$ and $\mathscr{R}_L$. For a (φ_q,Γ)-module over R, we mean a free R-module D of finite rank together with continuous semilinear actions of φ_q and Γ commuting with each other such that φ_q sends a basis of D to a basis of D. When $R=\mathscr{E}_L$, we say that D is *étale* if D has a φ_q-stable $\mathcal{O}_{\mathscr{E}_L}$-lattice M such that the linear map $\varphi_q^*M\to M$ is an isomorphism. When $R=\mathscr{E}^\dagger_L$, we say that D is *étale* if $\mathscr{E}_L\otimes_{\mathscr{E}^\dagger_L}D$ is étale. When $R=\mathscr{R}_L$, we say that D is *étale* or *of slope* 0 if there exists an étale (φ_q,Γ)-module Δ over $\mathscr{E}^\dagger_L$ such that $D=\mathscr{R}_L\otimes_{\mathscr{E}^\dagger_L}\Delta$. Let $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/R}$ be the category of étale (φ_q,Γ)-modules over R.

Put $\widetilde{\mathrm{B}}=\mathrm{W}(\widetilde{\mathrm{E}})_F[1/\pi]$. Let B be the completion of the maximal unramified extension of $\mathscr{E}$ in $\widetilde{\mathrm{B}}$ for the π-adic topology. Both $\widetilde{\mathrm{B}}$ and B admit actions of φ_q and G_F. We have $\mathrm{B}^{G_{F_\infty}}=\mathscr{E}$.

For any $V\in\mathrm{Rep}_L G_F$, put $\mathrm{D}_{\mathscr{E}}(V)=(\mathrm{B}\otimes_F V)^{G_{F_\infty}}$. For any $D\in\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{E}_L}$, put $\mathrm{V}(D)=(\mathrm{B}\otimes_{\mathscr{E}}D)^{\varphi_q=1}$.

Theorem 1.3. (*Kisin-Ren [17, Theorem 1.6]*) *The functors* V *and* $\mathrm{D}_{\mathscr{E}}$ *are quasi-inverse equivalences of categories between* $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{E}_L}$ *and* $\mathrm{Rep}_L G_F$.

As usual, let $\widetilde{\mathrm{B}}^\dagger$ be the subring of $\widetilde{\mathrm{B}}$ consisting of overconvergent elements, and put $\mathrm{B}^\dagger=\mathrm{B}\cap\widetilde{\mathrm{B}}^\dagger$. Then $(\mathrm{B}^\dagger)^{G_{F_\infty}}=\mathscr{E}^\dagger$.

Definition 1.4. If V is an L-representation of G_F, we say that V is *overconvergent* if $\mathrm{D}_{\mathscr{E}^\dagger}(V):=(\mathrm{B}^\dagger\otimes_F V)^{G_{F_\infty}}$ contains a basis of $\mathrm{D}_{\mathscr{E}}(V)$.

When $F=\mathbb{Q}_p$, according to Cherbonnier-Colmez theorem [6], all L-representations are overconvergent. But in general this is not true. For details see Remark 5.21.

Proposition 1.5. (a) *If* Δ *is an étale* (φ_q,Γ)*-module over* $\mathscr{E}^\dagger_L$*, then* $\mathrm{V}(\mathscr{E}_L\otimes_{\mathscr{E}^\dagger_L}\Delta)=(\mathrm{B}^\dagger\otimes_{\mathscr{E}^\dagger}\Delta)^{\varphi_q=1}$.

(b) *The functor* $\Delta\mapsto\mathscr{E}_L\otimes_{\mathscr{E}^\dagger_L}\Delta$ *is a fully faithful functor from the category* $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{E}^\dagger_L}$ *to the category* $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{E}_L}$.

(c) *The functor* $\mathrm{D}_{\mathscr{E}^\dagger}$ *is an equivalence of categories between the category of overconvergent* L*-representations of* G_F *and* $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{E}^\dagger_L}$.

Proof. Without loss of generality we may assume that $L=F$. Put $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}=\mathrm{W}(\widetilde{\mathrm{E}})[1/p]$ and $\widetilde{\mathrm{B}}^\dagger_{\mathbb{Q}_p}=\widetilde{\mathrm{B}}_{\mathbb{Q}_p}\cap\widetilde{\mathrm{B}}^\dagger$. The technics of almost étale descent as in Berger-Colmez [4] allows us to show that the functor $\Delta\mapsto\widetilde{\mathrm{B}}_{\mathbb{Q}_p}\otimes_{\widetilde{\mathrm{B}}^\dagger_{\mathbb{Q}_p}}\Delta$ from the category of étale (φ,G_F)-modules over $\widetilde{\mathrm{B}}^\dagger_{\mathbb{Q}_p}$ to the category of étale (φ,G_F)-modules over $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}$ is an equivalence. For any (φ_q,G_F)-module D over $\widetilde{\mathrm{B}}^\dagger$ (resp. $\widetilde{\mathrm{B}}$), we can attach a (φ,G_F)-module $\bar{D}$ over $\widetilde{\mathrm{B}}^\dagger_{\mathbb{Q}_p}$ (resp. $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}$) to D by letting $\bar{D}=\oplus_{i=0}^{f-1}\varphi^{i*}(D)$ with the map

$$\varphi^*(\bar{D})=\oplus_{i=1}^{f}\varphi^{i*}(D)\to\oplus_{i=0}^{f-1}\varphi^{i*}(D)=\bar{D}$$

that sends $\varphi^{i*}(D)$ identically to $\varphi^{i*}(D)$ for $i=1,\cdots,f-1$, and sends $\varphi^{f*}(D)=\varphi_q^*(D)$ to D using φ_q. Here $f=\log_p q$. Thus the functor $\alpha:\Delta\mapsto \widetilde{\mathrm{B}}\otimes_{\widetilde{\mathrm{B}}^\dagger}\Delta$ from the category of étale (φ_q,G_F)-modules over $\widetilde{\mathrm{B}}^\dagger$ to the category of étale (φ_q,G_F)-modules over $\widetilde{\mathrm{B}}$ is an equivalence. Now let Δ be an étale (φ_q,Γ)-module over $\mathscr{E}^\dagger$, and put $V=\mathrm{V}(\mathscr{E}\otimes_{\mathscr{E}^\dagger}\Delta)$. As $\alpha(\widetilde{\mathrm{B}}^\dagger\otimes_F V)=\widetilde{\mathrm{B}}\otimes_F V=\widetilde{\mathrm{B}}\otimes_{\mathscr{E}^\dagger}\Delta=\alpha(\widetilde{\mathrm{B}}^\dagger\otimes_{\mathscr{E}^\dagger}\Delta)$, we have $\widetilde{\mathrm{B}}^\dagger\otimes_F V=\widetilde{\mathrm{B}}^\dagger\otimes_{\mathscr{E}^\dagger}\Delta$. Thus V is contained in $\widetilde{\mathrm{B}}^\dagger\otimes_{\mathscr{E}^\dagger}\Delta\cap\mathrm{B}\otimes_{\mathscr{E}^\dagger}\Delta=\mathrm{B}^\dagger\otimes_{\mathscr{E}^\dagger}\Delta$, and $V=(\mathrm{B}^\dagger\otimes_{\mathscr{E}^\dagger}\Delta)^{\varphi_q=1}$. This proves (a).

Next we prove (b). Let Δ_1 and Δ_2 be two objects in $\mathrm{Mod}_{/\mathscr{E}^\dagger}^{\varphi_q,\Gamma,\mathrm{\acute{e}t}}$. What we have to show is that the natural map

$$\mathrm{Hom}_{\mathrm{Mod}_{/\mathscr{E}^\dagger}^{\varphi_q,\Gamma,\mathrm{\acute{e}t}}}(\Delta_1,\Delta_2)\to\mathrm{Hom}_{\mathrm{Mod}_{/\mathscr{E}}^{\varphi_q,\Gamma,\mathrm{\acute{e}t}}}(\mathscr{E}\otimes_{\mathscr{E}^\dagger}\Delta_1,\mathscr{E}\otimes_{\mathscr{E}^\dagger}\Delta_2)$$

is an isomorphism. For this we reduce to show that

$$\Big(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2\Big)^{\varphi_q=1,\Gamma=1}\to\Big(\mathscr{E}\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\Big)^{\varphi_q=1,\Gamma=1}$$

is an isomorphism. Here $\check{\Delta}_1$ is the $\mathscr{E}^\dagger$-module of $\mathscr{E}^\dagger$-linear maps from Δ_1 to $\mathscr{E}^\dagger$, which is equipped with a natural étale (φ_q,Γ)-module structure. We have

$$\begin{aligned}\Big(\mathscr{E}\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\Big)^{\varphi_q=1,\Gamma=1} &= \Big(\mathrm{B}\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\Big)^{\varphi_q=1,G_F=1}\\ &= \mathrm{V}(\mathscr{E}\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2))^{G_F=1}\\ &= \Big(\mathrm{B}^\dagger\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\Big)^{\varphi_q=1,G_F=1}\\ &= (\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)^{\varphi_q=1,\Gamma=1}.\end{aligned}$$

Finally, (c) follows from (a), (b) and Theorem 1.3. □

Proposition 1.6. *The functor $\Delta\mapsto\mathscr{R}_L\otimes_{\mathscr{E}_L^\dagger}\Delta$ is an equivalence of categories between $\mathrm{Mod}_{/\mathscr{E}_L^\dagger}^{\varphi_q,\Gamma,\mathrm{\acute{e}t}}$ and $\mathrm{Mod}_{/\mathscr{R}_L}^{\varphi_q,\Gamma,\mathrm{\acute{e}t}}$.*

Proof. Let D be an étale (φ_q,Γ)-module over $\mathscr{R}_L$. By Kedlaya's slope filtration theorem [16], there exists a unique φ_q-stable $\mathscr{E}_L^\dagger$-submodule Δ of D that is étale as a φ_q-module such that $D=\mathscr{R}_L\otimes_{\mathscr{E}_L^\dagger}\Delta$. For any $\gamma\in\Gamma$, $\gamma(\Delta)$ also has this property. Thus, by uniqueness of Δ, we have $\gamma(\Delta)=\Delta$. This means that Δ is Γ-invariant. □

1.3 $\mathscr{O}_F$-analytic (φ_q,Γ)-modules

For any $r\geq s>0$, let $v^{[s,r]}$ be the valuation defined by $v^{[s,r]}(f)=\inf_{r'\in[s,r]}v^{\{r'\}}(f)$. Note that $v^{[s,r]}(f)=\inf\limits_{z\in\mathbb{C}_p,s\leq v_p(z)\leq r}v_p(f(z))$.

Lemma 1.7. *For any $r>s>0$, there exists a sufficiently large integer $n=n(s,r)$ such that, if $\gamma\in\Gamma_n$, then we have $v^{[s,r]}((1-\gamma)z)\geq v^{[s,r]}(z)+1$ for all $z\in\mathscr{E}_L^{]0,r]}$.*

Proof. It suffices to consider $z=u_{\mathcal{F}}^k$, $k\in\mathbb{Z}$. If $k\geq 0$, then

$$\gamma(u_{\mathcal{F}}^k)-u_{\mathcal{F}}^k=u_{\mathcal{F}}^k(\frac{\gamma(u_{\mathcal{F}})}{u_{\mathcal{F}}}-1)(\frac{\gamma(u_{\mathcal{F}}^{k-1})}{u_{\mathcal{F}}^{k-1}}+\cdots+1)$$

and

$$\gamma(u_{\mathcal{F}}^{-k})-u_{\mathcal{F}}^{-k}=u_{\mathcal{F}}^{-k}(\frac{u_{\mathcal{F}}}{\gamma(u_{\mathcal{F}})}-1)(\frac{u_{\mathcal{F}}^{k-1}}{\gamma(u_{\mathcal{F}}^{k-1})}+\cdots+1).$$

As $v^{[s,r]}(yz)\geq v^{[s,r]}(y)+v^{[s,r]}(z)$, the lemma follows from the fact that $\frac{\gamma(u_{\mathcal{F}})}{u_{\mathcal{F}}}-1\to 0$ when $\gamma\to 1$. □

Let D be an object in $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{R}_L}$. We choose a basis $\{e_1,\cdots,e_d\}$ of D and write $D^{]0,r]}=\oplus_{i=1}^d\mathscr{E}_L^{]0,r]}\cdot e_i$. Note that our definition of $D^{]0,r]}$ depends on the choice of $\{e_1,\cdots,e_d\}$. However, if $\{e'_1,\cdots,e'_d\}$ is another basis, then $\oplus_{i=1}^d\mathscr{E}_L^{]0,r]}\cdot e_i=\oplus_{i=1}^d\mathscr{E}_L^{]0,r]}\cdot e'_i$ for sufficiently small $r>0$. When $r>0$ is sufficiently small, $D^{]0,r]}$ is stable under Γ. By Lemma 1.7 and the continuity of the Γ-action on $D^{]0,r]}$, the series

$$\log\gamma=\sum_{i=1}^{\infty}(\gamma-1)^i(-1)^{i-1}/i$$

converges on $D^{]0,r]}$ when $\gamma\to 1$. It follows that the map

$$\mathrm{d}\Gamma:\mathrm{Lie}\Gamma\to\mathrm{End}_L D^{]0,r]},\quad \beta\mapsto\log(\exp\beta)$$

is well defined for sufficiently small β, and we extend it to all of $\mathrm{Lie}\Gamma$ by $\mathbb{Z}_p$-linearity. As a result, we obtain a $\mathbb{Z}_p$-linear map $\mathrm{d}\Gamma_D:\mathrm{Lie}\Gamma\to\mathrm{End}_L D$. For any $\beta\in\mathrm{Lie}\Gamma$, $\mathrm{d}\Gamma_{\mathscr{R}_L}(\beta)$ is a derivation of $\mathscr{R}_L$ and $\mathrm{d}\Gamma_D(\beta)$ is a differential operator over $\mathrm{d}\Gamma_{\mathscr{R}_L}(\beta)$, which means that for any $a\in\mathscr{R}_L$, $m\in D$ and $\beta\in\mathrm{Lie}\Gamma$ we have

$$\mathrm{d}\Gamma_D(\beta)(am)=\mathrm{d}\Gamma_{\mathscr{R}_L}(\beta)(a)m+a\cdot\mathrm{d}\Gamma_D(\beta)(m). \tag{1.2}$$

The isomorphism $\chi_{\mathcal{F}}:\Gamma\to\mathcal{O}_F^\times$ induces an $\mathcal{O}_F$-linear isomorphism $\mathrm{Lie}\Gamma\to\mathcal{O}_F$. We will identify $\mathrm{Lie}\Gamma$ with $\mathcal{O}_F$ via this isomorphism.

We say that D is *$\mathcal{O}_F$-analytic* if the map $\mathrm{d}\Gamma_D$ is not only $\mathbb{Z}_p$-linear, but also $\mathcal{O}_F$-linear. If D is $\mathcal{O}_F$-analytic, the operator $\mathrm{d}\Gamma_D(\beta)/\beta$, $\beta\in\mathcal{O}_F$, $\beta\neq 0$, does not depend on the choice of β. The resulting operator is denoted by ∇_D or just ∇ if there is no confusion. Note that the Γ-action on $\mathscr{R}_L$ is $\mathcal{O}_F$-analytic and by [17, Lemma 2.1.4]

$$\nabla=t_{\mathcal{F}}\cdot\frac{\partial F_{\mathcal{F}}}{\partial Y}(u_{\mathcal{F}},0)\cdot\mathrm{d}/\mathrm{d}u_{\mathcal{F}}, \tag{1.3}$$

where $F_{\mathcal{F}}(X,Y)$ is the formal group law of $\mathcal{F}$. Put $\partial=\frac{\partial F_{\mathcal{F}}}{\partial Y}(u_{\mathcal{F}},0)\cdot\mathrm{d}/\mathrm{d}u_{\mathcal{F}}$. From the relation $\sigma_a(t_{\mathcal{F}})=at_{\mathcal{F}}$ we obtain $\nabla t_{\mathcal{F}}=t_{\mathcal{F}}$ and $\partial\, t_{\mathcal{F}}=1$. When $\mathcal{F}=\mathbb{G}_{\mathrm{m}}$, ∇ and ∂ are already defined in [2]. In this case $F_{\mathcal{F}}(X,Y)=X+Y+XY$ and so $\partial=(1+u_{\mathcal{F}})\mathrm{d}/\mathrm{d}u_{\mathcal{F}}$.

We end this section by classification of (φ_q,Γ)-modules over $\mathscr{R}_L$ of rank 1.

Let $\mathscr{I}(L)$ be the set of continuous characters $\delta:F^\times\to L^\times$, $\mathscr{I}_{\mathrm{an}}(L)$ the subset of locally F-analytic characters. If δ is in $\mathscr{I}_{\mathrm{an}}(L)$, then $\frac{\log\delta(a)}{\log(a)}$, $a\in\mathcal{O}_F^\times$, which makes sense when $\log(a)\neq 0$, does not depend on a. This number, denoted by w_δ, is called the *weight* of δ. Clearly $w_\delta=0$ if and only if δ is locally constant; w_δ is in $\mathbb{Z}$ if and only if δ is locally algebraic.

If $\delta\in\mathscr{I}(L)$, let $\mathscr{R}_L(\delta)$ be the (φ_q,Γ)-module over $\mathscr{R}_L$ (of rank 1) that has a basis e_δ such that $\varphi_q(e_\delta)=\delta(\pi)e_\delta$ and $\sigma_a(e_\delta)=\delta(a)e_\delta$. It is easy to check that, if $\delta\in\mathscr{I}_{\mathrm{an}}(L)$, then $\mathscr{R}_L(\delta)$ is $\mathcal{O}_F$-analytic and $\nabla_\delta=\nabla_{\mathscr{R}_L(\delta)}=t_{\mathcal{F}}\partial+w_\delta$ (more precisely $\nabla_\delta(ze_\delta)=(t_{\mathcal{F}}\partial z+w_\delta z)e_\delta$). If $\mathscr{R}_L(\delta)$ is étale, i.e. $v_p(\delta(\pi))=0$, we will use $L(\delta)$ to denote the Galois representation attached to $\mathscr{R}_L(\delta)$.

Remark 1.8. All of 1-dimensional L-representations of G_F are overconvergent. In fact, such a representation comes from a character of $F^\times$ and thus is of the form $L(\delta)$.

Proposition 1.9. *Let D be a (φ_q,Γ)-module over $\mathscr{R}_L$ of rank 1. Then there exists a character $\delta\in\mathscr{I}(L)$ such that D is isomorphic to $\mathscr{R}_L(\delta)$. Furthermore D is $\mathcal{O}_F$-analytic if and only if $\delta\in\mathscr{I}_{\mathrm{an}}(L)$.*

Proof. The argument is similar to the proof of [9, Proposition 3.1]. We first reduce to the case that D is étale. Then by Proposition 1.6 there exists an étale (φ_q,Γ)-module Δ over $\mathscr{E}_L^\dagger$ such that $D=\mathscr{R}_L\otimes_{\mathscr{E}_L^\dagger}\Delta$. Now the first assertion follows from Proposition 1.5 and Remark 1.8. The second assertion is obvious. ∎

2 The operators ψ and ∂

2.1 The operator ψ

We define an operator ψ and study its properties.

Note that $\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$ is a basis of $\mathscr{E}_L$ over $\varphi_q(\mathscr{E}_L)$. So $\mathscr{E}_L$ is a field extension of $\varphi_q(\mathscr{E}_L)$ of degree q. Put $\mathrm{tr}=\mathrm{tr}_{\mathscr{E}_L/\varphi_q(\mathscr{E}_L)}$.

Lemma 2.1.

(a) *There is a unique operator $\psi:\mathscr{E}_L\to\mathscr{E}_L$ such that $\varphi_q\circ\psi=q^{-1}\mathrm{tr}$.*

(b) *For any $a,b\in\mathscr{E}_L$ we have $\psi(\varphi_q(a)b)=a\psi(b)$. In particular, $\psi\circ\varphi_q=\mathrm{id}$.*

(c) *ψ commutes with Γ.*

Proof. Assertion (a) follows from the fact that φ_q is injective. Assertion (b) follows from the relation

$$\varphi_q(\psi(\varphi_q(a)b))=\mathrm{tr}(\varphi_q(a)b)/q=\varphi_q(a)\mathrm{tr}(b)/q=\varphi_q(a)\varphi_q(\psi(b))=\varphi_q(a\psi(b))$$

and the injectivity of φ_q. As φ_q commutes with Γ, $\varphi_q(\mathscr{E}_L)$ is stable under Γ. Thus $\gamma\circ\mathrm{tr}\circ\gamma^{-1}=\mathrm{tr}$ for all $\gamma\in\Gamma$. This ensures that ψ commutes with Γ. Assertion (c) follows. □

We first compute ψ in the case of the special Lubin-Tate group.

Proposition 2.2. *Suppose that $\mathcal{F}$ is the special Lubin-Tate group.*

(a) *If $\ell\geq 0$, then $\psi(u_{\mathcal{F}}^\ell)=\sum_{i=0}^{[\ell/q]}a_{\ell,i}u_{\mathcal{F}}^i$ with $v_\pi(a_{\ell,i})\geq[\ell/q]+1-i-v_\pi(q)$.*

(b) *If $\ell<0$, then $\psi(u_{\mathcal{F}}^\ell)=\sum_{i=\ell}^{[\ell/q]}b_{\ell,i}u_{\mathcal{F}}^i$ with $v_\pi(b_{\ell,i})\geq[\ell/q]+1-i-v_\pi(q)$.*

Proof. First we prove (a) by induction on ℓ. As the minimal polynomial of $u_{\mathcal{F}}$ is $X^q+\pi X-(u_{\mathcal{F}}^q+\pi u_{\mathcal{F}})$, by Newton formula we have

$$\mathrm{tr}(u_{\mathcal{F}}^i)=\begin{cases}0 & \text{if } 1\leq i\leq q-2,\\ (1-q)\pi & \text{if } i=q-1.\end{cases}$$

It follows that

$$\psi(u_{\mathcal{F}}^i)=\begin{cases}0 & \text{if } 1\leq i\leq q-2,\\ (1-q)\pi/q & \text{if } i=q-1.\end{cases}$$

Thus the assertion holds when $0\leq\ell\leq q-1$. Now we assume that $\ell=j\geq q$ and the assertion holds when $0\leq\ell\leq j-1$. We have

$$\begin{aligned}\psi(u_{\mathcal{F}}^\ell) &= \psi((u_{\mathcal{F}}^q+\pi u_{\mathcal{F}})u_{\mathcal{F}}^{\ell-q})-\psi(\pi u_{\mathcal{F}}^{\ell-q+1})=u_{\mathcal{F}}\psi(u_{\mathcal{F}}^{\ell-q})-\pi\psi(u_{\mathcal{F}}^{\ell-q+1})\\ &= \sum_{i=1}^{[\ell/q]}a_{\ell-q,i-1}u_{\mathcal{F}}^i-\sum_{i=0}^{[(\ell+1)/q]-1}\pi a_{\ell-q+1,i}u_{\mathcal{F}}^i.\end{aligned}$$

Thus $a_{\ell,i}=a_{\ell-q,i-1}-\pi a_{\ell-q+1,i}$. By the inductive assumption we have

$$v_\pi(a_{\ell-q,i-1})\geq[(\ell-q)/q]+1-(i-1)-v_\pi(q)=[\ell/q]+1-i-v_\pi(q)$$

and

$$v_\pi(a_{\ell-q+1,i})\geq[(\ell-q+1)/q]+1-i-v_\pi(q)\geq[\ell/q]-i-v_\pi(q)$$

It follows that $v_\pi(a_{\ell,i})\geq[\ell/q]+1-i-v_\pi(q)$.

Next we prove (b). We have

$$\psi(u_{\mathcal{F}}^\ell) = \psi\Big(\frac{(u_{\mathcal{F}}^{q-1}+\pi)^{-\ell}}{\varphi_q(u_{\mathcal{F}})^{-\ell}}\Big)=\frac{\psi\Big(\sum_{j=0}^{-\ell}\begin{bmatrix}-\ell\\ j\end{bmatrix}u_{\mathcal{F}}^{j(q-1)}\pi^{-\ell-j}\Big)}{u_{\mathcal{F}}^{-\ell}}$$

$$= \sum_{i=0}^{[-\ell(q-1)/q]} \sum_{j=0}^{-\ell} \begin{bmatrix} -\ell \\ j \end{bmatrix} \pi^{-\ell-j} a_{j(q-1),i} \cdot u_{\mathcal{F}}^{i+\ell} = \sum_{i=\ell}^{[\ell/q]} \sum_{j=0}^{-\ell} \begin{bmatrix} -\ell \\ j \end{bmatrix} \pi^{-\ell-j} a_{j(q-1),i-\ell} \cdot u_{\mathcal{F}}^{i}$$

Here, $\begin{bmatrix} -\ell \\ j \end{bmatrix} = \frac{(-\ell)!}{j!(-\ell-j)!}$. Thus $b_{\ell,i} = \sum_{j=0}^{-\ell} \begin{bmatrix} -\ell \\ j \end{bmatrix} \pi^{-\ell-j} a_{j(q-1),i-\ell}$. As

$$\begin{aligned} v_\pi(\pi^{-\ell-j} a_{j(q-1),i-\ell}) &\geq -\ell - j + ([\frac{j(q-1)}{q}] + 1 - (i-\ell) - v_\pi(q)) \\ &= [-j/q] + 1 - i - v_\pi(q) \geq [\ell/q] + 1 - i - v_\pi(q), \end{aligned}$$

we obtain $v_\pi(b_{\ell,i}) \geq [\ell/q] + 1 - i - v_\pi(q)$. □

Let $\mathscr{E}_L^-$ be the subset of $\mathscr{E}_L$ consisting of elements of the form $\sum_{i\leq -1} a_i u_{\mathcal{F}}^i$.

Corollary 2.3. *Suppose that $\mathcal{F}$ is the special Lubin-Tate group. Then $\psi(\mathscr{E}_L^-) \subset \mathscr{E}_L^-$.*

Proof. This follows directly from Proposition 2.2. □

Proposition 2.4. (a) *We have $\psi(\mathscr{E}_L^+) = \mathscr{E}_L^+$, $\psi(\mathcal{O}_{\mathscr{E}_L^+}) \subset \frac{\pi}{q}\mathcal{O}_{\mathscr{E}_L^+}$ and $\psi(\mathcal{O}_{\mathscr{E}_L}) \subset \frac{\pi}{q}\mathcal{O}_{\mathscr{E}_L}$.*

(b) *ψ is continuous for the weak topology on $\mathscr{E}_L$.*

(c) *$\mathscr{E}_L^\dagger$ is stable under ψ, and the restriction of ψ on $\mathscr{E}_L^\dagger$ is continuous for the weak topology of $\mathscr{E}_L^\dagger$.*

(d) *If $f \in \mathscr{E}_L^{(0,r]}$, then the sequence $(\frac{q}{\pi}\psi)^n(f)$, $n \in \mathbb{N}$, is bounded in $\mathscr{E}_L^{(0,r]}$ for the weak topology.*

Proof. Let $\mathcal{F}_0$ be the special Lubin-Tate group over F corresponding to π. Observe that $\psi_{\mathcal{F}} = \eta_{\mathcal{F}_0,\mathcal{F}}^{-1}\psi_{\mathcal{F}_0}\eta_{\mathcal{F}_0,\mathcal{F}}$. As $\eta_{\mathcal{F}_0,\mathcal{F}}(u_{\mathcal{F}_0}) = u_{\mathcal{F}}\times$ a unit in $\mathcal{O}_F[[u_{\mathcal{F}}]]$, for any $r > 0$ we have that $\eta_{\mathcal{F}_0,\mathcal{F}}(\mathcal{O}^{(0,r]}_{\mathscr{E}_{\mathcal{F}_0,L}}[1/u_{\mathcal{F}_0}]) = \mathcal{O}^{(0,r]}_{\mathscr{E}_{\mathcal{F},L}}[1/u_{\mathcal{F}}]$ and that $\eta_{\mathcal{F}_0,\mathcal{F}}$ respects the valuation $v^{[0,r]}$. Thus $\eta_{\mathcal{F}_0,\mathcal{F}} : \mathscr{E}^{(0,r]}_{\mathcal{F}_0,L} \to \mathscr{E}^{(0,r]}_{\mathcal{F},L}$ is a topological isomorphism. It follows that $\mathscr{E}^\dagger_{\mathcal{F}_0,L} \to \mathscr{E}^\dagger_{\mathcal{F},L}$ and its inverse are continuous for the weak topology. Similarly $\eta_{\mathcal{F}_0,\mathcal{F}} : \mathscr{E}_{\mathcal{F}_0,L} \to \mathscr{E}_{\mathcal{F},L}$ and its inverse are continuous for the weak topology. Hence we only need to consider the case of the special Lubin-Tate group. Assertions (a) and (b) follow from Proposition 2.2. For (c) we only need to show that, for any $r > 0$ we have $\psi(\mathscr{E}_L^{(0,r]}) \subset \mathscr{E}_L^{(0,r]}$ and the restriction $\psi : \mathscr{E}_L^{(0,r]} \to \mathscr{E}_L^{(0,r]}$ is continuous. By (b) the restriction of ψ to $\mathscr{E}_L^+$ is continuous. By Proposition 2.2 (b) and Corollary 2.3, if f is in $\mathscr{E}_L^- \cap \mathscr{E}_L^{(0,r]}$, then $\psi(f)$ is in $\mathscr{E}_L^-$ and $v^{\{r\}}(\psi(f)) \geq v^{\{r\}}(f) + v_p(\pi/q)$. Thus $\psi : \mathscr{E}_L^- \cap \mathscr{E}_L^{(0,r]} \to \mathscr{E}_L^- \cap \mathscr{E}_L^{(0,r]}$ is continuous, which proves (c). As $\frac{q}{\pi}\psi(\mathcal{O}_{\mathscr{E}_L^+}) \subset \mathcal{O}_{\mathscr{E}_L^+}$ and $v^{\{r\}}(\frac{q}{\pi}\psi(f)) \geq v^{\{r\}}(f)$ for any $f \in \mathscr{E}_L^- \cap \mathscr{E}_L^{(0,r]}$, (d) follows. □

Next we extend ψ to $\mathscr{R}_L$.

Proposition 2.5. *We can extend* tr *continuously to $\mathscr{R}_L$. The resulting operator* tr *satisfies* $\mathrm{tr}|_{\varphi_q(\mathscr{R}_L)} = q \cdot \mathrm{id}$ *and* $\mathrm{tr}(\mathscr{R}_L) = \varphi_q(\mathscr{R}_L)$.

Proof. Let $\mathscr{E}_L^{\gg -\infty}$ denote the subset of $\mathscr{E}_L$ consisting of $f \in \mathscr{E}_L$ of the form $\sum_{n \gg -\infty} a_n u_{\mathcal{F}}^n$. If $f \in \mathscr{E}_L^{\gg -\infty}$, then

$$\mathrm{tr}(f) = \sum_{\eta \in \ker[\pi]_{\mathcal{F}}} f(u_{\mathcal{F}} +_{\mathcal{F}} \eta).$$

If η is in $\ker[\pi]_{\mathcal{F}}$, then $v_p(\eta) \geq \frac{1}{(q-1)e_F}$ where $e_F = [F : F_0]$. Thus, if r and $s \in \mathbb{R}_+$ satisfy $\frac{1}{(q-1)e_F} > r \geq s$, the morphisms $u_{\mathcal{F}} \mapsto u_{\mathcal{F}} +_{\mathcal{F}} \eta$ ($\eta \in \ker[\pi]_{\mathcal{F}}$) keep the annulus $\{z \in \mathbb{C}_p : p^{-r} \leq |z| \leq p^{-s}\}$ stable. So for any $f \in \mathscr{E}_L^{\gg -\infty}$ we have $v^{[s,r]}(f(u_{\mathcal{F}} +_{\mathcal{F}} \eta)) = v^{[s,r]}(f)$ and $v^{[s,r]}(\mathrm{tr}(f)) \geq v^{[s,r]}(f)$. Hence there exists a unique continuous operator $\mathrm{Tr} : \mathscr{R}_L \to \mathscr{R}_L$ such that $\mathrm{Tr}(f) = \mathrm{tr}(f)$ for any $f \in \mathscr{E}_L^{\gg -\infty}$. (For any $f \in \mathscr{R}_L$, choosing a positive real number r such that $f \in \mathscr{E}_L^{]0,r]}$, we can find a sequence $\{f_i\}_{i\geq 1}$ in $\mathscr{E}_L^{\gg -\infty}$ such that $f_i \to f$ in $\mathscr{E}^{]0,r]}$;

then $\{\mathrm{tr}(f_i)\}_{i\geq 1}$ is a Cauchy sequence in $\mathscr{E}_L^{[s,r]}$ for any s satisfying $0 < s \leq r$, and we let $\mathrm{Tr}(f)$ be their limit in $\mathscr{E}^{]0,r]}$; it is easy to show that $\mathrm{Tr}(f)$ does not depend on any choice.) From the continuity of Tr we obtain that $\mathrm{Tr}|_{\mathscr{E}_L^\dagger} = \mathrm{tr}$ and $\mathrm{Tr}|_{\varphi_q(\mathscr{R}_L)} = q\cdot \mathrm{id}$. By Lemma 2.6 below, $\varphi_q : \mathscr{R}_L \to \mathscr{R}_L$ is strict and thus has a closed image. Since $\mathscr{E}_L^\dagger$ is dense in $\mathscr{R}_L$ and $\mathrm{Tr}(\mathscr{E}_L^\dagger) = \varphi_q(\mathscr{E}_L^\dagger) \subset \varphi_q(\mathscr{R}_L)$, we have $\mathrm{Tr}(\mathscr{R}_L) \subseteq \varphi_q(\mathscr{R}_L)$. ◻

Lemma 2.6. *If $\frac{q}{(q-1)e_F} > r \geq s > 0$ and $f \in \mathscr{E}_L^{]0,r]}$, then we have*

- $v^{[s,r]}(\gamma(f)) = v^{[s,r]}(f)$ *for all* $\gamma \in \Gamma$;
- $v^{[s,r]}(\varphi_q(f)) = v^{[qs,qr]}(f)$ *if* $r < \frac{1}{(q-1)e_F}$.

Proof. Since $[\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(u_{\mathcal{F}}) \in u_{\mathcal{F}}\mathcal{O}_F[[u_{\mathcal{F}}]]$, we have $v_p([\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(z)) \geqslant v_p(z)$ for all $z \in \mathbb{C}_p$ such that $v_p(z) > 0$. By the same reason we have $v_p([\chi_{\mathcal{F}}(\gamma^{-1})]_{\mathcal{F}}(z)) \geqslant v_p(z)$ and thus $v_p([\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(z)) \leqslant v_p(z)$. So $v_p([\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(z)) = v_p(z)$.

If $z \in \mathbb{C}_p$ satisfies $p^{-\frac{1}{(q-1)e_F}} < p^{-r} \leqslant |z| \leqslant p^{-s} < 1$, then $v_p([\pi]_{\mathcal{F}}(z)) = qv_p(z)$. Thus, the image by $z \mapsto [\pi]_{\mathcal{F}}(z)$ of the annulus $\{z \in \mathbb{C}_p : p^{-r} \leqslant |z| \leqslant p^{-s}\}$ is inside the annulus $\{z \in \mathbb{C}_p : p^{-qr} \leqslant |z| \leqslant p^{-qs}\}$. Conversely, if $w \in \mathbb{C}_p$ is such that $p^{-qr} \leqslant |w| \leqslant p^{-qs}$, then $v_p(w) < \frac{q}{(q-1)e_F}$. The Newton polygon of the polynomial $-w + [\pi]_{\mathcal{F}}(u_{\mathcal{F}})$ shows that this polynomial has q roots of valuation $\frac{1}{q}v_p(w)$. If $z \in \mathbb{C}_p$ is such a root, we have $p^{-r} \leqslant |z| \leqslant p^{-s}$. Thus, the image of the annulus $p^{-r} \leqslant |z| \leqslant p^{-s}$ is the annulus $p^{-qr} \leqslant |z| \leqslant p^{-qs}$. ◻

We define $\psi : \mathscr{R}_L \to \mathscr{R}_L$ by $\psi = \frac{1}{q}\ \varphi_q^{-1} \circ \mathrm{tr}$.

Lemma 2.7. *If $\frac{q}{(q-1)e_F} > r \geq s > 0$ and $f \in \mathscr{E}_L^{]0,r]}$, then $v^{[s,r]}(\psi(f)) \geqslant v^{[s/q,r/q]}(f) - v_p(q)$.*

Proof. By Lemma 2.6 it suffices to show that

$$v^{[s/q,r/q]}(\varphi_q(\psi(f))) = v^{[s/q,r/q]}(q^{-1}\mathrm{tr}(f)) \geq v^{[s/q,r/q]}(f) - v_p(q).$$

But this follows from Proposition 2.5 and its proof. ◻

As a consequence, $\psi : \mathscr{R}_L \to \mathscr{R}_L$ is continuous.

Corollary 2.8. (a) *$\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$ is a basis of $\mathscr{E}_L^\dagger$ over $\varphi_q(\mathscr{E}_L^\dagger)$, and $\mathrm{tr}|_{\mathscr{E}_L^\dagger} = \mathrm{tr}_{\mathscr{E}_L^\dagger/\varphi_q(\mathscr{E}_L^\dagger)}$.*

(b) *$\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$ is a basis of $\mathscr{R}_L$ over $\varphi_q(\mathscr{R}_L)$.*

Proof. Let $\{b_i\}_{0\leq i\leq q-1}$ be the dual basis of $\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$ relative to $\mathrm{tr}_{\mathscr{E}_L/\varphi_q(\mathscr{E}_L)}$. Let B be the inverse of the matrix $(\mathrm{tr}(u_{\mathcal{F}}^{i+j}))_{i,j}$. Then $B \in \mathrm{GL}_q(\mathscr{E}_L^\dagger)$ and $(b_0, b_1, \cdots, b_{q-1})^t = B(1, u_{\mathcal{F}}, \cdots, u_{\mathcal{F}}^{q-1})^t$. So $b_0, b_1, \cdots, b_{q-1}$ are in $\mathscr{E}_L^\dagger$. Then $f = \sum_{i=0}^{q-1} u_{\mathcal{F}}^i \psi(b_i f)$ for any $f \in \mathscr{E}_L$, $\mathscr{E}_L^\dagger$ or $\mathscr{R}_L$. (For the former two cases, this follows from the definition of $\{b_i\}_{0\leq i\leq q-1}$; for the last case, we apply the continuity of ψ.) Thus $\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$ generate $\mathscr{E}_L^\dagger$ (resp. $\mathscr{R}_L$) over $\varphi_q(\mathscr{E}_L^\dagger)$ (resp. $\varphi_q(\mathscr{R}_L)$). In either case, to prove the independence of $\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$, we only need to use the fact $\psi(b_i u_{\mathcal{F}}^j) = \delta_{ij}$ $(i, j \in \{0, 1, \cdots, q-1\})$, where δ_{ij} is the Kronecker sign. Finally we note that the second assertion of (a) follows from the first one. ◻

We apply the above to (φ_q, Γ)-modules.

Proposition 2.9. *If D is a (φ_q, Γ)-module over R where $R = \mathscr{E}_L$, $\mathscr{E}_L^\dagger$ or $\mathscr{R}_L$, then there is a unique operator $\psi : D \to D$ such that*

$$\psi(a\varphi_q(x)) = \psi(a)x \text{ and } \psi(\varphi_q(a)x) = a\psi(x) \tag{2.1}$$

for any $a \in R$ and $x \in D$. Moreover ψ commutes with Γ.

Proof. Let $\{e_1, e_2, \cdots, e_d\}$ be a basis of D over R. By the definition of (φ_q, Γ)-modules, $\{\varphi_q(e_1), \varphi_q(e_2), \cdots, \varphi_q(e_d)\}$ is also a basis of D. For any $m \in D$ writing $m = a_1\varphi_q(e_1) + a_2\varphi_q(e_2) + \cdots + a_d\varphi_q(e_d)$, we put $\psi(m) = \psi(a_1)e_1 + \psi(a_2)e_2 + \cdots + \psi(a_d)e_d$. Then ψ satisfies (2.1). It is easy to prove the uniqueness of ψ. Observe that for any $\gamma \in \Gamma$, $\gamma\psi\gamma^{-1}$ also satisfies (2.1). Thus $\gamma\psi\gamma^{-1} = \psi$ by uniqueness of ψ. This means that ψ commutes with Γ. ◻

2.2 The operator ∂ and the map Res

Recall that $\partial = \frac{\partial F_{\mathcal{F}}}{\partial Y}(u_{\mathcal{F}},0)\cdot \mathrm{d}/\mathrm{d}u_{\mathcal{F}}$. So $\mathrm{d}t_{\mathcal{F}} = \frac{\partial F_{\mathcal{F}}}{\partial Y}(u_{\mathcal{F}},0)\mathrm{d}u_{\mathcal{F}}$ and $\frac{\mathrm{d}t_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}} = (\frac{\partial F_{\mathcal{F}}}{\partial Y}(u_{\mathcal{F}},0))^{-1}$.

Lemma 2.10. *If $r \geq s > 0$ and $f \in \mathscr{R}_L^{]0,r]}$, then $v^{[s,r]}(\partial f) \geqslant v^{[s,r]}(f) - r$.*

Proof. Observe that $v_p(\frac{\partial F_{\mathcal{F}}}{\partial Y}(z,0)) = 0$ for all z in the disk $|z| < 1$. Thus $v^{[s,r]}(\partial f) = v^{[s,r]}\left(\frac{\mathrm{d}f}{\mathrm{d}u_{\mathcal{F}}}\right)$. Write $f = \sum_{n\in\mathbb{Z}} a_n u_{\mathcal{F}}^n$. Then we have

$$\begin{aligned} v^{[s,r]}\left(\frac{\mathrm{d}f}{\mathrm{d}u_{\mathcal{F}}}\right) &= \inf_{\substack{r\geqslant v_p(z)\geqslant s\\ n\in\mathbb{Z}}} v_p\left(na_nz^{n-1}\right) \\ &\geqslant \inf_{\substack{r\geqslant v_p(z)\geqslant s\\ n\in\mathbb{Z}}} (v_p(a_n)+nv_p(z)-v_p(z)) \\ &\geqslant \inf_{\substack{r\geqslant v_p(z)\geqslant s\\ n\in\mathbb{Z}}} (v_p(a_n)+nv_p(z)) - r \\ &\geqslant v^{[s,r]}(f) - r, \end{aligned}$$

as desired. □

Lemma 2.11. *We have*

$$\partial\cdot\sigma_a = a\sigma_a\cdot\partial,\quad \partial\cdot\varphi_q = \pi\varphi_q\cdot\partial,\quad \partial\circ\psi = \pi^{-1}\psi\circ\partial.$$

Proof. From the definition of ∇ we see that $\nabla = t_{\mathcal{F}}\partial$ commutes with Γ, φ_q and ψ. So the equalities

$$\sigma_a(t_{\mathcal{F}}) = at_{\mathcal{F}},\ \varphi_q(t_{\mathcal{F}}) = \pi t_{\mathcal{F}},\ \psi(t_{\mathcal{F}}) = \psi(\pi^{-1}\varphi_q(t_{\mathcal{F}})) = \pi^{-1}t_{\mathcal{F}}$$

imply the lemma. □

Let $\mathrm{res}: \mathscr{R}_L\mathrm{d}u_{\mathcal{F}} \to L$ be the residue map $\mathrm{res}(\sum_{i\in\mathbb{Z}} a_iu_{\mathcal{F}}^i\mathrm{d}u_{\mathcal{F}}) = a_{-1}$, and let $\mathrm{Res}: \mathscr{R}_L \to L$ be the map defined by $\mathrm{Res}(f) = \mathrm{res}(f\mathrm{d}t_{\mathcal{F}})$.

Proposition 2.12. *We have the following exact sequence*

$$0 \longrightarrow L \longrightarrow \mathscr{R}_L \xrightarrow{\partial} \mathscr{R}_L \xrightarrow{\mathrm{Res}} L \longrightarrow 0$$

where $L \to \mathscr{R}_L$ is the inclusion map.

Proof. The kernel of ∂ is just the kernel of $\mathrm{d}/\mathrm{d}u_{\mathcal{F}}$ and thus is L. For any $a \in L$ we have $\mathrm{Res}(\frac{a}{u_{\mathcal{F}}}\cdot(\frac{\mathrm{d}t_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}})^{-1}) = a$, which implies that Res is surjective. If $f = \partial g$, then $f\mathrm{d}t_{\mathcal{F}} = \mathrm{d}g$ and so $\mathrm{Res}(f) = \mathrm{res}(\mathrm{d}g) = 0$. It follows that $\mathrm{Res}\circ\partial = 0$. Conversely, if $f \in \mathscr{R}_L$ satisfies $\mathrm{Res}(f) = 0$, then f can be written as $f = (\frac{\mathrm{d}t_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}})^{-1}\cdot\sum_{i\neq -1} a_iu_{\mathcal{F}}^i$. Put $g = \sum_{i\neq-1}\frac{a_i}{i+1}u_{\mathcal{F}}^{i+1}$. Then $f = \partial g$. □

Proposition 2.13.

(a) $\mathrm{Res}\circ\sigma_a = a^{-1}\mathrm{Res}$.

(b) $\mathrm{Res}\circ\varphi_q = \frac{q}{\pi}\mathrm{Res}$ *and* $\mathrm{Res}\circ\psi = \frac{\pi}{q}\mathrm{Res}$.

Proof. First we prove (a). Let g be in $\mathscr{R}_L$ and put $f = \partial g$. By Lemma 2.11 we have

$$\sigma_a(f) = \sigma_a\circ\partial(g) = a^{-1}\partial(\sigma_a(g)),\quad \psi(f) = \psi\circ\partial(g) = \pi\partial(\psi(g)).$$

Thus by Proposition 2.12 we have $\mathrm{Res}\circ\sigma_a = a^{-1}\mathrm{Res} = 0$ and $\mathrm{Res}\circ\psi = \frac{\pi}{q}\mathrm{Res} = 0$ on $\partial\mathscr{R}_L$. From

$$\sigma_a(1/u_{\mathcal{F}}) = \frac{1}{[a]_{\mathcal{F}}(u_{\mathcal{F}})} \equiv \frac{1}{au_{\mathcal{F}}} \mod \mathscr{R}_L^+,$$

we see that $\mathrm{Res}\circ\sigma_a(\frac{1}{u_{\mathcal{F}}}) = a^{-1}\mathrm{Res}(\frac{1}{u_{\mathcal{F}}})$. Assertion (a) follows.

To prove $\mathrm{Res}\circ\psi = \frac{\pi}{q}\mathrm{Res}$, without loss of generality we suppose that $\mathcal{F}$ is the special Lubin-Tate group. In this case $\psi(\frac{1}{u_{\mathcal{F}}}) = \frac{\pi}{qu_{\mathcal{F}}}$, and so $\mathrm{Res}(\psi(1/u_{\mathcal{F}})) = \frac{\pi}{q}\mathrm{Res}(1/u_{\mathcal{F}})$. It follows that $\mathrm{Res}\circ\psi = \frac{\pi}{q}\mathrm{Res}$. Finally we have $\mathrm{Res}(\varphi_q(z)) = \frac{q}{\pi}\mathrm{Res}(\psi(\varphi_q(z))) = \frac{q}{\pi}\mathrm{Res}(z)$ for any $z \in \mathscr{R}_L$. In other words, $\mathrm{Res}\circ\varphi_q = \frac{q}{\pi}\mathrm{Res}$. □

Using Res we can define a pairing $\{\cdot,\cdot\} : \mathscr{R}_L \times \mathscr{R}_L \to L$ by $\{f, g\} = \mathrm{Res}(fg)$.

Proposition 2.14. (a) *The pairing $\{\cdot,\cdot\}$ is perfect and induces a continuous isomorphism from $\mathscr{R}_L$ to its dual.*

(b) *We have*

$$\{\sigma_a(f), \sigma_a(g)\} = a^{-1}\{f, g\}, \quad \{\varphi_q(f), \varphi_q(g)\} = \frac{q}{\pi}\{f, g\}, \quad \{\psi(f), \psi(g)\} = \frac{\pi}{q}\{f, g\}.$$

Proof. Assertion (a) follows from [12, Remark I.1.5]. Assertion (b) follows from Proposition 2.13. □

3 Operators on $\mathscr{R}_{\mathbb{C}_p}$

3.1 The operator ψ on $\mathscr{R}_{\mathbb{C}_p}$

First we define $\mathscr{R}_{\mathbb{C}_p}$. For any $r \geq 0$, let $\mathscr{E}^{]0,r]}_{\mathbb{C}_p} := \mathscr{E}^{]0,r]}\widehat{\otimes}_F\mathbb{C}_p$ be the topological tensor product, i.e. the Hausdorff completion of the projective tensor product $\mathscr{E}^{]0,r]} \otimes_F \mathbb{C}_p$ (cf. [23]). Then $\mathscr{E}^{]0,r]}_{\mathbb{C}_p}$ is the ring of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i$ with coefficients in $\mathbb{C}_p$ that are convergent on the annulus $0 < v_p(u_{\mathcal{F}}) \leq r$. We also write $\mathscr{R}^+_{\mathbb{C}_p}$ for $\mathscr{E}^{]0,+\infty]}_{\mathbb{C}_p}$. Then we define $\mathscr{R}_{\mathbb{C}_p}$ to be the inductive limit $\varinjlim_{r\to 0} \mathscr{E}^{]0,r]}_{\mathbb{C}_p}$.

The p-adic Fourier theory of Schneider and Teitelbaum [24] shows that $\mathscr{R}^+_{\mathbb{C}_p}$ is isomorphic to the ring $\mathscr{D}(\mathcal{O}_F, \mathbb{C}_p)$ of $\mathbb{C}_p$-valued locally F-analytic distributions on $\mathcal{O}_F$. We recall this below.

By [24] there exists a rigid analytic group variety $\mathfrak{X}$ such that $\mathfrak{X}(L)$, for any extension $L \subseteq \mathbb{C}_p$ of F, is the set of L-valued locally F-analytic characters. For $\lambda \in \mathscr{D}(\mathcal{O}_F, L)$, put $F_\lambda(\chi) = \lambda(\chi)$, $\chi \in \mathfrak{X}(L)$. Then F_λ is in $\mathcal{O}(\mathfrak{X}/L)$, and the map $\mathscr{D}(\mathcal{O}_F, L) \to \mathcal{O}(\mathfrak{X}/L)$, $\lambda \mapsto F_\lambda$, is an isomorphism of L-Fréchet algebras.

Let $\mathcal{F}'$ be the p-divisible group dual to $\mathcal{F}$, $T\mathcal{F}'$ the Tate module of $\mathcal{F}'$. Then $T\mathcal{F}'$ is a free $\mathcal{O}_F$-module of rank 1; the Galois action on $T\mathcal{F}'$ is given by the continuous character $\tau := \chi_{\mathrm{cyc}}\cdot\chi_{\mathcal{F}}^{-1}$, where χ_{cyc} is the cyclotomic character. By Cartier duality, we obtain a Galois equivariant pairing $\langle\ ,\ \rangle : \mathcal{F}(\mathbb{C}_p) \otimes_{\mathcal{O}_F} T\mathcal{F}' \to \mathrm{B}_1(\mathbb{C}_p)$, where $\mathrm{B}_1(\mathbb{C}_p)$ is the multiplicative group $\{z \in \mathbb{C}_p : |z - 1| < 1\}$. Fixing a generator t' of $T\mathcal{F}'$, we obtain a map $\mathcal{F}(\mathbb{C}_p) \to \mathrm{B}_1(\mathbb{C}_p)$. As a formal series, this morphism can be written as $\beta_{\mathcal{F}}(X) := \exp(\Omega \log_{\mathcal{F}}(X))$ for some $\Omega \in \mathbb{C}_p$, and it lies in $1 + X\mathscr{O}_{\mathbb{C}_p}[[X]]$. Moreover, we have $v_p(\Omega) = \frac{1}{p-1} - \frac{1}{(q-1)e_F}$ (cf. the appendix of [24] or [7]) and $\sigma(\Omega) = \tau(\sigma)\Omega$ for all $\sigma \in G_F$. Using $\langle\cdot,\cdot\rangle$ we obtain an isomorphism of rigid analytic group varieties

$$\kappa : \mathcal{F}(\mathbb{C}_p) \xrightarrow{\sim} \mathfrak{X}(\mathbb{C}_p), \quad z \mapsto \kappa_z(i) := \langle t', [i]_{\mathcal{F}}(z)\rangle = \beta_{\mathcal{F}}([i]_{\mathcal{F}}(z)).$$

Passing to global sections, we obtain the desired isomorphism $\mathscr{D}(\mathcal{O}_F, \mathbb{C}_p) \cong \mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \cong \mathscr{R}^+_{\mathbb{C}_p}$.

We extend φ_q, ψ and the Γ-action $\mathbb{C}_p$-linear and continuously to $\mathscr{R}_{\mathbb{C}_p}$. By continuity we have $\psi(\varphi_q(f)g) = f\psi(g)$ for any $f, g \in \mathscr{R}_{\mathbb{C}_p}$. All of these actions keep $\mathscr{R}^+_{\mathbb{C}_p}$ invariant.

Lemma 3.1. *We have*

$$\begin{aligned}
\sigma_a(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= \beta_{\mathcal{F}}([ai]_{\mathcal{F}}),\\
\varphi_q(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= \beta_{\mathcal{F}}([\pi i]_{\mathcal{F}}),\\
\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= \begin{cases} 0 & \text{if } i\notin \pi\mathcal{O}_F \\ \beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}}) & \text{if } i\in\pi\mathcal{O}_F, \end{cases}\\
\partial(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= i\Omega\beta_{\mathcal{F}}([i]_{\mathcal{F}}).
\end{aligned}$$

Proof. The formulae for σ_a and φ_q are obvious. The formula for ∂ follows from that

$$\partial\exp(i\Omega\log_{\mathcal{F}}(u_{\mathcal{F}})) = \exp(i\Omega\log_{\mathcal{F}}(u_{\mathcal{F}}))\cdot\partial(i\Omega t_{\mathcal{F}}) = i\Omega\exp(i\Omega\log_{\mathcal{F}}(u_{\mathcal{F}})).$$

If $i\in\pi\mathcal{O}_F$, then $\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \psi\circ\varphi_q(\beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}})) = \beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}})$. For any $i\notin\pi\mathcal{O}_F$, we have

$$\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \frac{1}{q}\varphi_q^{-1}\left(\sum_{\eta\in\ker[\pi]_{\mathcal{F}}}\beta_{\mathcal{F}}([i]_{\mathcal{F}}(u_{\mathcal{F}}+_{\mathcal{F}}\eta))\right) = \frac{1}{q}\varphi_q^{-1}\left(\beta_{\mathcal{F}}([i]_{\mathcal{F}})\sum_{\eta\in\ker[\pi]_{\mathcal{F}}}\beta_{\mathcal{F}}([i]_{\mathcal{F}}(\eta))\right) = 0 \tag{3.1}$$

because $\{\beta_{\mathcal{F}}([i]_{\mathcal{F}}(\eta)) : \eta\in\ker[\pi]_{\mathcal{F}}\} = \{\beta_{\mathcal{F}}(\eta) : \eta\in\ker[\pi]_{\mathcal{F}}\}$ take values in the set of p-th roots of unity and each of these p-th roots of unity appears q/p times. □

The isomorphism $\mathscr{R}^+_{\mathbb{C}_p}\cong\mathscr{D}(\mathcal{O}_F,\mathbb{C}_p)$ transfers the actions of φ_q, ψ and Γ to $\mathscr{D}(\mathcal{O}_F,\mathbb{C}_p)$.

Lemma 3.2. *For any $\mu\in\mathscr{D}(\mathcal{O}_F,\mathbb{C}_p)$, we have*

$$\sigma_a(\mu)(f) = \mu(f(a\cdot)),\quad \varphi_q(\mu)(f) = \mu(f(\pi\cdot)).$$

Proof. Note that the action of φ_q and Γ on $\mathscr{R}^+_{\mathbb{C}_p}$ comes, by passing to global sections, from the (φ_q,Γ)-action on $\mathcal{F}(\mathbb{C}_p)$ with $\varphi_q = [\pi]_{\mathcal{F}}$ and $\sigma_a = [a]_{\mathcal{F}}$. The isomorphism κ transfers the action to $\mathfrak{X}(\mathbb{C}_p)$: $\varphi_q(\chi)(x) = \chi(\pi x)$ and $\sigma_a(\chi)(x) = \chi(ax)$. Passing to global sections yields what we want. □

Lemma 3.3. *The family $\Big(\beta_{\mathcal{F}}([i]_{\mathcal{F}})\Big)_{\bar{i}\in\mathcal{O}_F/\pi}$ is a basis of $\mathscr{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathscr{R}_{\mathbb{C}_p})$. Moreover, if*

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi}\beta_{\mathcal{F}}([i]_{\mathcal{F}})\varphi_q(f_i),$$

then the terms of the sum do not depend on the choice of the liftings i, and we have

$$f_i = \psi\Big(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})f\Big).$$

Proof. What we need to show is that

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi}\beta_{\mathcal{F}}([i]_{\mathcal{F}})\cdot\varphi_q\circ\psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})f) \tag{3.2}$$

for all $f\in\mathscr{R}_{\mathbb{C}_p}$. Indeed, (3.2) implies that $\{\beta_{\mathcal{F}}([i]_{\mathcal{F}})\}_{\bar{i}\in\mathcal{O}_F/\pi}$ generate $\mathscr{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathscr{R}_{\mathbb{C}_p})$. On the other hand, if $f = \sum\limits_{\bar{i}\in\mathcal{O}_F/\pi}\beta_{\mathcal{F}}([i]_{\mathcal{F}})\varphi_q(f_i)$, using (3.1) we obtain $f_i = \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})f)$, which implies the linear independence of $\{\beta_{\mathcal{F}}([i]_{\mathcal{F}})\}_{\bar{i}\in\mathcal{O}_F/\pi}$ over $\varphi_q(\mathscr{R}_{\mathbb{C}_p})$. As the map $f\mapsto\sum_{\bar{i}\in\mathcal{O}_F/\pi}\beta_{\mathcal{F}}([i]_{\mathcal{F}})\cdot\varphi_q\circ\psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})f)$ is $\varphi_q(\mathscr{R}_{\mathbb{C}_p})$-linear and continuous, we only need to prove (3.2) for a subset that topologically generates $\mathscr{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathscr{R}_{\mathbb{C}_p})$. For example, $\{u^i_{\mathcal{F}}\}_{0\leq i\leq q-1}$ is such a subset. So it is sufficient to prove (3.2) for $f\in\mathscr{R}^+_{\mathbb{C}_p}$. For any $i\in\mathcal{O}_F$, let δ_i be the Dirac distribution such that $\delta_i(f) = f(i)$. Then $\kappa^*(\delta_i) = \beta_{\mathcal{F}}([i]_{\mathcal{F}})$. Indeed, we have

$$\kappa^*(\delta_i)(z) = \delta_i(z) = \kappa_z(i) = \beta_{\mathcal{F}}([i]_{\mathcal{F}}(z)).$$

It is easy to see that $(\delta_i)_{\bar{i}\in\mathcal{O}_F/\pi}$ is a basis of $\mathscr{D}(\mathcal{O}_F,\mathbb{C}_p)$ over $\varphi_q(\mathscr{D}(\mathcal{O}_F,\mathbb{C}_p))$. Thus every $f\in\mathscr{R}^+_{\mathbb{C}_p}$ can be written uniquely in the form $f = \sum_{\bar{i}\in\mathcal{O}_F/\pi}\beta_{\mathcal{F}}([i]_{\mathcal{F}})\varphi_q(f_i)$ with $f_i\in\mathscr{R}^+_{\mathbb{C}_p}$. As is observed above, from (3.1) we deduce that $f_i = \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})f)$. □

Next we define operators Res_U. These are analogues of the operators defined in [12].

For any $f \in \mathscr{R}_{\mathbb{C}_p}$, $i \in \mathcal{O}_F$ and integer $m \geqslant 0$, put

$$\mathrm{Res}_{i+\pi^m\mathcal{O}_F}(f) = \beta_{\mathcal{F}}([i]_{\mathcal{F}})(\varphi_q^m \circ \psi^m)\Big(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})f\Big).$$

Lemma 3.3 says that

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi} \mathrm{Res}_{i+\pi\mathcal{O}_F}(f),$$

This implies that the operators $\mathrm{Res}_{i+\pi^m\mathcal{O}_F}$ are well defined (i.e. $\mathrm{Res}_{i+\pi^m\mathcal{O}_F}$ does not depend on the choice of i in the ball $i+\pi^m\mathcal{O}_F$). Applying Lemma 3.3 recursively we get

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi^m} \mathrm{Res}_{i+\pi^m\mathcal{O}_F}(f).$$

Finally, if U is a compact open subset of $\mathcal{O}_F$, it is a finite disjoint union of balls $i_k + \pi^{m_k}\mathcal{O}_F$. Define $\mathrm{Res}_U = \sum_k \mathrm{Res}_{i_k+\pi^{m_k}\mathcal{O}_F}$. The map $\mathrm{Res}_U\colon \mathscr{R}_{\mathbb{C}_p} \to \mathscr{R}_{\mathbb{C}_p}$ does not depend on the choice of these balls, and we have $\mathrm{Res}_{\mathcal{O}_F} = 1$, $\mathrm{Res}_\emptyset = 0$ and $\mathrm{Res}_{U\cup U'} + \mathrm{Res}_{U\cap U'} = \mathrm{Res}_U + \mathrm{Res}_{U'}$.

3.2 The operator m_α

Let $\alpha\colon \mathcal{O}_F \to \mathbb{C}_p$ be a locally (F-)analytic function. In this subsection, we define an operator $m_\alpha\colon \mathscr{R}_{\mathbb{C}_p} \to \mathscr{R}_{\mathbb{C}_p}$ similar to the one defined in [10, V.2].

Since α is a locally analytic function on $\mathcal{O}_F$, there is an integer $m \geqslant 0$ such that

$$\alpha(x) = \sum_{n=0}^{+\infty} a_{i,n}(x-i)^n \qquad \text{for all } x \in i+\pi^m\mathcal{O}_F,$$

with $a_{i,n} = \frac{1}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\alpha(x)\big|_{x=i}$. Let $\ell \geqslant m$ be an integer. Define

$$m_\alpha(f) = \sum_{\bar{i}\in\mathcal{O}_F/\pi^\ell} \beta_{\mathcal{F}}([i]_{\mathcal{F}})\left(\varphi_q^\ell \circ \left(\sum_{n=0}^{+\infty} a_{i,n}\pi^{\ell n}\Omega^{-n}\partial^n\right)\circ\psi^\ell\right)\Big(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})\cdot f\Big).$$

(Formally, this definition can be seen as "$m_\alpha = \alpha(\Omega^{-1}\partial)$"). According to Lemmas 2.6, 2.7 and 2.10, if $r < \frac{1}{q^{\ell-1}(q-1)e_F}$ then we have

$$v^{[s,r]}\left((\varphi_q^\ell \circ \Omega^{-n}\partial^n \circ \psi^\ell)(g)\right) \geqslant -nq^\ell r - nv_p(\Omega) + v^{[s,r]}(g) - \ell v_p(q),$$

and thus $\sum_{n=0}^{+\infty} a_{n,i}\pi^{\ell n}(\varphi_q^\ell \circ \Omega^{-n}\partial^n \circ \psi^\ell)(g)$ converges when ℓ and r satisfy

$$\frac{\ell}{e_F} - q^\ell r - \frac{1}{p-1} + \frac{1}{(q-1)e_F} \geqslant \frac{m}{e_F}.$$

If we choose $\ell > m + \frac{e_F}{p-1} - \frac{1}{q-1}$ and r close enough to 0, then this condition is satisfied. Hence, we have indeed defined a continuous operator $m_\alpha\colon \mathscr{R}_{\mathbb{C}_p} \to \mathscr{R}_{\mathbb{C}_p}$.

Now, let us prove that $m_\alpha(f)$ neither depend on the choice of ℓ, nor on that of the liftings i for $\bar{i} \in \mathcal{O}_F/\pi^\ell$. By linearity and continuity, we may assume that $f = 1_{i+\pi^m\mathcal{O}_F}(x-i)^k$. Remark that we have

$$a_{i+\pi^m v,n} = \begin{bmatrix} k \\ n \end{bmatrix} \pi^{(k-n)m} v^{k-n}.$$

It suffices to show that,

$$\sum_{\overline{v}\in\mathcal{O}_F/\pi^{\ell-m}} \beta_{\mathcal{F}}([\pi^m v]_{\mathcal{F}}) \left(\varphi_q^{\ell}\circ\left(\sum_{n=0}^{k} a_{i+\pi^m v,n}\pi^{\ell n}\Omega^{-n}\partial^n\right)\circ\psi^{\ell}\right)\left(\beta_{\mathcal{F}}([-\pi^m v]_{\mathcal{F}})\cdot f\right)$$
$$= \left(\varphi_q^m\circ\left(\pi^{mk}\Omega^{-k}\partial^k\right)\circ\psi^m\right) f.$$

and for this it is sufficient to prove that

$$\sum_{\overline{v}\in\mathcal{O}_F/\pi^{\ell-m}} \beta_{\mathcal{F}}([v]_{\mathcal{F}}) \left(\varphi_q^{\ell-m}\circ\left(\sum_{n=0}^{k} a_{i+\pi^m v,n}\pi^{\ell n}\Omega^{-n}\partial^n\right)\circ\psi^{\ell-m}\right)\left(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})\cdot f\right) = \pi^{mk}\Omega^{-k}\partial^k f.$$

As

$$\sum_{n=0}^{k} a_{i+\pi^m v,n}\pi^{\ell n}\Omega^{-n}\partial^n = \sum_{n=0}^{k}\begin{bmatrix} k\\ n\end{bmatrix}\pi^{(k-n)m}v^{k-n}\cdot\pi^{\ell n}\Omega^{-n}\partial^n = \pi^{mk}\left(\pi^{\ell-m}\Omega^{-1}\partial+v\right)^k,$$

it suffices to prove that

$$\Omega^{-k}\partial^k f = \sum_{\overline{v}\in\mathcal{O}_F/\pi^{\ell-m}} \beta_{\mathcal{F}}([v]_{\mathcal{F}})\left(\varphi_q^{\ell-m}\circ\left(\pi^{\ell-m}\Omega^{-1}\partial+v\right)^k\circ\psi^{\ell-m}\right)\left(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})f\right).$$

Since $\left(\pi^{\ell-m}\Omega^{-1}\partial+v\right)^k\circ\psi^{\ell-m} = \psi^{\ell-m}\circ\left(\Omega^{-1}\partial+v\right)^k$ and

$$\left(\Omega^{-1}\partial+v\right)\left(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})f\right) = \beta_{\mathcal{F}}([-v]_{\mathcal{F}})\Omega^{-1}\partial f$$

(which follows from Lemma 3.1), the problem reduces to proving

$$f = \sum_{\overline{v}\in\mathcal{O}_F/\pi^{\ell-m}} \beta_{\mathcal{F}}([v]_{\mathcal{F}})\left(\varphi^{\ell-m}\circ\psi^{\ell-m}\right)\left(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})f\right).$$

But this can be deduced from Lemma 3.1 and Lemma 3.3.

Lemma 3.4. *If $\alpha,\beta\colon\mathcal{O}_F\to\mathbb{C}_p$ are locally analytic functions, then $m_\alpha\circ m_\beta = m_{\alpha\beta}$.*

Proof. We can choose ℓ sufficiently large, so that the same value can be used to define $m_\alpha(f)$ and $m_\beta(f)$. Since $\psi^\ell\circ\varphi_q^\ell = 1$, the equality in the lemma reduces to the expression of the product of two power series. □

Lemma 3.5. *We have:*

- $m_1 = \mathrm{id}$
- *If U is a compact open subset of $\mathcal{O}_F$, then* $\mathrm{Res}_U = m_{\mathbf{1}_U}$.
- *If $\lambda\in\mathbb{C}_p$, then $m_{\lambda\alpha} = \lambda m_\alpha$.*
- $\varphi_q\circ m_\alpha = m_{x\mapsto\mathbf{1}_{\pi\mathcal{O}_F}(x)\alpha(\pi^{-1}x)}\circ\varphi_q$
- $\psi\circ m_\alpha = m_{x\mapsto\alpha(\pi x)}\circ\psi$
- *For any $a\in\mathcal{O}_F^\times$, we have $\sigma_a\circ m_\alpha = m_{x\mapsto\alpha(a^{-1}x)}\circ\sigma_a$*
- *$\mathscr{R}^+_{\mathbb{C}_p}$ is stable under m_α.*

Proof. These are easy consequences of the definition of m_α. □

Remark 3.6. The notation m_α stands for "multiply by α": for any $\mu\in\mathscr{D}(\mathcal{O}_F,\mathbb{C}_p)$ we have $m_\alpha\kappa^*(F_\mu) = \kappa^*(F_{\alpha\mu})$, where $\alpha\mu$ is the distribution such that $(\alpha\mu)(f) = \mu(\alpha f)$ for any locally F-analytic function f.

The operator m_α has been defined over $\mathscr{R}_{\mathbb{C}_p}$, using a period $\Omega \in \mathbb{C}_p$ that is transcendental over F. However, in some cases, it is possible to construct related operators over $\mathscr{R}_L$, for L smaller than $\mathbb{C}_p$. This is done using the following lemma.

Lemma 3.7. *Let σ be in G_L. Consider the action of σ over $\mathscr{R}_{\mathbb{C}_p}$ given by*

$$f^\sigma(u_{\mathcal{F}}) = \sum_{n\in\mathbb{Z}} \sigma(a_n) u_{\mathcal{F}}^n \qquad \text{if} \qquad f(u_{\mathcal{F}}) = \sum_{n\in\mathbb{Z}} a_n u_{\mathcal{F}}^n \in \mathscr{R}_{\mathbb{C}_p}.$$

Then, we have $m_\alpha(f)^\sigma = m_\beta(f^\sigma)$, for $\beta(x) = \sigma\left(\alpha\left(\frac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)}x\right)\right)$.

Proof. This can be deduced easily from the definition of m_α and the action of σ on Ω. □

3.3 The $L[\Gamma]$-module $\mathscr{R}_L(\delta)^{\psi=0}$

Let $\delta\colon F^\times \to L^\times$ be a locally F-analytic character. Then the map $x \mapsto \mathbf{1}_{\mathcal{O}_F^\times}(x)\delta(x)$ is locally analytic on $\mathcal{O}_F$. Thus, we have an operator $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}$ on $\mathscr{R}_{\mathbb{C}_p}$.

Lemma 3.8. *Let f be in $\mathscr{R}_L$. If $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f) = \sum_{n\in\mathbb{Z}} a_n u_{\mathcal{F}}^n \in \mathscr{R}_{\mathbb{C}_p}$, then the coefficients a_n are all on the same line of the L-vector space $\mathbb{C}_p$. Moreover, this line does not depend on f.*

Proof. Let σ be in G_L. From Lemma 3.7 and Lemma 3.5 we see that

$$m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f)^\sigma = \delta\left(\frac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)}\right) m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f),$$

and thus $\sigma(a_n) = \delta\left(\frac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)}\right) a_n$ for all n.

Ax-Sen-Tate's theorem (see e.g. [1] or [18]) says that $\mathbb{C}_p^{G_L} = L$. Hence,

$$\left\{ z \in \mathbb{C}_p : \ \sigma(z) = \delta\left(\frac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)}\right) z \ \forall\, \sigma \in G_L \right\}$$

is an L-vector subspace of $\mathbb{C}_p$ with dimension 0 or 1, which proves the lemma. □

Since $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta} \circ m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta^{-1}} = \mathrm{Res}_{\mathcal{O}_F^\times} = 1 - \varphi_q \circ \psi$ is not null, there is a unique L-line in $\mathbb{C}_p$ (which depends only on δ) in which all the coefficients of the series $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f)$, for $f \in \mathscr{R}_L$, lie. Choose some non-zero a_δ on this line.

As

$$\varphi_q \circ \psi \circ m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta} = m_{\mathbf{1}_{\pi\mathcal{O}_F}\mathbf{1}_{\mathcal{O}_F^\times}\delta} = 0$$

and φ_q is injective, $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f)$ is in $\mathscr{R}_{\mathbb{C}_p}^{\psi=0}$.

Lemma 3.9. *Define:*

$$\begin{array}{rccc} M_\delta\colon & \mathscr{R}_L^{\psi=0} & \longrightarrow & \mathscr{R}_L^{\psi=0}, \\ & f & \longmapsto & a_\delta^{-1} m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f). \end{array}$$

(These maps are defined up to homothety, with ratio in L, because of the choice of constants a_δ). Then:

- *M_1 is a homothety (with ratio in $L^\times$) of $\mathscr{R}_L^{\psi=0}$;*
- *$M_{\delta_1} \circ M_{\delta_2} = M_{\delta_1\delta_2}$, up to homothety;*
- *M_δ is a bijection, and its inverse is $M_{\delta^{-1}}$ up to homothety;*

- *for all $\gamma\in\Gamma$, we have $\delta(\gamma)\gamma\circ M_\delta = M_\delta\circ\gamma$;*
- *$(\mathscr{R}_L^+)^{\psi=0}$ is stable under M_δ.*

Proof. This follows from Lemma 3.5 and the fact that $\mathrm{Im}(\mathrm{Res}_{\mathcal{O}_F^\times}) = \mathrm{Ker}(\mathrm{Res}_{\pi\mathcal{O}_F}) = \mathscr{R}_{\mathbb{C}_p}^{\psi=0}$. ☐

If δ is in $\mathscr{I}_{\mathrm{an}}(L)$, we put $\mathscr{R}_L^-(\delta) = \mathscr{R}_L(\delta)/\mathscr{R}_L^+(\delta)$. Since $\mathscr{R}_L^+(\delta)$ is φ_q,ψ,Γ-stable, $\mathscr{R}_L^-(\delta)$ also has φ_q, ψ, Γ-actions.

Lemma 3.10. *We have an exact sequence*

$$0 \longrightarrow \mathscr{R}_L^+(\delta)^{\psi=0} \longrightarrow \mathscr{R}_L(\delta)^{\psi=0} \longrightarrow \mathscr{R}_L^-(\delta)^{\psi=0} \longrightarrow 0.$$

Proof. This follows from the snake lemma and the surjectivity of the map $\psi:\mathscr{R}_L^+(\delta)\to\mathscr{R}_L^+(\delta)$. ☐

Observe that $\mathscr{R}_L(\delta)^{\psi=0} = \mathscr{R}_L^{\psi=0}\cdot e_\delta$ and $\mathscr{R}_L^+(\delta)^{\psi=0} = (\mathscr{R}_L^+)^{\psi=0}\cdot e_\delta$. As ψ commutes with Γ, $\mathscr{R}_L(\delta)^{\psi=0}$, $\mathscr{R}_L^+(\delta)^{\psi=0}$ and $\mathscr{R}_L^-(\delta)^{\psi=0}$ are all Γ-invariant.

Proposition 3.11. *Let δ_1 and δ_2 be two locally F-analytic characters $F^\times\to L^\times$. Then as $L[\Gamma]$-modules, $\mathscr{R}_L(\delta_1)^{\psi=0}$ is isomorphic to $\mathscr{R}_L(\delta_2)^{\psi=0}$, $\mathscr{R}_L^+(\delta_1)^{\psi=0}$ is isomorphic to $\mathscr{R}_L^+(\delta_2)^{\psi=0}$, and $\mathscr{R}_L^-(\delta_1)^{\psi=0}$ is isomorphic to $\mathscr{R}_L^-(\delta_2)^{\psi=0}$.*

Proof. All of the isomorphisms in question are induced by $M_{\delta_1^{-1}\delta_2}$. ☐

Proposition 3.12. *The map ∂ induces Γ-equivariant isomorphisms $(\mathscr{R}_L(\delta))^{\psi=0}\to(\mathscr{R}_L(x\delta))^{\psi=0}$, $(\mathscr{R}_L^+(\delta))^{\psi=0}\to(\mathscr{R}_L^+(x\delta))^{\psi=0}$ and $(\mathscr{R}_L^-(\delta))^{\psi=0}\to(\mathscr{R}_L^-(x\delta))^{\psi=0}$.*

Proof. We first show that the maps in question are bijective. For this we only need to consider the case of $\delta=1$. As $\mathrm{Ker}(\partial)=L$, ∂ is injective on $\mathscr{R}_L^{\psi=0}$. For any $z\in\mathscr{R}_L^{\psi=0}$, $\mathrm{Res}(z)=\frac{q}{\pi}\mathrm{Res}(\psi(z))=0$. Thus by Proposition 2.12 there exists $z'\in\mathscr{R}_L$ such that $\partial z'=z$. As $\partial(\psi(z'))=\frac{1}{\pi}\psi(\partial z')=0$, $\psi(z)=c$ for some $c\in L$. Then $z'-c\in\mathscr{R}_L^{\psi=0}$ and $\partial(z'-c)=z$. This shows that the map $\mathscr{R}_L^{\psi=0}\to\mathscr{R}_L^{\psi=0}$ is bijective. It is clear that, for any $z\in\mathscr{R}_L^{\psi=0}$, $\partial z\in\mathscr{R}_L^+$ if and only if $z\in\mathscr{R}_L^+$. Thus the restriction $\partial:(\mathscr{R}_L^+)^{\psi=0}\to(\mathscr{R}_L^+)^{\psi=0}$ and the induced map $\partial:(\mathscr{R}_L^-)^{\psi=0}\to(\mathscr{R}_L^-)^{\psi=0}$ are also bijective.

That these isomorphisms are Γ-equivariant follows from Lemma 2.11. ☐

Put

$$S_\delta := \mathscr{R}_L^-(\delta)^{\Gamma=1,\psi=0}. \tag{3.3}$$

As before, let ∇_δ be the operator on $\mathscr{R}_L^+$ or $\mathscr{R}_L$ such that $(\nabla_\delta a)e_\delta = \nabla(ae_\delta)$, i.e. $\nabla_\delta = t_{\mathcal{F}}\partial + w_\delta$. The set $\mathscr{R}_L^+(\delta)/\nabla_\delta\mathscr{R}_L^+(\delta)$ also admits actions of Γ, φ_q and ψ. Put

$$T_\delta := (\mathscr{R}_L^+(\delta)/\nabla_\delta\mathscr{R}_L^+(\delta))^{\Gamma=1,\psi=0}.$$

Both S_δ and T_δ are L-vector spaces and only depend on $\delta|_{\mathcal{O}_F^\times}$.

Lemma 3.13. *$S_\delta = \mathscr{R}_L^-(\delta)^{\psi=0,\nabla_\delta=0,\ \Gamma=1}$, i.e. S_δ coincides with the set of Γ-invariant solutions of $\nabla_\delta z=0$ in $\mathscr{R}_L^-(\delta)^{\psi=0}$.*

Proof. In fact, if $z\in\mathscr{R}_L^-(\delta)^{\Gamma=1}$, then $\nabla_\delta z=0$. ☐

Corollary 3.14. $\dim_L S_\delta = \dim_L S_1$ *and* $\dim_L T_\delta = \dim_L T_1$ *for all $\delta\in\mathscr{I}_{\mathrm{an}}(L)$.*

Proof. This follows directly from Proposition 3.11. ☐

Corollary 3.15. *The map $z\mapsto\partial^n z$ induces isomorphisms $S_\delta\to S_{x^n\delta}$ and $T_\delta\to T_{x^n\delta}$.*

Proof. This follows directly from Proposition 3.12. ☐

We determine $\dim_L S_\delta$ and $\dim_L T_\delta$ below.

Lemma 3.16. *The map ∇_δ induces an injection $\bar{\nabla}_\delta : S_\delta \to T_\delta$.*

Proof. By Proposition 3.11 we only need to consider the case of $\delta = 1$.

Let z be an element of S_1. Let $\tilde{z} \in \mathscr{R}_L^{\psi=0}$ be a lifting of z. By Lemma 3.13, $\nabla\tilde{z}$ is in $\mathscr{R}_L^+$. We show that the image of $\nabla\tilde{z}$ in $\mathscr{R}_L^+/\nabla\mathscr{R}_L^+$ belongs to T_1. Since $\psi(\tilde{z}) = 0$, $\psi(\nabla\tilde{z}) = \nabla(\psi(\tilde{z})) = 0$. For any $\gamma \in \Gamma$ there exists $a_\gamma \in \mathscr{R}_L^+$ such that $\gamma\tilde{z} = \tilde{z} + a_\gamma$. Thus $\gamma(\nabla\tilde{z}) = \nabla\tilde{z} + \nabla a_\gamma$. Hence the image of $\tilde{z}$ in $\mathscr{R}_L^+/\nabla\mathscr{R}_L^+(\delta)$ is fixed by Γ. Furthermore the image only depends on z. Indeed, if $\tilde{z}' \in \mathscr{R}_L^{\psi=0}$ is another lifting of z, then $\nabla(\tilde{z}' - \tilde{z})$ is in $\nabla\mathscr{R}_L^+$. Therefore we obtain a map $\bar{\nabla} : S_1 \to T_1$.

We prove that $\bar{\nabla}$ is injective. Suppose that $z \in S_1$ satisfies $\bar{\nabla}z = 0$. Let $\tilde{z} \in \mathscr{R}_L^{\psi=0}$ be a lifting of z. Since $\nabla\tilde{z}$ is in $\nabla\mathscr{R}_L^+$, there exists $y \in \mathscr{R}_L^+$ such that $\nabla y = \nabla\tilde{z}$. Thus $\nabla(\tilde{z} - y) = 0$. Then $\tilde{z} - y$ is in L, which implies that $\tilde{z} \in \mathscr{R}_L^+$ or equivalently $z = 0$. ∎

Lemma 3.17. $\dim_L T_1 = 1$.

Proof. Note that $T_1 = (\mathscr{R}_L^+/\mathscr{R}_L^+ t_{\mathcal{F}})^{\Gamma=1,\psi=0}$. As $\mathscr{R}_L^+$ is a Fréchet-Stein algebra, from the decomposition (1.1) of the ideal $(t_{\mathcal{F}})$ we obtain an isomorphism

$$\jmath : \mathscr{R}_L^+/\mathscr{R}_L^+ t_{\mathcal{F}} \xrightarrow{\sim} \mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})) \times \prod_{n\geq 1} \mathscr{R}_L^+/(\varphi_q^n(Q)). \tag{3.4}$$

The operator ψ induces maps $\psi_0 : \mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})) \to \mathscr{R}_L^+/\mathscr{R}_L^+ u_{\mathcal{F}}$ and $\psi_n : \mathscr{R}_L^+/(\varphi_q^n(Q)) \to \mathscr{R}_L^+/(\varphi_q^{n-1}(Q))$, $n \geq 1$. Thus $\jmath((\mathscr{R}_L^+/\mathscr{R}_L^+ t_{\mathcal{F}})^{\Gamma=1,\psi=0})$ is exactly the subset

$$\{(y_n)_{n\geq 0} : y_0 \in (\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^\Gamma, \psi_0(y_0) = 0,\ y_n \in (\mathscr{R}_L^+/(\varphi_q^n(Q)))^\Gamma, \psi_n(y_n) = 0\ \forall n \geq 1\}$$

of $\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})) \times \prod_{n\geq 1} \mathscr{R}_L^+/(\varphi_q^n(Q))$.

If $n \geq 1$, then $\mathscr{R}_F^+/\varphi_q^n(Q)$ is a finite extension of F and the action of Γ factors through the whole Galois group of this extension. Thus $(\mathscr{R}_F^+/(\varphi_q^n(Q)))^\Gamma = F$ and $(\mathscr{R}_L^+/(\varphi_q^n(Q)))^\Gamma = L$. Since $\psi_n(a) = a$ for any $a \in L$, $(\mathscr{R}_F^+/(\varphi_q^n(Q)))^\Gamma \cap \ker(\psi_n) = 0$ for any $n \geq 1$. Similarly $(\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^\Gamma = (\mathscr{R}_L^+/(u_{\mathcal{F}}))^\Gamma \times (\mathscr{R}_L^+/(Q))^\Gamma$ is 2-dimensional over L. As $\psi_0(1) = 1$ and the image of ψ_0, i.e. $\mathscr{R}_L^+/\mathscr{R}_L^+ u_{\mathcal{F}}$, is 1-dimensional over L, the kernel of $\psi_0|_{(\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^\Gamma}$ is of dimension 1. It follows that $T_1 = (\mathscr{R}_L^+/\mathscr{R}_L^+ t_{\mathcal{F}})^{\Gamma=1,\psi=0}$ is of dimension 1. ∎

Corollary 3.18. $\dim_L S_1 = 1$.

Proof. The map ∇ injects S_1 into T_1 with image of dimensional 1. ∎

Remark 3.19. If $z \in T_1$ is non-zero, then any lifting $\tilde{z} \in \mathscr{R}_L^+$ of z is not in $u_{\mathcal{F}}\mathscr{R}_L^+$ or equivalently $\tilde{z}|_{u_{\mathcal{F}}=0} \neq 0$. We only need to verify this for the special Lubin-Tate group. In this case, $\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})) = \oplus_{i=0}^{q-1} L u_{\mathcal{F}}^i$. We have $(\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^\Gamma = L \oplus L u_{\mathcal{F}}^{q-1}$. Indeed, an element of $\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}}))$ is fixed by Γ if and only if it is fixed by the operators $z \mapsto \sigma_\xi(z)$ with $\xi \in \mu_{q-1}$; but $\sigma_\xi(u_{\mathcal{F}}) = [\xi]_{\mathcal{F}}(u_{\mathcal{F}}) = \xi u_{\mathcal{F}}$ and so $\sigma_\xi(u_{\mathcal{F}}^i) = \xi^i u_{\mathcal{F}}^i$ for any $i \in \mathbb{N}$. Then $\left(\mathscr{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}}))\right)^{\Gamma=1,\psi=0} = L \cdot (u_{\mathcal{F}}^{q-1} - (1-q)\pi/q)$.

Proposition 3.20. *For any $\delta \in \mathscr{I}_{\mathrm{an}}(L)$, $\dim_L S_\delta = \dim_L T_\delta = 1$ and the map $\bar{\nabla}_\delta$ is an isomorphism.*

Proof. This follows from Corollary 3.14, Lemma 3.16, Lemma 3.17 and Corollary 3.18. ∎

4 Cohomology theories for (φ_q, Γ)-modules

For a (φ_q, Γ)-module D over $\mathscr{R}_L$, the (φ_q, Γ)-module structure induces an action of the semi-group $G^+ := \varphi_q^{\mathbb{N}} \times \Gamma$ on D. Following [13] we define $H^\bullet(D)$ as the cohomology of the semi-group G^+. Let $C^\bullet(G^+, D)$ be the complex

$$0 \longrightarrow C^0(G^+, D) \xrightarrow{d_1} C^1(G^+, D) \xrightarrow{d_2} \cdots,$$

where $C^0(G^+,D)=D$, $C^n(G^+,D)$ for $n\geq 1$ is the set of continuous functions from $(G^+)^n$ to D, and d_{n+1} is the differential

$$d_{n+1}c(g_0,\cdots,g_n)=g_0\cdot c(g_1,\cdots,g_n)+\sum_{i=1}^{n-1}(-1)^{i+1}c(g_0,\cdots,g_ig_{i+1},\cdots g_n)+(-1)^{n+1}c(g_0,\cdots g_{n-1}).$$

Then $H^i(D)=H^i(C^\bullet(G^+,D))$.

If D_1 and D_2 are two (φ_q,Γ)-modules over $\mathscr{R}_L$, we use $\mathrm{Ext}(D_1,D_2)$ to denote the set, in fact an L-vector space, of extensions of D_1 by D_2 in the category of (φ_q,Γ)-modules over $\mathscr{R}_L$.

We construct a natural map $\Theta^D:\mathrm{Ext}(\mathscr{R}_L,D)\to H^1(D)$ for any (φ_q,Γ)-module D. Let $\widetilde{D}$ be an extension of $\mathscr{R}_L$ by D. Let $e\in\widetilde{D}$ be a lifting of $1\in\mathscr{R}_L$. Then $g\mapsto g(e)-e$, $g\in G^+$, is a 1-cocycle, and induces an element of $H^1(D)$ independent of the choice of e. Thus we obtain the desired map

$$\Theta^D:\mathrm{Ext}(\mathscr{R}_L,D)\to H^1(D).$$

Proposition 4.1. *For any (φ_q,Γ)-module D over $\mathscr{R}_L$, Θ^D is an isomorphism.*

Proof. Let $\widetilde{D}$ be an extension of $\mathscr{R}_L$ by D in the category of (φ_q,Γ)-modules whose image under Θ^D is zero. Let $e\in\widetilde{D}$ be a lifting of $1\in\mathscr{R}_L$. As the image of $g\mapsto g(e)-e$, $g\in G^+$, in $H^1(D)$ is zero, there exists some $d\in D$ such that $(g-1)e=(g-1)d$ for all $g\in G^+$. Then $g(e-d)=e-d$ for all $g\in G^+$. Thus $\widetilde{D}=D\oplus\mathscr{R}_L(e-d)$ as a (φ_q,Γ)-module. This proves the injectivity of Θ^D. Next we prove the surjectivity of Θ^D. Given a 1-cocycle $g\mapsto c(g)\in D$, correspondingly we can extend the (φ_q,Γ)-module structure on D to the $\mathscr{R}_L$-module $\widetilde{D}=D\oplus\mathscr{R}_Le$ such that $\varphi_q(e)=e+c(\varphi_q)$ and $\gamma(e)=e+c(\gamma)$ for $\gamma\in\Gamma$. □

If D_1 and D_2 are two $\mathcal{O}_F$-analytic (φ_q,Γ)-modules over $\mathscr{R}_L$, we use $\mathrm{Ext}_{\mathrm{an}}(D_1,D_2)$ to denote the L-vector space of extensions of D_1 by D_2 in the category of $\mathcal{O}_F$-analytic (φ_q,Γ)-modules over $\mathscr{R}_L$. We will introduce another cohomology theory $H^*_{\mathrm{an}}(-)$, wherein for any $\mathcal{O}_F$-analytic (φ_q,Γ)-module D the first cohomology group $H^1_{\mathrm{an}}(D)$ coincides with $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L,D)$.

If D is $\mathcal{O}_F$-analytic, we consider the following complex

$$C^\bullet_{\varphi_q,\nabla}(D):\qquad 0\longrightarrow D\xrightarrow{f_1}D\oplus D\xrightarrow{f_2}D\longrightarrow 0\ ,$$

where $f_1:D\to D\oplus D$ is the map $m\mapsto((\varphi_q-1)m,\nabla m)$ and $f_2:D\oplus D\to D$ is $(m,n)\mapsto\nabla m-(\varphi_q-1)n$. As f_1 and f_2 are Γ-equivariant, Γ acts on the cohomology groups $H^i_{\varphi_q,\nabla}(D):=H^i(C^\bullet_{\varphi_q,\nabla}(D))$, $i=0,1,2$. Put $H^i_{\mathrm{an}}(D):=H^i_{\varphi_q,\nabla}(D)^\Gamma$.

By a simple calculation we obtain

$$H^0(D)=H^0_{\mathrm{an}}(D)=D^{\varphi_q=1,\Gamma=1}.$$

Note that $D^{\varphi_q=1}$ is finite dimensional over L, and so is $H^0(D)$. If D is étale and if V is the L-linear Galois representation of G_F attached to D, then

$$H^0(D)=H^0_{\mathrm{an}}(D)=H^0(G_F,V)=V^{G_F}.$$

For our convenience we introduce some notations. Put $Z^1_{\varphi_q,\nabla}(D):=\ker(f_2)$ and $B^1(D):=\mathrm{im}(f_1)$. For any (m_1,n_1) and (m_2,n_2) in $Z^1_{\varphi_q,\nabla}(D)$, we write $(m_1,n_1)\sim(m_2,n_2)$ if $(m_1-m_2,n_1-n_2)\in B^1(D)$. Put

$$Z^1(D):=\{(m,n)\in Z^1_{\varphi_q,\nabla}(D):(m,n)\sim\gamma(m,n)\text{ for any }\gamma\in\Gamma\}.$$

Then $H^1_{\mathrm{an}}(D)=Z^1(D)/B^1(D)$.

Let $\widetilde{D}$ be an $\mathcal{O}_F$-analytic extension of $\mathscr{R}_L$ by D. Let $e\in\widetilde{D}$ be a lifting of $1\in\mathscr{R}_L$. Then $((\varphi_q-1)e,\nabla_{\widetilde{D}}e)$ belongs to $Z^1(D)$ and induces an element of $H^1_{\mathrm{an}}(D)$ independent of the choice of e. In this way we obtain a map

$$\Theta^D_{\mathrm{an}}:\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L,D)\to H^1_{\mathrm{an}}(D).$$

Theorem 4.2. *(= Theorem 0.1) For any $\mathcal{O}_F$-analytic (φ_q,Γ)-module D over $\mathscr{R}_L$, Θ_{an}^D is an isomorphism.*

The proof below is due to the referee and much simpler than the proof in our original version.

Proof. First we show that Θ^D_{an} is injective. Let $\widetilde{D}$ be an $\mathcal{O}_F$-analytic extension of $\mathscr{R}_L$ by D whose image under Θ^D_{an} is zero. Let $e\in\widetilde{D}$ be a lifting of $1\in\mathscr{R}_L$. As the image of $((\varphi_q-1)e,\nabla_{\widetilde{D}}e)$ in $H^1_{\varphi_q,\nabla}(D)$ is zero, there exists some $d\in D$ such that $(\varphi_q-1)e=(\varphi_q-1)d$ and $\nabla_{\widetilde{D}}e=\nabla_{\widetilde{D}}d$. Then $e-d$ is in $\widetilde{D}^{\varphi_q=1,\nabla=0}$. The Γ-action on $\widetilde{D}^{\varphi_q=1,\nabla=0}$ is locally constant and thus is semisimple. So $1\in\mathscr{R}_L$ has a lifting $e'\in\widetilde{D}^{\varphi_q=1,\nabla=0}$ fixed by Γ. This proves the injectivity of Θ^D_{an}.

Next we prove the surjectivity of Θ^D_{an}.

Let z be in $H^1_{\mathrm{an}}(D)$ and let (x,y) represent z, so that $\nabla x=(\varphi_q-1)y$. The invariance of z by Γ ensures the existence of $y_\sigma\in D$ for each $\sigma\in\Gamma$ such that $(\sigma-1)(x,y)=((\varphi_q-1)y_\sigma,\nabla y_\sigma)$. As y_σ is unique up to an element of $D^{\varphi_q=1,\nabla=0}$, the 2-cocycle $y_{\sigma,\tau}=y_{\sigma\tau}-\sigma y_\tau-y_\sigma$ takes values in $D^{\varphi_q=1,\nabla=0}$. If $z=0$, then there exists $a\in D$ such that $x=(\varphi_q-1)a$ and $y=\nabla a$. We have $\nabla(y_\sigma-(\sigma-1)a)=0$. In other words, we can write $y_\sigma=(\sigma-1)a+a_\sigma$ with $a_\sigma\in D^{\varphi_q=1,\nabla=0}$. Then $y_{\sigma,\tau}=a_{\sigma\tau}-\sigma a_\tau-a_\sigma$ and thus $y_{\bullet,\bullet}$ is a coboundary. So we obtain a map $H^1_{\mathrm{an}}(D)\to H^2(\Gamma,D^{\varphi_q=1,\nabla=0})$.

We will show that the image of z by this map is zero. Fix a basis $\{e_1,\cdots,e_d\}$ of D over $\mathscr{R}_L$. Let $r>0$ be sufficiently small such that the matrices of φ_q and $\sigma\in\Gamma$ relative to $\{e_i\}_{i=1}^d$ are all in $\mathrm{GL}_d(\mathscr{E}_L^{]0,r]})$. Put $D^{]0,r]}=\oplus_{i=1}^d\mathscr{E}_L^{]0,r]}e_i$; if $s\in(0,r]$ put $D^{[s,r]}=\oplus_{i=1}^d\mathscr{E}_L^{[s,r]}e_i$. Then $D^{]0,r]}$ and $D^{[s,r]}$ are stable by Γ. As the matrix of φ_q is invertible in $\mathrm{M}_d(\mathscr{E}_L^{]0,r]})$, $\{\varphi_q(e_i)\}_{i=1}^d$ is also a basis of $D^{]0,r]}$. Shrinking r if necessarily we may assume that φ_q maps $D^{[s,r]}$ to $D^{[s/q,r/q]}$; we may also suppose that x and y are in $D^{]0,r]}$, and that $t_{\mathcal{F}}\in\mathscr{E}_L^{]0,r]}$. By the relation $\nabla=t_{\mathcal{F}}\partial$ on $\mathscr{E}_L^{[s,r]}$, Lemma 2.10 and the fact that ∇ is a differential operator i.e. satisfies a relation similar to (1.2), we can show that the action of Γ induces a bounded infinitesimal action ∇ on the Banach space $D^{[s,r]}$. We leave this to the reader. Let us denote $\ell(\sigma)=\log(\chi_{\mathcal{F}}(\sigma))$. For σ close enough to 1 (depending on D and s,r) the series of operators

$$E(\sigma)=\ell(\sigma)+\frac{\ell(\sigma)^2}{2}\nabla+\frac{\ell(\sigma)^3}{3!}\nabla^2+\cdots$$

converges on $D^{[s,r]}$ and also on $D^{[s/q,r/q]}$. Note that, for σ close enough to 1 we have $\sigma=\exp(\ell(\sigma)\nabla)$ on $D^{[s/q,r/q]}$. Let Γ' be an open subgroup of Γ such that for $\sigma\in\Gamma'$ the above two facts hold. Then for $\sigma\in\Gamma'$ we have

$$(\varphi_q-1)(E(\sigma)y)=E(\sigma)(\varphi_q-1)y=E(\sigma)\nabla x=\nabla E(\sigma)x=(\sigma-1)x. \tag{4.1}$$

Note that $\varphi_q(E(\sigma)y)$ is in $D^{[s/q,r/q]}$. So by (4.1) we have $E(\sigma)y\in D^{[s/q,r/q]}\cap D^{[s,r]}=D^{[s/q,r]}$ if s is chosen such that $s<r/q$. Doing this repeatedly we will obtain $E(\sigma)y\in D^{]0,r]}$. Taking $y_\sigma=E(\sigma)y$ for $\sigma\in\Gamma'$ we will have $y_{\sigma,\tau}=0$ for $\sigma,\tau\in\Gamma'$. In other words, the restriction to Γ' of the image of z in $H^2(\Gamma,D^{\varphi_q=1,\nabla=0})$ is 0. Since Γ/Γ' is finite and $D^{\varphi_q=1,\nabla=0}$ is a $\mathbb{Q}$-vector space, the image of z is itself 0. So we can modify y_σ by an element of $D^{\varphi_q=1,\nabla=0}$ so that $y_{\sigma,\tau}$ is identically 0. But this means that $(\sigma-1)y_\tau=(\tau-1)y_\sigma$, so the 1-cocycle $\varphi_q\mapsto x$, $\sigma\mapsto y_\sigma$ defines an element of $H^1(D)$ hence also an extension of $\mathscr{R}_L$ by D.

We will show that the resulting extension in fact belongs to $\mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_L,D)$. As Γ is locally constant on $D^{\varphi_q=1,\nabla=0}$, shrinking Γ' if necessary we may assume that Γ' acts trivially on $D^{\varphi_q=1,\nabla=0}$. Then $\sigma\mapsto y_\sigma-E(\sigma)y$ is a continuous homomorphism from Γ' to $D^{\varphi_q=1,\nabla=0}$. Note that any homomorphism from Γ' to $D^{\varphi_q=1,\nabla=0}$ can be extended to Γ. Thus $y_\sigma-E(\sigma)y=\lambda(\sigma)$ for some $\lambda\in\mathrm{Hom}(\Gamma,D^{\varphi_q=1,\nabla=0})$ and all $\sigma\in\Gamma'$. If S is a set of representatives of Γ/Γ' in Γ, the map $T_S=\frac{1}{|\Gamma:\Gamma'|}\sum_{\sigma\in S}\sigma$ is the identity on $H^1_{\mathrm{an}}(D)$ and a projection from $D^{\varphi_q=1,\nabla=0}$ to $H^0(D)$; moreover it commutes with φ_q, ∇ and Γ. This means that we can apply T_S to (x,y) and y_σ; then we have $y_\sigma-E(\sigma)y=\lambda(\sigma)$ for some $\lambda\in\mathrm{Hom}(\Gamma,H^0(D))$ and all $\sigma\in\Gamma'$. As $\sigma\mapsto E(\sigma)y$ is analytic, the extension in question is $\mathcal{O}_F$-analytic. □

As above, let $\mathrm{Hom}(\Gamma,H^0(D))$ be the set of homomorphisms of groups from Γ to $H^0(D)$. A homomorphism $h:\Gamma\to H^0(D)$ is said to be *locally analytic* if $h(\exp(a\beta))=ah(\exp(\beta))$ for all $a\in\mathcal{O}_F$ and $\beta\in\mathrm{Lie}\Gamma$. Let $\mathrm{Hom}_{\mathrm{an}}(\Gamma,H^0(D))$ be the subset of $\mathrm{Hom}(\Gamma,H^0(D))$ consisting of locally analytic homomorphisms.

Note that we have natural injections

$$\mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D)) \to \mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_L, D) \quad \text{and} \quad \mathrm{Hom}(\Gamma, H^0(D)) \to \mathrm{Ext}^1(\mathscr{R}_L, D).$$

Theorem 4.3. *Assume that D is an $\mathcal{O}_F$-analytic (φ_q, Γ)-module over $\mathscr{R}_L$. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D)) \longrightarrow \mathrm{Hom}(\Gamma, H^0(D)) \oplus \mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_L, D) \longrightarrow \mathrm{Ext}^1(\mathscr{R}_L, D) \longrightarrow 0.$$

For the proof we introduce an auxiliary cohomology theory. Let γ be an element of Γ of infinite order, i.e. $\log(\chi_{\mathcal{F}}(\gamma)) \neq 0$. We consider the complex

$$C^\bullet_{\varphi_q,\gamma}(D): \quad 0 \longrightarrow D \xrightarrow{g_1} D \oplus D \xrightarrow{g_2} D \longrightarrow 0\ ,$$

where $g_1 : D \to D \oplus D$ is the map $m \mapsto ((\varphi_q - 1)m, (\gamma - 1)m)$ and $g_2 : D \oplus D \to D$ is $(m, n) \mapsto (\gamma - 1)m - (\varphi_q - 1)n$. As g_1 and g_2 are Γ-equivariant, Γ acts on $H^i_{\varphi_q,\gamma}(D) := H^i(C^\bullet_{\varphi_q,\gamma}(D))$, $i = 0, 1, 2$. Put $H^i_{\mathrm{an},\gamma}(D) := H^i_{\varphi_q,\gamma}(D)^\Gamma$. A simple calculation shows that $H^0_{\mathrm{an},\gamma}(D) = H^0_{\mathrm{an}}(D)$.

For any $\gamma \in \Gamma$ we use $\overline{\langle\gamma\rangle}$ to denote the closed subgroup of Γ topologically generated by γ. If γ is of infinite order and if D is an $\mathscr{R}_L$-module together with a semilinear $\overline{\langle\gamma\rangle}$-action, let ∇_γ be the operator on D that can be written as $\varinjlim_{\gamma'} \frac{\log(\gamma')}{\log(\chi_{\mathcal{F}}(\gamma'))}$ formally, where γ' runs through all elements of $\overline{\langle\gamma\rangle}$ with $\log \chi_{\mathcal{F}}(\gamma') \neq 0$. (For a precise definition we only need to imitate the definition of ∇.)

Let $\widetilde{D}$ be an $\mathcal{O}_F$-analytic extension of $\mathscr{R}_L$ by D. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathscr{R}_L$. Then $((\varphi_q - 1)e, (\gamma - 1)e)$ induces an element of $H^1_{\mathrm{an},\gamma}(D)$ independent of the choice of e. This yields a map $\Theta^D_{\mathrm{an},\gamma} : \mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D) \to H^1_{\mathrm{an},\gamma}(D)$. Given an element of $H^1_{\mathrm{an},\gamma}(D)$, we can attach to it an extension $\widetilde{D}$ of $\mathscr{R}_L$ by D in the category of free $\mathscr{R}_L$-modules of finite rank together with semilinear actions of φ_q and $\overline{\langle\gamma\rangle}$. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathscr{R}_L$. Then $\big((\varphi_q - 1)e, \nabla_\gamma e\big)$ belongs to $Z^1(D)$ and induces an element of $H^1_{\mathrm{an}}(D)$ independent of the choice of e. This gives a map $\Upsilon^D_{\mathrm{an},\gamma} : H^1_{\mathrm{an},\gamma}(D) \to H^1_{\mathrm{an}}(D)$. Observe that $\Upsilon^D_{\mathrm{an},\gamma} \circ \Theta^D_{\mathrm{an},\gamma} = \Theta^D_{\mathrm{an}}$. By an argument similar to the proof of the injectivity of Θ^D_{an}, we can show that both $\Theta^D_{\mathrm{an},\gamma}$ and $\Upsilon^D_{\mathrm{an},\gamma}$ are injective. Hence it follows from Theorem 4.2 that $\Theta^D_{\mathrm{an},\gamma}$ and $\Upsilon^D_{\mathrm{an},\gamma}$ are isomorphisms.

If c is a 1-cocycle representing an element z of $H^1(D)$, then $(c(\varphi_q), c(\gamma))$ induces an element in $H^1_{\mathrm{an},\gamma}(D)$ which only depends on z. This yields a map $\Upsilon^D_\gamma : H^1(D) \to H^1_{\mathrm{an},\gamma}(D)$. Hence, $\Theta^D_{\mathrm{an},\gamma} : \mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D) \to H^1_{\mathrm{an},\gamma}(D)$ extends to a map $\mathrm{Ext}(\mathscr{R}_L, D) \to H^1_{\mathrm{an},\gamma}(D)$, which will also be denoted by $\Theta^D_{\mathrm{an},\gamma}$. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(\mathscr{R}_L, D) & \xrightarrow[\sim]{\Theta^D} & H^1(D) \\ \big\uparrow & \searrow^{\Theta^D_{\mathrm{an},\gamma}} & \big\downarrow \Upsilon^D_\gamma \\ \mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D) & \xrightarrow[\Theta^D_{\mathrm{an},\gamma}]{\sim} & H^1_{\mathrm{an},\gamma}(D). \end{array} \tag{4.2}$$

The composition $(\Theta^D_{\mathrm{an},\gamma^{-1}})^{-1} \circ \Upsilon^D_\gamma \circ \Theta^D$ is a projection from $\mathrm{Ext}(\mathscr{R}_L, D)$ to $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D)$, which depends on γ.

Proof of Theorem 4.3. The only nontrivial thing to be proved is the surjectivity of $\mathrm{Hom}(\Gamma, H^0(D)) \oplus \mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_L, D) \to \mathrm{Ext}^1(\mathscr{R}_L, D)$. Let $\widetilde{D}$ be in $\mathrm{Ext}^1(\mathscr{R}_L, D)$. Without loss of generality we may assume that the image of $\widetilde{D}$ by the projection $(\Theta^D_{\mathrm{an},\gamma^{-1}})^{-1} \circ \Upsilon^D_\gamma \circ \Theta^D$ is zero. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathscr{R}_L$. Then let c be the 1-cocycle defined by $\varphi_q \mapsto (\varphi_q - 1)e$, $\sigma \mapsto (\sigma - 1)e$ for $\sigma \in \Gamma$, so that $\bar{c}$, the class of c in $H^1(D)$, corresponds to $\widetilde{D}$. So the image of $\bar{c}$ by the map Υ^D_γ is zero. This means that there exists $d \in D$ such that $(\varphi_q - 1)d = c(\varphi_q)$ and $(\gamma - 1)d = c(\gamma)$. Replacing e by $e - d$, we may assume that $c(\varphi_q) = c(\gamma) = 0$. Then for any $\sigma \in \Gamma$, we have $(\varphi_q - 1)c(\sigma) = (\sigma - 1)c(\varphi_q) = 0$ and $(\gamma - 1)c(\sigma) = (\sigma - 1)c(\gamma) = 0$. This means that $c(\sigma) \in D^{\varphi_q = 1, \gamma = 1}$. Note that $M := D^{\varphi_q = 1, \gamma = 1}$ is of finite rank over L. We write $M = H^0(D) \oplus \oplus_j M_j$ as a Γ-module, where each of M_j is an irreducible Γ-module. Write $c = c' + \sum_j c_j$ by this decomposition. Observe that c' and c_j are all 1-cocycles. As M_j is irreducible and the Γ-action on M_j is nontrivial, there exists some $\gamma_j \in \Gamma$ such that $\gamma_j - 1$ is invertible on M_j. Then there exists $m_j \in M_j$ such that $c_j(\gamma_j) = (\gamma_j - 1)m_j$. A simple calculation shows that $c_j(\sigma) = (\sigma - 1)m_j$ for all $\sigma \in \Gamma$. Replacing e by $e - \sum_j m_j$, we may assume that $c = c'$. Then $c(\varphi_q) = 0$ and $c|_\Gamma$ is a homomorphism from Γ to $H^0(D)$. □

Corollary 4.4. *(=Theorem 0.2)* $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D)$ *is of codimension* $([F : \mathbb{Q}_p] - 1)\dim_L H^0(D)$ *in* $\mathrm{Ext}(\mathscr{R}_L, D)$. *In particular, if* $H^0(D) = 0$, *then* $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D) = \mathrm{Ext}(\mathscr{R}_L, D)$; *in other words, all extensions of* $\mathscr{R}_L$ *by* D *are* $\mathcal{O}_F$*-analytic.*

Proof. This follows from Theorem 4.3 and the equalities $\dim_L \mathrm{Hom}(\Gamma, H^0(D)) = [F : \mathbb{Q}_p]\dim_L H^0(D)$ and $\dim_L \mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D)) = \dim_L H^0(D)$. □

5 Computation of $H^1_{\mathrm{an}}(\delta)$ and $H^1(\delta)$

In the case of $F = \mathbb{Q}_p$, Colmez [9] computed H^1 for not necessarily étale (φ, Γ)-modules of rank 1 over the Robba ring. In this case, Liu [20] computed H^2 for this kind of (φ, Γ)-modules, and used it and Colmez's result to build analogues, for not necessarily étale (φ, Γ)-modules over the Robba ring, of the Euler-Poincaré characteristic formula and Tate local duality. Later, Chenevier [5] obtained the Euler-Poincaré characteristic formula for families of trianguline (φ, Γ)-modules and some related results.

In this section we compute $H^1_{\mathrm{an}}(\delta) = H^1_{\mathrm{an}}(\mathscr{R}_L(\delta))$ (for $\delta \in \mathscr{I}_{\mathrm{an}}(L)$) and $H^1(\delta) = H^1(\mathscr{R}_L(\delta))$ (for $\delta \in \mathscr{I}(L)$) following Colmez's approach. In Sections 5.2 and 5.5 we assume that δ is in $\mathscr{I}(L)$, and in Sections 5.3, 5.4 and 5.6 we assume that δ is in $\mathscr{I}_{\mathrm{an}}(L)$.

5.1 Preliminary lemmas

Lemma 5.1. (a) *If* $\alpha \in L^\times$ *is not of the form* π^{-i}, $i \in \mathbb{N}$, *then* $\alpha\varphi_q - 1 : \mathscr{R}^+_L \to \mathscr{R}^+_L$ *is an isomorphism.*

(b) *If* $\alpha = \pi^{-i}$ *with* $i \in \mathbb{N}$, *then the kernel of* $\alpha\varphi_q - 1 : \mathscr{R}^+_L \to \mathscr{R}^+_L$ *is* $L \cdot t^i_{\mathcal{F}}$, *and* $a \in \mathscr{R}^+_L$ *is in the image of* $\alpha\varphi_q - 1$ *if and only if* $\partial^i a|_{u_{\mathcal{F}}=0} = 0$. *Further,* $\alpha\varphi_q - 1$ *is bijective on the subset* $\{a \in \mathscr{R}^+_L : \partial^i a|_{u_{\mathcal{F}}=0} = 0\}$.

Proof. The argument is similar to the proof of [9, Lemma A.1]. If $k > -v_\pi(\alpha)$, then $-\sum_{n=0}^{+\infty}(\alpha\varphi_q)^n$ is the continuous inverse of $\alpha\varphi_q - 1$ on $u^k_{\mathcal{F}}\mathscr{R}^+_L$. The assertions follows from the fact that $\mathscr{R}^+_L = \oplus_{i=0}^{k-1} L \cdot t^i_{\mathcal{F}} \oplus u^k_{\mathcal{F}}\mathscr{R}^+_L$ and the formula $\varphi_q(t^i_{\mathcal{F}}) = \pi^i t^i_{\mathcal{F}}$. We just need to remark that $\partial^i a|_{u_{\mathcal{F}}=0} = 0$ if and only if a is in $\oplus_{j=0}^{i-1} L t^j_{\mathcal{F}} \oplus u^{i+1}_{\mathcal{F}}\mathscr{R}^+_L$. □

Lemma 5.2. *If* $\alpha \in L$ *satisfies* $v_\pi(\alpha) < 1 - v_\pi(q)$, *then for any* $b \in \mathscr{E}^\dagger_L$ *there exists* $c \in \mathscr{E}^\dagger_L$ *such that* $b' = b - (\alpha\varphi_q - 1)c$ *is in* $(\mathscr{E}^\dagger_L)^{\psi=0}$.

Proof. By Proposition 2.4 (d), $c = \sum_{k=1}^{+\infty} \alpha^{-k}\psi^k(b)$ is convergent in $\mathscr{E}^\dagger_L$. It is easy to check that $\alpha c - \psi(c) = \psi(b)$, which proves the lemma. □

Corollary 5.3. *If* $\alpha \in L$ *satisfies* $v_\pi(\alpha) < 1 - v_\pi(q)$, *then for any* $b \in \mathscr{R}_L$ *there exists* $c \in \mathscr{R}_L$ *such that* $b' = b - (\alpha\varphi_q - 1)c$ *is in* $(\mathscr{E}^\dagger_L)^{\psi=0}$.

Proof. Let k be an integer $> -v_\pi(\alpha)$. By Lemma 5.1, there exists $c_1 \in \mathscr{R}_L$ such that $b - (\alpha\varphi_q - 1)c_1$ is of the form $\sum_{i<k} a_i u^i_{\mathcal{F}}$ and thus is in $\mathscr{E}^\dagger_L$. Then we apply Lemma 5.2. □

Lemma 5.4. *If* $\alpha \in L$ *satisfies* $v_\pi(\alpha) < 1 - v_\pi(q)$, *and if* $z \in \mathscr{R}_L$ *satisfies* $\psi(z) - \alpha z \in \mathscr{R}^+_L$, *then* $z \in \mathscr{R}^+_L$.

Proof. Write z in the form $\sum_{k\in\mathbb{Z}} a_k u^k_{\mathcal{F}}$ and put $y = \sum_{k\leq -1} a_k u^k_{\mathcal{F}} \in \mathscr{E}^\dagger_L$. If $y \neq 0$, multiplying z by a scalar in L we may suppose that $\inf_{k\leq -1} v_p(a_k) = 0$. Then

$$y - \alpha^{-1}\psi(y) = \alpha^{-1}(\alpha z - \psi(z)) + \sum_{k\geq 0} a_k(\alpha^{-1}\psi(u^k_{\mathcal{F}}) - u^k_{\mathcal{F}})$$

belongs to $\mathcal{O}_{\mathscr{E}_L^\dagger} \cap \mathscr{R}_L^+ = \mathcal{O}_L[[u_{\mathcal{F}}]]$. But this is a contradiction since $y - \alpha^{-1}\psi(y) \equiv y \mod \pi$. Hence $y = 0$. ☐

Corollary 5.5. *If $\alpha \in L$ satisfies $v_\pi(\alpha) < 1 - v_\pi(q)$, and if $z \in \mathscr{R}_L$ satisfies $(\alpha\varphi_q - 1)z \in \mathscr{R}_L^{\psi=0}$, then z is in $\mathscr{R}_L^+$.*

Proof. We have $\psi(z) - \alpha z = \psi(z - \alpha\varphi_q(z)) = 0$. Then we apply Lemma 5.4. ☐

5.2 Computation of $H^0(\delta)$

Recall that, if $\delta \in \mathscr{I}_{\mathrm{an}}(L)$, then $H^0_{\mathrm{an}}(\delta) = H^0(\delta)$.

Proposition 5.6. *Let δ be in $\mathscr{I}(L)$.*

(a) *If δ is not of the form x^{-i} with $i \in \mathbb{N}$, then $H^0(\delta) = 0$.*

(b) *If $i \in \mathbb{N}$, then $H^0(x^{-i}) = Lt^i_{\mathcal{F}}$.*

Proof. Observe that $\mathscr{R}_L^-(\delta)^{\varphi_q=1} = (\mathscr{R}_L^-)^{\delta(\pi)\varphi_q=1}\cdot e_\delta = 0$, where $\mathscr{R}_L^-(\delta) = \mathscr{R}_L(\delta)/\mathscr{R}_L^+(\delta)$. Thus $\mathscr{R}_L(\delta)^{\varphi_q=1,\Gamma=1} = \mathscr{R}_L^+(\delta)^{\varphi_q=1,\Gamma=1}$. If $\delta(\pi)$ is not of the form π^{-i} with $i \in \mathbb{N}$, by Lemma 5.1 (a) we have $\mathscr{R}_L^+(\delta)^{\varphi_q=1} = 0$ and so $\mathscr{R}_L^+(\delta)^{\varphi_q=1,\Gamma=1} = 0$. If $\delta(\pi) = \pi^{-i}$, then

$$\mathscr{R}_L^+(\delta)^{\varphi_q=1,\Gamma=1} = (Lt^i_{\mathcal{F}}\cdot e_\delta)^{\Gamma=1} = \begin{cases} Lt^i_{\mathcal{F}}\cdot e_\delta & \text{if } \delta = x^{-i}, \\ 0 & \text{otherwise,} \end{cases}$$

as desired. ☐

Corollary 5.7. *If δ_1 and δ_2 are two different characters in $\mathscr{I}(L)$, then $\mathscr{R}_L(\delta_1)$ is not isomorphic to $\mathscr{R}_L(\delta_2)$.*

Proof. We only need to show that $\mathscr{R}_L(\delta_1\delta_2^{-1})$ is not isomorphic to $\mathscr{R}_L$. By Proposition 5.6, $\mathscr{R}_L(\delta_1\delta_2^{-1})$ is not generated by $H^0(\delta_1\delta_2^{-1})$, but $\mathscr{R}_L$ is generated by $H^0(1)$. Thus $\mathscr{R}_L(\delta_1\delta_2^{-1})$ is not isomorphic to $\mathscr{R}_L$. ☐

5.3 Computation of $H^1_{\mathrm{an}}(\delta)$ for $\delta \in \mathscr{I}_{\mathrm{an}}(L)$ with $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$

Until the end of Section 5 we will write $\mathscr{R}_L(\delta)$ by $\mathscr{R}_L$ with the twisted (φ_q, Γ)-action given by

$$\varphi_{q;\delta}(x) = \delta(\pi)\varphi_q(x), \quad \sigma_{a;\delta}(x) = \delta(a)\sigma_a(x).$$

Recall that $\nabla_\delta = t_{\mathcal{F}}\partial + w_\delta$. Write $\delta(\sigma_a) = \delta(a)$.

Lemma 5.8. *Suppose that $\delta \in \mathscr{I}_{\mathrm{an}}(L)$ satisfies $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$. For any $(a, b) \in Z^1_{\varphi_q,\nabla}(\delta)$, there exists $(m, n) \in Z^1_{\varphi_q,\nabla}(\delta)$ with $m \in (\mathscr{E}_L^\dagger)^{\psi=0}$ and $n \in \mathscr{R}_L^+$ such that $(a, b) \sim (m, n)$.*

Proof. As $v_\pi(\delta(\pi)) < 1-v_\pi(q)$, by Corollary 5.3 there exists $c \in \mathscr{R}_L$ such that $m = a-(\delta(\pi)\varphi_q-1)c$ is in $(\mathscr{E}_L^\dagger)^{\psi=0}$. Put $n = b - \nabla_\delta c$. Then (m, n) is in $Z^1_{\varphi_q,\nabla}(\delta)$ and $(m, n) \sim (a, b)$. As $(\delta(\pi)\varphi_q - 1)n = \nabla_\delta m = t_{\mathcal{F}}\partial m + w_\delta m$ is in $\mathscr{R}_L^{\psi=0}$, by Corollary 5.5, n is in $\mathscr{R}_L^+$. ☐

Lemma 5.9. *Suppose that $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$ and δ is not of the form x^{-i}. Let (m, n) be in $Z^1_{\varphi_q,\nabla}(\delta)$ with $m \in (\mathscr{E}_L^\dagger)^{\psi=0}$ and $n \in \mathscr{R}_L^+$. Then (m, n) is in $B^1(\delta)$ if and only if*

- *$m \in (\mathscr{E}_L^+)^{\psi=0}$ when $\delta(\pi)$ is not of the form π^{-i}, $i \in \mathbb{N}$;*
- *$m \in (\mathscr{E}_L^+)^{\psi=0}$ and $\partial^i m|_{u_{\mathcal{F}}=0} = 0$ when $\delta(\pi) = \pi^{-i}$ and $w_\delta \neq -i$ for some $i \in \mathbb{N}$.*
- *$m \in (\mathscr{E}_L^+)^{\psi=0}$ and $\partial^i m|_{u_{\mathcal{F}}=0} = \partial^i n|_{u_{\mathcal{F}}=0} = 0$ when $\delta(\pi) = \pi^{-i}$ and $w_\delta = -i$ for some $i \in \mathbb{N}$.*

Proof. We only prove the assertion for the case that $\delta(\pi)=\pi^{-i}$ and $w_\delta\neq -i$ for some $i\in\mathbb{N}$. The arguments for the other two cases are similar.

If (m,n) is in $B^1(\delta)$, then there exists $z\in\mathscr{R}_L$ such that $(\delta(\pi)\varphi_q-1)z=m$ and $\nabla_\delta z=n$. Since m is in $\mathscr{R}_L^{\psi=0}$, by Corollary 5.5 we have $z\in\mathscr{R}_L^+$. It follows that m is in $\mathscr{R}_L^+\cap\mathscr{E}_L^\dagger=\mathscr{E}_L^+$. By Lemma 5.1 (b), we have $\partial^i m|_{u_\mathcal{F}=0}=0$.

Now we assume that m is in $\mathscr{E}_L^+$ and $\partial^i m|_{u_\mathcal{F}=0}=0$. By Lemma 5.1 (b), there exists $z\in\mathscr{R}_L^+$ with $\partial^i z|_{u_\mathcal{F}=0}=0$ such that $(\delta(\pi)\varphi_q-1)z=m$. Then $(\delta(\pi)\varphi_q-1)(\nabla_\delta z-n)=\nabla_\delta(\delta(\pi)\varphi_q-1)z-(\delta(\pi)\varphi_q-1)n=\nabla_\delta m-(\delta(\pi)\varphi_q-1)n=0$. Again by Lemma 5.1 (b), we have $\nabla_\delta z-n=c\,t_\mathcal{F}^i$ for some $c\in L$. Put $z'=z-\frac{c\,t_\mathcal{F}^i}{w_\delta+i}$. Then $(\delta(\pi)\varphi_q-1)z'=m$ and $\nabla_\delta z'=n$. Hence (m,n) is in $B^1(\delta)$. ◻

Recall that $S_\delta=\mathscr{R}_L^-(\delta)^{\Gamma=1,\psi=0}$.

Proposition 5.10. *Suppose that $v_\pi(\delta(\pi))<1-v_\pi(q)$.*

(a) *If δ is not of the form x^{-i}, then $H^1_{\mathrm{an}}(\delta)$ is isomorphic to the L-vector space S_δ and is 1-dimensional.*

(b) *If $\delta=x^{-i}$, then $H^1_{\mathrm{an}}(\delta)$ is 2-dimensional over L and is generated by the images of $(t_\mathcal{F}^i,0)$ and $(0,t_\mathcal{F}^i)$.*

Proof. For (a) we only consider the case that $\delta(\pi)=\pi^{-i}$ and $w_\delta=-i$ for some $i\in\mathbb{N}$. The arguments for the other cases are similar. As $\delta\neq x^{-i}$, there exists an element $\gamma_1\in\Gamma$ of infinite order such that $\delta(\gamma_1)\neq\chi_\mathcal{F}(\gamma_1)^{-i}$.

We give two useful facts: for any $z\in\mathscr{R}_L^+$, $\partial^i z|_{u_\mathcal{F}=0}=0$ if and only if $\partial^i(\delta(\gamma_1)\gamma_1-1)z|_{u_\mathcal{F}=0}=0$; if $\partial^i z|_{u_\mathcal{F}=0}=0$, then $\partial^i(\delta(\gamma)\gamma-1)z|_{u_\mathcal{F}=0}=0$ for any $\gamma\in\Gamma$. Both of these two facts follow from Lemma 5.1 (b). We will use them freely below.

Let (m,n) be in $Z^1(\delta)$ with $m\in(\mathscr{E}_L^\dagger)^{\psi=0}$ and $n\in\mathscr{R}_L^+$. For any $\gamma\in\Gamma$, since $\gamma(m,n)-(m,n)\in B^1(\delta)$, by Lemma 5.9, $(\delta(\gamma)\gamma-1)m$ is in $\mathscr{R}_L^+$, i.e. the image of m in $\mathscr{R}_L^-(\delta)$ belongs to S_δ.

We will show that, for any $\bar{m}\in S_\delta$, there exists a lifting $m\in(\mathscr{E}_L^\dagger)^{\psi=0}$ of $\bar{m}$ such that $\partial^i(\delta(\gamma)\gamma-1)m|_{u_\mathcal{F}=0}=0$ for all $\gamma\in\Gamma$. Let $m'\in(\mathscr{E}_L^\dagger)^{\psi=0}$ be an arbitrary lifting of $\bar{m}$. Assume that $\partial^i(\delta(\gamma_1)\gamma_1-1)m'|_{u_\mathcal{F}=0}=c$. Put $m=m'-\frac{1}{i!}\frac{c\,t_\mathcal{F}^i}{\delta(\gamma_1)\chi_\mathcal{F}(\gamma_1)^i-1}$. Then $\partial^i(\delta(\gamma_1)\gamma_1-1)m|_{u_\mathcal{F}=0}=0$ and thus $\partial^i\nabla_\delta m|_{u_\mathcal{F}=0}=0$. Hence, by Lemma 5.1 (b) there exists $n\in\mathscr{R}_L^+$ with $\partial^i n|_{u_\mathcal{F}=0}=0$ such that $(\delta(\pi)\varphi_q-1)n=\nabla_\delta m$. This means that $(m,n)\in Z^1_{\varphi_q,\nabla}(\delta)$. For any $\gamma\in\Gamma$, as $\partial^i(\delta(\gamma_1)\gamma_1-1)(\delta(\gamma)\gamma-1)m|_{u_\mathcal{F}=0}=\partial^i(\delta(\gamma)\gamma-1)(\delta(\gamma_1)\gamma_1-1)m|_{u_\mathcal{F}=0}=0$, we have $\partial^i(\delta(\gamma)\gamma-1)m|_{u_\mathcal{F}=0}=0$. In a word, for any $\gamma\in\Gamma$, $(\delta(\gamma)\gamma-1)m$ is in $\mathscr{R}_L^+$ and $\partial^i(\delta(\gamma)\gamma-1)m|_{u_\mathcal{F}=0}=\partial^i(\delta(\gamma)\gamma-1)n|_{u_\mathcal{F}=0}=0$. This means that $\gamma(m,n)-(m,n)$ is in $B^1(\delta)$ for any $\gamma\in\Gamma$. In other words, (m,n) is in $Z^1(\delta)$.

Now let (m_1,n_1) and (m_2,n_2) be two elements of $Z^1(\delta)$ with $m_1,m_2\in(\mathscr{E}_L^\dagger)^{\psi=0}$ and $n_1,n_2\in\mathscr{R}_L^+$. By Lemma 5.9,

$$\partial^i(\delta(\gamma_1)\gamma_1-1)m_1|_{u_\mathcal{F}=0}=\partial^i(\delta(\gamma_1)\gamma_1-1)m_2|_{u_\mathcal{F}=0}=\partial^i(\delta(\gamma_1)\gamma_1-1)n_1|_{u_\mathcal{F}=0}=\partial^i(\delta(\gamma_1)\gamma_1-1)n_2|_{u_\mathcal{F}=0}=0.$$

Suppose that the image of m_1 in S_δ coincides with that of m_2, which implies that $m_1-m_2\in\mathscr{E}_L^+$. From

$$\partial^i(\delta(\gamma_1)\gamma_1-1)(m_1-m_2)|_{u_\mathcal{F}=0}=\partial^i(\delta(\gamma_1)\gamma_1-1)(n_1-n_2)|_{u_\mathcal{F}=0}=0$$

we obtain $\partial^i(m_1-m_2)|_{u_\mathcal{F}=0}=\partial^i(n_1-n_2)|_{u_\mathcal{F}=0}=0$. This means that $(m_1,n_1)\sim(m_2,n_2)$.

Combining all of the above discussions, we obtain an isomorphism $S_\delta\xrightarrow{\sim}H^1_{\mathrm{an}}(\delta)$. Then by Proposition 3.20, $\dim_L H^1_{\mathrm{an}}(\delta)=\dim_L S_\delta=1$.

Next we prove (b). Again let (m,n) be in $Z^1(\delta)$ with $m\in(\mathscr{E}_L^\dagger)^{\psi=0}$ and $n\in\mathscr{R}_L^+$. Then the image of m in $\mathscr{R}_L^-(\delta)$, denoted by $\bar{m}$, is in S_δ. We show that m in fact belongs to $(\mathscr{R}_L^+)^{\psi=0}$, i.e. $\bar{m}=0$. By Corollary 3.15, $\partial^i:S_\delta\to S_1$ is an isomorphism. So we only need to prove that the image of $\partial^i m$ in S_1 is zero. By Remark 3.19, it suffices to show that $\nabla\partial^i m|_{u_\mathcal{F}=0}=0$. But $\nabla\partial^i m=\partial^i\nabla_\delta m$. Since $\nabla_\delta m=(\delta(\pi)\varphi_q-1)n$, by Lemma 5.1 (b) we have $\partial^i\nabla_\delta m|_{u_\mathcal{F}=0}=0$.

Write $m=at_\mathcal{F}^i+m'$ with $a\in L$ and $m'\in\mathscr{R}_L^+$ satisfying $\partial^i m'|_{u_\mathcal{F}=0}=0$. By Lemma 5.1 (b) there exists $z\in\mathscr{R}_L^+$ such that $(\delta(\pi)\varphi_q-1)z=m'$. Then $(m,n)\sim(at_\mathcal{F}^i,n-\nabla_\delta z)$. Thus we may suppose that $m=at_\mathcal{F}^i$. Then $(\delta(\pi)\varphi_q-1)n=\nabla_\delta(at_\mathcal{F}^i)=0$. So, by Lemma 5.1 (b), we have $n=bt_\mathcal{F}^i$ for some $b\in L$.

Suppose $(at^i_{\mathcal{F}}, bt^i_{\mathcal{F}})$ is in $B^1(\delta)$. Then there exists $z \in \mathscr{R}_L$ such that $(\delta(\pi)\varphi_q - 1)z = at^i_{\mathcal{F}}$ and $\nabla_\delta z = bt^i_{\mathcal{F}}$. So $\psi(z) - \delta(\pi)z = \psi((1-\delta(\pi)\varphi_q)z) = \psi(-at^i_{\mathcal{F}}) \in \mathscr{R}^+_L$. By Lemma 5.4 we get $z \in \mathscr{R}^+_L$. By Lemma 5.1 (b) again we have $a = 0$ and $z \in Lt^i_{\mathcal{F}}$. Then $bt^i_{\mathcal{F}} = \nabla_\delta z = 0$. □

5.4 $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to H^1_{\varphi_q,\nabla}(\delta)$ and $\partial : H^1_{\mathrm{an}}(x^{-1}\delta) \to H^1_{\mathrm{an}}(\delta)$

Observe that, if (m, n) is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ (resp. $B^1(x^{-1}\delta)$), then $(\partial m, \partial n)$ is in $Z^1_{\varphi_q,\nabla}(\delta)$ (resp. $B^1(\delta)$). Thus we have a map $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to H^1_{\varphi_q,\nabla}(\delta)$. Further, the map is Γ-equivariant and it induces a map $\partial : H^1_{\mathrm{an}}(x^{-1}\delta) \to H^1_{\mathrm{an}}(\delta)$.

Put $\bar{Z}^1_{\varphi_q,\nabla}(\delta) := \{(m, n) \in Z^1_{\varphi_q,\nabla}(\delta) : \mathrm{Res}(m) = \mathrm{Res}(n) = 0\}$ and $\bar{B}^1(\delta) := \{(m, n) \in B^1(\delta) : \mathrm{Res}(m) = \mathrm{Res}(n) = 0\}$. Then $\bar{H}^1_{\varphi_q,\nabla}(\delta) := \bar{Z}^1_{\varphi_q,\nabla}(\delta)/\bar{B}^1_{\varphi_q,\nabla}(\delta)$ is a subspace of $H^1_{\varphi_q,\nabla}(\delta)$.

Lemma 5.11. *If $\delta(\pi) \neq \pi/q$ or $w_\delta \neq 1$, then for any $(m, n) \in Z^1_{\varphi_q,\nabla}(\delta)$, there exists $(m_1, n_1) \in \bar{Z}^1_{\varphi_q,\nabla}(\delta)$ such that $(m, n) \sim (m_1, n_1)$, and so $H^1_{\varphi_q,\nabla}(\delta) = \bar{H}^1_{\varphi_q,\nabla}(\delta)$.*

Proof. Let (m, n) be in $Z^1_{\varphi_q,\nabla}(\delta)$. Then $\nabla_\delta m = (\delta(\pi)\varphi_q - 1)n$. If $\delta(\pi) \neq \frac{\pi}{q}$, by Proposition 2.13 and the definition of Res we have

$$\mathrm{Res}\Big(m - (\delta(\pi)\varphi_q - 1)\big(\mathrm{Res}(m)\frac{(\frac{\mathrm{d}t_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}})^{-1}}{(\delta(\pi)\frac{q}{\pi} - 1)u_{\mathcal{F}}})\Big) = 0.$$

Replacing (m, n) by

$$\Big(m - (\delta(\pi)\varphi_q - 1)\big(\mathrm{Res}(m)\frac{(\frac{\mathrm{d}t_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}})^{-1}}{(\delta(\pi)\frac{q}{\pi} - 1)u_{\mathcal{F}}}), n - \nabla_\delta\big(\mathrm{Res}(m)\frac{(\frac{\mathrm{d}t_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}})^{-1}}{(\delta(\pi)\frac{q}{\pi} - 1)u_{\mathcal{F}}})\Big),$$

we may assume that $\mathrm{Res}(m) = 0$. Then

$$(\frac{q}{\pi}\delta(\pi) - 1)\mathrm{Res}(n) = \mathrm{Res}((\delta(\pi)\varphi_q - 1)n) = \mathrm{Res}(\nabla_\delta m) = \mathrm{Res}(\partial(t_{\mathcal{F}} m) + (w_\delta - 1)m) = (w_\delta - 1)\mathrm{Res}(m) = 0,$$

and so $\mathrm{Res}(n) = 0$.

The argument for the case of $w_\delta \neq 1$ is similar. □

As $\mathrm{Res} \circ \partial = 0$, the map $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to H^1_{\varphi_q,\nabla}(\delta)$ factors through $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to \bar{H}^1_{\varphi_q,\nabla}(\delta)$.

Lemma 5.12. (a) *If $\delta(\pi) \neq \pi$ or $w_\delta \neq 1$, then $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to \bar{H}^1_{\varphi_q,\nabla}(\delta)$ is surjective.*

(b) *If $\delta(\pi) = \pi$ and $w_\delta = 1$, then we have an exact sequence of Γ-modules*

$$H^1_{\varphi_q,\nabla}(x^{-1}\delta) \xrightarrow{\partial} \bar{H}^1_{\varphi_q,\nabla}(\delta) \longrightarrow L(x^{-1}\delta) \longrightarrow 0.$$

Proof. Let (m, n) be in $\bar{Z}^1_{\varphi_q,\nabla}(\delta)$. Then there exist m' and n' such that $\partial m' = m$ and $\partial n' = n$. Then $\nabla_{x^{-1}\delta} m' - (\pi^{-1}\delta(\pi)\varphi_q - 1)n' = c$ is in L. If $\delta(\pi) \neq \pi$, we replace n' by $n' + \frac{c}{\pi^{-1}\delta(\pi)-1}$. If $w_\delta \neq 1$, we replace m' by $m' - \frac{c}{w_\delta - 1}$. Then (m', n') is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$. This proves (a). When $\delta(\pi) = \pi$ and $w_\delta = 1$, $\nabla m' - (\varphi_q - 1)n'$ does not depend on the choice of m' and n'. This induces a map $\bar{H}^1_{\varphi_q,\nabla}(\delta) \to L$ whose kernel is exactly $\partial H^1_{\varphi_q,\nabla}(x^{-1}\delta)$. We show that $\bar{H}^1_{\varphi_q,\nabla}(\delta) \to L$ is surjective. Put $m' = \log \frac{\varphi_q(u_{\mathcal{F}})}{u^q_{\mathcal{F}}}$. A simple calculation shows that

$$\nabla m' = (\frac{t_{\mathcal{F}} \cdot [\pi]'_{\mathcal{F}}(u_{\mathcal{F}})}{[\pi]_{\mathcal{F}}(u_{\mathcal{F}})} - q\frac{t_{\mathcal{F}}}{u_{\mathcal{F}}})\partial u_{\mathcal{F}} \equiv (1-q) \mod u_{\mathcal{F}}\mathscr{R}^+_L.$$

Thus by Lemma 5.1 (b) there exists $n' \in u_{\mathcal{F}}\mathscr{R}^+_L$ such that $(\varphi_q - 1)n' = \nabla m' - (1-q)$. Put $m = \partial m'$ and $n = \partial n'$. Then (m, n) is in $\bar{Z}^1_{\varphi_q,\nabla}(\delta)$ whose image in L is nonzero. The Γ-action on $\bar{H}^1_{\varphi_q,\nabla}(\delta)$ induces an action on L. From

$$(\delta(a)\sigma_a(m), \delta(a)\sigma_a(n)) = (\partial(a^{-1}\delta(a)\sigma_a(m')), \partial(a^{-1}\delta(a)\sigma_a(n')))$$

and

$$\nabla(a^{-1}\delta(a)\sigma_a(m')) - (\varphi_q - 1)(a^{-1}\delta(a)\sigma_a(n')) = a^{-1}\delta(a)\sigma_a(\nabla m' - (\varphi_q - 1)n') \equiv a^{-1}\delta(a)(1-q) \mod u_{\mathcal{F}}\mathscr{R}_L^+$$

we see that the induced action comes from the character $x^{-1}\delta$. ☐

Sublemma 5.13. *Let a, b be in L. If (a,b) is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ but not in $B^1(x^{-1}\delta)$, then $\delta(\pi) = \pi$ and $w_\delta = 1$.*

Proof. If $\delta(\pi) \neq \pi$, then $(a,b) \sim (0, b - \frac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi)-1}a)$. So

$$(\pi^{-1}\delta(\pi) - 1)(b - \frac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi)-1}a) = (\pi^{-1}\delta(\pi)\varphi_q - 1)(b - \frac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi)-1}a) = 0.$$

As $\delta(\pi) \neq \pi$, we have $b - \frac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi)-1}a = 0$. Similarly, if $w_\delta \neq 1$, then $(a,b) \in Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ if and only if $(a,b) \sim (0,0)$. ☐

Recall that δ_{unr} is the character of $F^\times$ such that $\delta_{\text{unr}}(\pi) = q^{-1}$ and $\delta_{\text{unr}}|_{\mathcal{O}_F^\times} = 1$.

Sublemma 5.14. *$(\frac{1}{q}\log\frac{\varphi_q(u_{\mathcal{F}})}{u_{\mathcal{F}}^q}, \frac{t_{\mathcal{F}}\partial u_{\mathcal{F}}}{u_{\mathcal{F}}})$ induces a nonzero element of $H^1_{\text{an}}(\delta_{\text{unr}})$.*

Proof. Write $(m,n) = (\frac{1}{q}\log\frac{\varphi_q(u_{\mathcal{F}})}{u_{\mathcal{F}}^q}, \frac{t_{\mathcal{F}}\partial u_{\mathcal{F}}}{u_{\mathcal{F}}})$. Note that $m = (\delta_{\text{unr}}(\pi)\varphi_q - 1)\log u_{\mathcal{F}}$ and $n = \nabla \log u_{\mathcal{F}}$. Thus (m,n) is in $Z^1_{\varphi_q,\nabla}(\delta_{\text{unr}})$. For any $\gamma \in \Gamma$ we have $\gamma(m,n) \sim (m,n)$. Indeed, $\gamma(m,n) - (m,n) = ((\delta_{\text{unr}}(\pi)\varphi_q - 1)\log\frac{\gamma(u_{\mathcal{F}})}{u_{\mathcal{F}}}, \nabla\log\frac{\gamma(u_{\mathcal{F}})}{u_{\mathcal{F}}})$. So (m,n) is in $Z^1(\delta_{\text{unr}})$. We show that (m,n) is not in $B^1(\delta_{\text{unr}})$. Otherwise there exists $z \in \mathscr{R}_L$ such that $m = (\delta_{\text{unr}}(\pi)\varphi_q - 1)z$ and $n = \nabla z$. This will implies that $\nabla(\log u_{\mathcal{F}} - z) = 0$ or equivalently $\log u_{\mathcal{F}} - z$ is in L, a contradiction. ☐

Corollary 5.15. *If $\delta(\pi) = \pi/q$ and $w_\delta = 1$, then $(\frac{1}{q}\log\frac{\varphi_q(u_{\mathcal{F}})}{u_{\mathcal{F}}^q}, \frac{t_{\mathcal{F}}\partial u_{\mathcal{F}}}{u_{\mathcal{F}}})$ is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ but not in $B^1(x^{-1}\delta)$.*

Lemma 5.16. (a) *If $\delta(\pi) \neq \pi, \pi/q$ or if $w_\delta \neq 1$, then $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to \bar{H}^1_{\varphi_q,\nabla}(\delta)$ is injective.*

(b) *If $\delta(\pi) = \pi$ and $w_\delta = 1$, then we have an exact sequence of Γ-modules*

$$0 \longrightarrow L(x^{-1}\delta) \oplus L(x^{-1}\delta) \longrightarrow H^1_{\varphi_q,\nabla}(x^{-1}\delta) \xrightarrow{\partial} \bar{H}^1_{\varphi_q,\nabla}(\delta).$$

(c) *If If $\delta(\pi) = \pi/q$ and $w_\delta = 1$, then we have an exact sequence of Γ-modules*

$$0 \longrightarrow L(x^{-1}\delta) \longrightarrow H^1_{\varphi_q,\nabla}(x^{-1}\delta) \xrightarrow{\partial} \bar{H}^1_{\varphi_q,\nabla}(\delta).$$

Proof. Let (m,n) be in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$, and suppose that $(\partial m, \partial n) \in \bar{B}^1(\delta)$. Let z be an element of $\mathscr{R}_L$ such that $(\delta(\pi)\varphi_q - 1)z = \partial m$ and $\nabla_\delta z = \partial n$. If $\text{Res}(z) = 0$, then there exists $z' \in \mathscr{R}_L$ such that $\partial z' = z$. Then $m - (\delta(\pi)\pi^{-1}\varphi_q - 1)z'$ and $n - \nabla_{x^{-1}\delta}z'$ are in $\{(a,b) : a, b \in L\}$, i.e. (m,n) is in $B^1(x^{-1}\delta) \oplus L(0,1) \oplus L(1,0)$.

If either $\delta(\pi) \neq \frac{\pi}{q}$ or $w_\delta \neq 1$, we always have $\text{Res}(z) = 0$. Indeed, this follows from

$$(\delta(\pi)\frac{q}{\pi} - 1)\text{Res}(z) = \text{Res}((\delta(\pi)\varphi_q - 1)z) = \text{Res}(\partial m) = 0$$

and

$$(w_\delta - 1)\text{Res}(z) = \text{Res}(\partial(t_{\mathcal{F}}z) + (w_\delta - 1)z) = \text{Res}(\nabla_\delta z) = \text{Res}(\partial n) = 0.$$

In the case of $\delta(\pi) = \frac{\pi}{q}$ and $w_\delta = 1$, if $z \in L\frac{\partial u_{\mathcal{F}}}{u_{\mathcal{F}}}$, then (m,n) is in $L(0,1) \oplus L(1,0) \oplus L(\frac{1}{q}\log\frac{\varphi_q(u_{\mathcal{F}})}{u_{\mathcal{F}}^q}, \frac{t_{\mathcal{F}}\partial u_{\mathcal{F}}}{u_{\mathcal{F}}})$. Now our lemma follows from Sublemma 5.13 and Corollary 5.15. ☐

Proposition 5.17. (a) *If $\delta(\pi) \neq \pi, \pi/q$ or if $w_\delta \neq 1$, then $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to H^1_{\varphi_q,\nabla}(\delta)$ is an isomorphism of Γ-modules.*

(b) *If $\delta(\pi) = \pi$ and $w_\delta = 1$, then we have an exact sequence of Γ-modules*

$$0 \longrightarrow L(x^{-1}\delta) \oplus L(x^{-1}\delta) \longrightarrow H^1_{\varphi_q,\nabla}(x^{-1}\delta) \xrightarrow{\partial} H^1_{\varphi_q,\nabla}(\delta) \longrightarrow L(x^{-1}\delta) \longrightarrow 0.$$

(c) *If $\delta(\pi) = \pi/q$ and $w_\delta = 1$, then we have an exact sequence of Γ-modules*

$$0 \longrightarrow L(x^{-1}\delta) \longrightarrow H^1_{\varphi_q,\nabla}(x^{-1}\delta) \xrightarrow{\partial} H^1_{\varphi_q,\nabla}(\delta) \longrightarrow L(x^{-1}\delta) \oplus L(x^{-1}\delta) \longrightarrow 0.$$

Proof. Assertions (a) and (b) follow from Lemma 5.11, Lemma 5.12 and Lemma 5.16. Based on these lemmas, for (c) we only need to show that, we have an exact sequence of Γ-modules

$$0 \longrightarrow \bar{H}^1_{\varphi_q,\nabla}(\delta) \longrightarrow H^1_{\varphi_q,\nabla}(\delta) \xrightarrow{\mathrm{Res}} L(x^{-1}\delta) \oplus L(x^{-1}\delta) \longrightarrow 0,$$

where Res is induced by $(m,n) \mapsto (\mathrm{Res}(m), \mathrm{Res}(n))$ which is Γ-equivariant by Proposition 2.13. Here we prove this under the assumption that q is not a power of π. We will see in Section 5.6 that it also holds without this assumption. Put $m_1 = 1/u_{\mathcal{F}}$. Then $\nabla_\delta m_1 = t_{\mathcal{F}}\partial(1/u_{\mathcal{F}}) + 1/u_{\mathcal{F}} = \partial(t_{\mathcal{F}}/u_{\mathcal{F}})$ is in $\mathscr{R}^+_L$. As q is not a power of π, the map $\frac{\pi}{q}\varphi_q - 1 : \mathscr{R}^+_L \to \mathscr{R}^+_L$ is an isomorphism. Let n_1 be the unique solution of $(\frac{\pi}{q}\varphi_q - 1)n_1 = t_{\mathcal{F}}\partial m_1 + m_1$ in $\mathscr{R}^+_L$. Then $c_1 = (m_1, n_1)$ is in $Z^1_{\varphi_q,\nabla}(\delta)$ and $\mathrm{Res}(m_1, n_1) = (1, 0) \neq 0$. For any $\ell \in \mathbb{N}$ we choose a root ξ_ℓ of $Q_\ell = \varphi_q^{\ell-1}(Q)$. For any $f(u_{\mathcal{F}}) \in \mathscr{R}^+_L$, the value of f at ξ_ℓ is an element $f(\xi_\ell)$ in $L \otimes_F F_\ell$. By (3.4) there exists an element $z \in \mathscr{R}^+_L$ whose value at ξ_ℓ is $1 \otimes \log \xi_\ell$. Put $m_2 = t_{\mathcal{F}}^{-1}(q^{-1}\varphi_q - 1)(\log u_{\mathcal{F}} - z)$ and $n_2 = \partial(\log u_{\mathcal{F}} - z)$. Then (m_2, n_2) is in $Z^1_{\varphi_q,\nabla}(\delta)$ and $\mathrm{Res}(n_2) = 1$. □

Proposition 5.18. (a) *If $\delta \neq x,\ x\delta_{\mathrm{unr}}$, then $\partial : H^1_{\mathrm{an}}(x^{-1}\delta) \to H^1_{\mathrm{an}}(\delta)$ is an isomorphism.*

(b) *If $\delta = x$, then $\partial : H^1_{\mathrm{an}}(x^{-1}\delta) \to H^1_{\mathrm{an}}(\delta)$ is zero, and $\dim_L H^1_{\mathrm{an}}(\delta) = 1$.*

(c) *If $\delta = x\delta_{\mathrm{unr}}$, then $\partial : H^1_{\mathrm{an}}(x^{-1}\delta) \to H^1_{\mathrm{an}}(\delta)$ is zero, and $\dim_L H^1_{\mathrm{an}}(\delta) = 2$.*

Proof. We apply Proposition 5.17. There is nothing to prove for the case that $\delta(\pi) \neq \pi, \pi/q$ or $w_\delta \neq 1$. Combining the assertions in this case and Proposition 5.10 we obtain that $\dim_L H^1_{\mathrm{an}}(\delta_{\mathrm{unr}}) = 1$. This fact is useful below.

Next we consider the case of $\delta(\pi) = \pi/q$ and $w_\delta = 1$. The argument for the case of $\delta(\pi) = \pi$ and $w_\delta = 1$ is similar.

Let M be the image of $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to H^1_{\varphi_q,\nabla}(x)$. Then we have two short exact sequences of Γ-modules

$$0 \longrightarrow L(x^{-1}\delta) \longrightarrow H^1_{\varphi_q,\nabla}(x^{-1}\delta) \xrightarrow{\partial} M \longrightarrow 0$$

and

$$0 \longrightarrow M \longrightarrow H^1_{\varphi_q,\nabla}(\delta) \longrightarrow L(x^{-1}\delta) \oplus L(x^{-1}\delta) \longrightarrow 0.$$

We will show that, taking Γ-invariants yields two exact sequences

$$0 \longrightarrow L(x^{-1}\delta)^\Gamma \longrightarrow H^1_{\mathrm{an}}(x^{-1}\delta) \xrightarrow{\partial} M^\Gamma \longrightarrow 0$$

and

$$0 \longrightarrow M^\Gamma \longrightarrow H^1_{\mathrm{an}}(\delta) \longrightarrow L(x^{-1}\delta)^\Gamma \oplus L(x^{-1}\delta)^\Gamma \longrightarrow 0.$$

If we have that the Γ-actions on $H^1_{\varphi_q,\nabla}(x^{-1}\delta)$ and $H^1_{\varphi_q,\nabla}(\delta)$ are semisimple, then there is nothing to prove. However we will avoid this by an alternative argument. It suffices to prove the surjectivity of $H^1_{\varphi_q,\nabla}(x^{-1}\delta)^\Gamma \to$

M^Γ and that of $H^1_{\varphi_q,\nabla}(\delta)^\Gamma \to L(x^{-1}\delta)^\Gamma \oplus L(x^{-1}\delta)^\Gamma$. The latter follows from the proof of Proposition 5.17. In fact, if $\delta = x\delta_{\rm unr}$, then (m_1, n_1) and (m_2, n_2) constructed there are in $Z^1(\delta)$. Now let c be any element of M^Γ, then the preimage $\partial^{-1}(Lc)$ is two dimensional over L and Γ-invariant. From the definition of $H^1_{\varphi_q,\nabla}$, we obtain that the induced ∇-action on $\partial^{-1}(Lc)$ is zero and thus $\partial^{-1}(Lc)$ is a semisimple Γ-module, as wanted.

If $\delta = x\delta_{\rm unr}$, then $\dim_L L(x^{-1}\delta)^\Gamma = \dim_L H^1_{\rm an}(x^{-1}\delta) = 1$, and so $M^\Gamma = 0$. Thus $\partial : H^1_{\rm an}(x^{-1}\delta) \to H^1_{\rm an}(\delta)$ is zero and $\dim_L H^1_{\rm an}(\delta) = 2$. If $\delta \neq x\delta_{\rm unr}$, then $\partial : H^1_{\rm an}(x^{-1}\delta) \to H^1_{\rm an}(\delta)$ is an isomorphism since both $H^1_{\rm an}(x^{-1}\delta) \to M^\Gamma$ and $M^\Gamma \to H^1_{\rm an}(\delta)$ are isomorphisms. □

5.5 Dimension of $H^1(\delta)$ for $\delta \in \mathscr{I}(L)$

Theorem 5.19. *(= Theorem 0.3) Let δ be in $\mathscr{I}_{\rm an}(L)$.*

(a) *If δ is not of the form x^{-i} with $i \in \mathbb{N}$, or the form $x^i\delta_{\rm unr}$ with $i \in \mathbb{Z}_+$, then $H^1_{\rm an}(\delta)$ and $H^1(\delta)$ are 1-dimensional over L.*

(b) *If $\delta = x^i\delta_{\rm unr}$ with $i \in \mathbb{Z}_+$, then $H^1_{\rm an}(\delta)$ and $H^1(\delta)$ are 2-dimensional over L.*

(c) *If $\delta = x^{-i}$ with $i \in \mathbb{N}$, then $H^1_{\rm an}(\delta)$ is 2-dimensional over L and $H^1(\delta)$ is $(d+1)$-dimensional over L, where $d = [F : \mathbb{Q}_p]$.*

Proof. The assertions for $H^1_{\rm an}(\delta)$ follow from Proposition 5.10 and Proposition 5.18. By Proposition 5.6 we have

$$\dim_L \mathscr{R}_L(\delta)^{\varphi_q=1,\Gamma=1} = \begin{cases} 1 & \text{if } \delta = x^{-i} \text{ with } i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

So the assertions for $H^1(\delta)$ come from the assertions for $H^1_{\rm an}(\delta)$ and Corollary 4.4. □

When $\delta = x^{-i}$ with $i \in \mathbb{N}$, $H^1_{\rm an}(\delta)$ is generated by the classes of $(t^i_{\mathcal{F}}, 0)$ and $(0, t^i_{\mathcal{F}})$. Let ρ_i $(i = 1, \cdots, d)$ be a basis of $\mathrm{Hom}(\Gamma, Lt^i_{\mathcal{F}})$. Then the class of the 1-cocycle c_0 with $c_0(\varphi_q) = t^i_{\mathcal{F}}$ and $c_0|\Gamma = 0$, and the classes of 1-cocycles c_i with $c_i(\varphi_q) = 0$ and $c_i|\Gamma = \rho_i$ $(i = 1, \cdots, d)$, form a basis of $H^1(\delta)$.

Theorem 5.20. *(=Theorem 0.4) If $\delta \in \mathscr{I}(L)$ is not locally F-analytic, then $H^1(\delta) = 0$.*

Proof. As the maps $\gamma - 1$, $\gamma \in \Gamma$, are null on $H^1(\delta)$, by definition of H^1, so are the maps $\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta)$, $\beta \in \mathrm{Lie}\Gamma$, and the differences $\beta^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta) - \beta'^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta')$. Note that $\beta^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta) - \beta'^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta')$ are $\mathscr{R}_L$-linear on $\mathscr{R}_L(\delta)$. So $\beta^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta) - \beta'^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta')$ are multiplications by scalars in L, since $\beta^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta)e_\delta - \beta'^{-1}\mathrm{d}\Gamma_{\mathscr{R}_L(\delta)}(\beta')e_\delta$ is in Le_δ. If the intersection of their kernels is null, then the cohomology $H^1(\delta)$ vanishes. Thus, either the intersection of their kernels is 0 and so the cohomology vanishes, or they are all null and δ is of form $x \mapsto x^w$ for x close to 1 with $w = \frac{\log \delta(\beta)}{\log \beta}$ for β close to 1 (i.e. δ is locally F-analytic). □

Remark 5.21. Suppose that $[F : \mathbb{Q}_p] \geq 2$. Let $\delta \neq 1$ be a character of $F^\times$ with $\delta(\pi) \in \mathcal{O}_L^\times$, and let $L(\delta)$ be the L-representation of G_F induced by δ. Suppose that $\delta \neq x^2\delta_{\rm unr}$ when $[F : \mathbb{Q}_p] = 2$. Combining Theorem 5.19 and the Euler-Poincaré characteristic formula [26] we obtain that, there exist Galois representations in $\mathrm{Ext}(L, L(\delta))$ that are not overconvergent. Theorem 5.20 tells us that, if further δ is not locally analytic, then there is no nontrivial overconvergent extension of L by $L(\delta)$.

5.6 The maps $\iota_k : H^1(\delta) \to H^1(x^{-k}\delta)$ and $\iota_{k,\rm an} : H^1_{\rm an}(\delta) \to H^1_{\rm an}(x^{-k}\delta)$

Let k be a positive integer.

Proposition 5.22. *Let δ be in $\mathscr{I}_{\rm an}(L)$.*

(a) *If $w_\delta \notin \{1-k, \cdots, 0\}$, then $H^0_{\rm an}(\mathscr{R}_L(\delta)/t^k_{\mathcal{F}}\mathscr{R}_L(\delta)) = 0$.*

(b) *If $w_\delta \in \{1-k, \cdots, 0\}$, then $H^0_{\rm an}(\mathscr{R}_L(\delta)/t^k_{\mathcal{F}}\mathscr{R}_L(\delta))$ is a 1-dimensional L-vector space.*

Proof. We have $\mathscr{R}_L^+/t_{\mathcal{F}}^k\mathscr{R}_L^+ = \mathscr{R}_L^+/(u_{\mathcal{F}}^k)\times\prod_{n=1}^\infty \mathscr{R}_L^+/(\varphi_q^{n-1}(Q))^k$. As Γ-modules, $\mathscr{R}_L^+/(u_{\mathcal{F}}^k) = \oplus_{i=0}^{k-1} Lt_{\mathcal{F}}^i$ and $\mathscr{R}_L^+/(\varphi_q^n(Q))^k = \bigoplus_{i=0}^{k-1}(L\otimes_F F_n)t_{\mathcal{F}}^i$. Thus as a Γ-module $\mathscr{R}_L^+/t_{\mathcal{F}}^k\mathscr{R}_L^+$ is isomorphic to $\bigoplus_{i=0}^{k-1}(\mathscr{R}_L^+/\mathscr{R}_L^+t_{\mathcal{F}})\otimes_L Lt_{\mathcal{F}}^i$. Note that the natural map $\mathscr{R}_L^+/\mathscr{R}_L^+t_{\mathcal{F}}^k \to \mathscr{R}_L/\mathscr{R}_Lt_{\mathcal{F}}^k$ is surjective. Furthermore, two sequences $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ in $\mathscr{R}_L^+/\mathscr{R}_L^+u_{\mathcal{F}}^k\times\prod_{n=1}^\infty\mathscr{R}_L^+/(\varphi_q^{n-1}(Q))^k$ have the same image in $\mathscr{R}_L/\mathscr{R}_Lt_{\mathcal{F}}^k$, if and only if there exists $N>0$ such that $y_n = z_n$ when $n\geq N$.

Since the action of Γ on $(\mathscr{R}_L^+/t_{\mathcal{F}}\mathscr{R}_L^+)t_{\mathcal{F}}^i$ twisted by the character x^{-i} is smooth, (a) follows.

For (b) we only need to consider the case of $w_\delta = 0$ and $k=1$. The operator φ_q induces an injection $\mathscr{R}_L^+/(\varphi_q^n(Q))\to\mathscr{R}_L^+/(\varphi_q^{n+1}(Q))$ which is denoted by $\varphi_{q,n}$. The action of φ_q on $\mathscr{R}_L/\mathscr{R}_Lt_{\mathcal{F}}$ is given by $\varphi_q(y_n)_n = (\varphi_{q,n}(y_n))_{n+1}$. For any $n\geq 0$, the Γ-action on $L\otimes_F F_n$ factors through Γ/Γ_n, and the resulting Γ/Γ_n-module $L\otimes_F F_n$ is isomorphic to the regular one. Thus for any discrete character δ of Γ, $\dim_L(L\otimes_F F_n)^{\Gamma=\delta^{-1}} = 1$ when n is sufficiently large. Then from the fact that $\varphi_{q,n}$ $(n\geq 1)$ are injective, we obtain $\dim_L\left(\mathscr{R}_L/t_{\mathcal{F}}\mathscr{R}_L\right)^{\Gamma=\delta^{-1},\varphi_q=\delta(\pi)^{-1}} = 1$. □

Corollary 5.23. *Let δ be in $\mathscr{I}_{\mathrm{an}}(L)$.*

(a) *If $w_\delta\notin\{1,\cdots,k\}$, then $H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta)) = 0$.*

(b) *If $w_\delta\in\{1,\cdots,k\}$, then $H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta))$ is a 1-dimensional L-vector space.*

Note that $\mathscr{R}_L(x^{-k}\delta)$ is canonically isomorphic to $t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)$. When $k\geq 1$, the inclusion $\mathscr{R}_L(\delta)\hookrightarrow t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)$ induces maps $\iota_{k,\mathrm{an}}: H^1_{\mathrm{an}}(\delta)\to H^1_{\mathrm{an}}(x^{-k}\delta)$ and $\iota_k: H^1(\delta)\to H^1(x^{-k}\delta)$. If $\gamma\in\Gamma$ is of infinite order, then we have the following commutative diagram

$$\begin{array}{ccc} H^1(\delta) & \xrightarrow{\iota_k} & H^1(x^{-k}\delta) \\ \Big\downarrow{\scriptstyle \Upsilon^\delta_{\mathrm{an},\gamma}\circ\Upsilon^\delta_\gamma} & & \Big\downarrow{\scriptstyle \Upsilon^{x^{-k}\delta}_{\mathrm{an},\gamma}\circ\Upsilon^{x^{-k}\delta}_\gamma} \\ H^1_{\mathrm{an}}(\delta) & \xrightarrow{\iota_{k,\mathrm{an}}} & H^1_{\mathrm{an}}(x^{-k}\delta). \end{array} \tag{5.1}$$

Lemma 5.24. *We have the following exact sequence*

$$0\to H^0_{\mathrm{an}}(\delta)\to H^0_{\mathrm{an}}(x^{-k}\delta)\to H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta))\to H^1_{\mathrm{an}}(\delta)\xrightarrow{\iota_{k,\mathrm{an}}} H^1_{\mathrm{an}}(x^{-k}\delta). \tag{5.2}$$

Proof. From the short exact sequence $0\to\mathscr{R}_L(\delta)\to\mathscr{R}_L(x^{-k}\delta)\to\mathscr{R}_L(x^{-k}\delta)/\mathscr{R}_L(\delta)\to 0$ we deduce an exact sequence

$$0\to H^0_{\varphi_q,\nabla}(\delta)\to H^0_{\varphi_q,\nabla}(x^{-k}\delta)\to H^0_{\varphi_q,\nabla}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta))\to H^1_{\varphi_q,\nabla}(\delta)\to H^1_{\varphi_q,\nabla}(x^{-k}\delta). \tag{5.3}$$

Being finite dimensional $H^0_{\varphi_q,\nabla}(\delta)$ and $H^0_{\varphi_q,\nabla}(x^{-k}\delta)$ are semisimple Γ-modules; since $t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta)$ is a semisimple Γ-module, so is $H^0_{\varphi_q,\nabla}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta))$. Hence, taking Γ-invariants of each term in (5.3), we obtain the desired exact sequence. □

Proposition 5.25. *Let δ be in $\mathscr{I}_{\mathrm{an}}(L)$, $k\in\mathbb{Z}_+$. If $w_\delta\notin\{1,\cdots,k\}$, then $\iota_{k,\mathrm{an}}$ and ι_k are isomorphisms.*

Proof. We only prove the assertion for $\iota_{k,\mathrm{an}}$. The proof of the assertion for ι_k is similar. By Theorem 5.19, $\dim_L H^1_{\mathrm{an}}(\delta) = \dim_L H^1_{\mathrm{an}}(x^{-k}\delta)$ when $w_\delta\notin\{1,\cdots,k\}$. Combining (5.2) with the facts that $H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta)) = 0$ and that $\dim_L H^1_{\mathrm{an}}(\delta) = \dim_L H^1_{\mathrm{an}}(x^{-k}\delta)$, we obtain the assertion. □

We assign to any nonzero $c\in H^1_{\mathrm{an}}(\delta)$ an $\mathscr{L}$-invariant in $\mathrm{P}^1(L) = L\cup\{\infty\}$. In the case of $\delta = x^{-k}$ with $k\in\mathbb{N}$, put $\mathscr{L}((at_{\mathcal{F}}^k, bt_{\mathcal{F}}^k)) = a/b$. If $\delta = x\delta_{\mathrm{unr}}$, then any $c\in H^1_{\mathrm{an}}(\delta)$ can be written as

$$c = t_{\mathcal{F}}^{-1}((q^{-1}\varphi_q - 1)(\lambda G(1,1)+\mu(\log u_{\mathcal{F}} - z)), t_{\mathcal{F}}\partial(\lambda G(1,1)+\mu(\log u_{\mathcal{F}} - z)))$$

with $\lambda,\mu\in L$. Here $G(1,1)$ is an element of $\mathscr{R}_L$ which induces a basis of $(\mathscr{R}_L/\mathscr{R}_Lt_{\mathcal{F}})^\Gamma$ and whose value at ξ_n is $1\otimes 1\in L\otimes_F F_n$ when n is large enough; z is an element of $\mathscr{R}_L$ whose value at ξ_n is $1\otimes\log(\xi_n)\in L\otimes_F F_n$

for any n. We put $\mathscr{L}(c)=-\frac{e_F(q-1)}{q}\cdot\frac{\lambda}{\mu}$. In the case of $\delta=x^k\delta_{\rm unr}$ with $k\geq 2$, for any $c\in H^1_{\rm an}(x^k\delta_{\rm unr})$, put $\mathscr{L}(c)=\mathscr{L}(\iota_{k-1}(c))$. In the case that δ is not of the form x^{-k} with $k\in\mathbb{N}$ or the form $x^k\delta_{\rm unr}$ with $k\in\mathbb{Z}_+$, we put $\mathscr{L}(c)=\infty$.

Proposition 5.26. *Let δ be in $\mathscr{I}_{\rm an}(L)$, $k\in\mathbb{Z}_+$.*

(a) *If $w_\delta\in\{1,\cdots,k\}$ and if $\delta\neq x^{w_\delta}$, $x^{w_\delta}\delta_{\rm unr}$, then $\iota_{k,\rm an}$ and ι_k are zero.*

(b) *If $\delta=x^{w_\delta}\delta_{\rm unr}$ with $1\leq w_\delta\leq k$, then $\iota_{k,\rm an}$ and ι_k are surjective, and the kernel of $\iota_{k,\rm an}$ is the 1-dimensional subspace $\{c\in H^1_{\rm an}(\delta): c=0$ or $\mathscr{L}(c)=\infty\}$.*

(c) *If $\delta=x^{w_\delta}$ with $1\leq w_\delta\leq k$, then $\iota_{k,\rm an}$ and ι_k are injective, and the image of $\iota_{k,\rm an}$ is $\{c\in H^1_{\rm an}(x^{-k}\delta): c=0$ or $\mathscr{L}(c)=\infty\}$.*

Proof. We will use the exact sequence (5.2) frequently without mentioning it.

First we prove (a). From the fact that $\dim_L H^0_{\rm an}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta))=\dim_L H^1_{\rm an}(\delta)=1$ and $H^0_{\rm an}(x^{-k}\delta)=0$, we obtain the assertion for $\iota_{k,\rm an}$. The assertion for ι_k follows from this and the commutative diagram (5.1) where the two vertical maps are isomorphisms.

Next we prove (b). From the fact that

$$H^0_{\rm an}(x^{-k}\delta)=0,\ \ \dim_L H^0_{\rm an}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta))=1,\ \ \dim_L H^1_{\rm an}(\delta)=2\ \text{ and }\ \dim_L H^1_{\rm an}(x^{-k}\delta)=1,$$

we obtain the surjectivity of $\iota_{k,\rm an}$. The surjectivity of ι_k follows from this and the commutative diagram (5.1) where the two vertical maps are isomorphisms. We show that, if $c\in H^1_{\rm an}(\delta)$ satisfies $\mathscr{L}(c)=\infty$, then $\iota_{k,\rm an}(c)=0$. As $\mathscr{L}(\iota_{w_\delta-1,\rm an}(c))=\infty$ and $\iota_{k,\rm an}=\iota_{k+1-w_\delta,\rm an}\iota_{w_\delta-1,\rm an}$, we reduce to the case of $\delta=x\delta_{\rm unr}$. In this case, $c=t_{\mathcal{F}}^{-1}\lambda((q^{-1}\varphi_q-1)G(1,1),\nabla G(1,1))$ with $\lambda\in L$. Thus $\iota_{1,\rm an}(c)=\lambda((q^{-1}\varphi_q-1)G(1,1),\nabla G(1,1))\sim(0,0)$. Hence $\iota_{k,\rm an}(c)=0$ for any integer $k\geq 1$.

Finally we prove (c). From the fact that

$$H^0_{\rm an}(\delta)=0\ \text{ and }\ \dim_L H^0_{\rm an}(x^{-k}\delta)=\dim_L H^0_{\rm an}(t_{\mathcal{F}}^{-k}\mathscr{R}_L(\delta)/\mathscr{R}_L(\delta))=1,$$

we obtain the injectivity of $\iota_{k,\rm an}$. The injectivity of ι_k follows from this and the commutative diagram (5.1) where the vertical map $\Upsilon^\delta_{\rm an,\gamma}\circ\Upsilon^\delta_\gamma$ is an isomorphism. For the second assertion, let (m,n) be in $Z^1(x^{w_\delta})$. Then $\iota_{w_\delta-1}(m,n)=(t_{\mathcal{F}}^{w_\delta-1}m,t_{\mathcal{F}}^{w_\delta-1}n)\in Z^1(x)$. In other words, $\partial(t_{\mathcal{F}}^{w_\delta}m)=\nabla_x(t_{\mathcal{F}}^{w_\delta-1}m)=(\pi\varphi_q-1)(t_{\mathcal{F}}^{w_\delta-1}n)$. Thus $\mathrm{Res}(t_{\mathcal{F}}^{w_\delta-1}n)=0$ and there exists $z\in\mathscr{R}_L$ such that $\partial z=t_{\mathcal{F}}^{w_\delta-1}n$ or equivalently $\nabla z=t_{\mathcal{F}}^{w_\delta}n$. It follows that $\nabla_{x^{w_\delta-k}}(t_{\mathcal{F}}^{k-w_\delta}z)=\big(\nabla+(w_\delta-k)\big)(t_{\mathcal{F}}^{k-w_\delta}z)=t_{\mathcal{F}}^{k-w_\delta}\nabla z=t_{\mathcal{F}}^k n$. Thus $\iota_{k,\rm an}(m,n)=(t_{\mathcal{F}}^k m,t_{\mathcal{F}}^k n)\sim\Big(t_{\mathcal{F}}^k m-(\pi^{w_\delta-k}\varphi_q-1)(t_{\mathcal{F}}^{k-w_\delta}z),0\Big)$. So we have $\iota_{k,\rm an}(m,n)=(at_{\mathcal{F}}^{k-w_\delta},0)$. If $\iota_{k,\rm an}(m,n)\neq 0$ or equivalently $a\neq 0$, then $\mathscr{L}(\iota_{k,\rm an}(m,n))=\infty$. ☐

6 Triangulable (φ_q,Γ)-modules of rank 2

In his paper [9], Colmez classified 2-dimensional trianguline representations of the Galois group $G_{\mathbb{Q}_p}$. Later Nakamura [22] classified 2-dimensional trianguline representations of the Galois group of a p-adic local field that is finite over $\mathbb{Q}_p$, generalizing Colmez's work.

In this section we classify triangulable $\mathcal{O}_F$-analytic (φ_q,Γ)-modules of rank 2 following Colmez's method [9]. First we recall the definition.

Definition 6.1. A (φ_q,Γ)-module over $\mathscr{R}_L$ is called *triangulable* if there exists a filtration of D consisting of (φ_q,Γ)-submodules $0=D_0\subset D_1\subset\cdots\subset D_d=D$ such that D_i/D_{i-1} is free of rank 1 over $\mathscr{R}_L$.

Note that, if D is $\mathcal{O}_F$-analytic, then so is D_i/D_{i-1} for any i.

If $\delta_1,\delta_2\in\mathscr{I}_{\rm an}(L)$, then $\mathrm{Ext}(\mathscr{R}_L(\delta_2),\mathscr{R}_L(\delta_1))$ is isomorphic to $\mathrm{Ext}(\mathscr{R}_L,\mathscr{R}_L(\delta_1\delta_2^{-1}))$, or $H^1(\delta_1\delta_2^{-1})$. The isomorphism only depends on the choices of e_{δ_1}, e_{δ_2} and $e_{\delta_1\delta_2^{-1}}$. Thus it is unique up to a nonzero multiple and induces an isomorphism from $\mathrm{Proj}(\mathrm{Ext}(\mathscr{R}_L(\delta_2),\mathscr{R}_L(\delta_1)))$ to $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$ independent of the choices of e_{δ_1}, e_{δ_2} and $e_{\delta_1\delta_2^{-1}}$. Similarly there is a natural isomorphism from $\mathrm{Proj}(\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L(\delta_2),\mathscr{R}_L(\delta_1)))$ to $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$. Hence the set of triangulable (resp. triangulable and $\mathcal{O}_F$-analytic) (φ_q,Γ)-modules D of rank 2 satisfying the following two properties is classified by $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$ (resp. $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$):

- $\mathscr{R}_L(\delta_1)$ is a saturated (φ_q,Γ)-submodule of D and $\mathscr{R}_L(\delta_2)$ is the quotient module,
- D is not isomorphic to $\mathscr{R}_L(\delta_1)\oplus\mathscr{R}_L(\delta_2)$.

Let $\mathscr{S}^{\mathrm{an}}=\mathscr{S}^{\mathrm{an}}(L)$ be the analytic variety obtained by blowing up $(\delta_1,\delta_2)\in\mathscr{I}_{\mathrm{an}}(L)\times\mathscr{I}_{\mathrm{an}}(L)$ along the subvarieties $\delta_1\delta_2^{-1}=x^i\delta_{\mathrm{unr}}$ for $i\in\mathbb{Z}_+$ and the subvarieties $\delta_1\delta_2^{-1}=x^{-i}$ for $i\in\mathbb{N}$. The fiber over the point (δ_1,δ_2) is isomorphic to $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$. Similarly let $\mathscr{S}=\mathscr{S}(L)$ be the analytic variety over $\mathscr{I}_{\mathrm{an}}(L)\times\mathscr{I}_{\mathrm{an}}(L)$ whose fiber over (δ_1,δ_2) is isomorphic to $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$. The inclusions $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L(\delta_1),\mathscr{R}_L(\delta_2))\hookrightarrow\mathrm{Ext}(\mathscr{R}_L(\delta_1),\mathscr{R}_L(\delta_2))$ for $\delta_1,\delta_2\in\mathscr{I}_{\mathrm{an}}(L)$ induce a natural injective map $\mathscr{S}^{\mathrm{an}}\hookrightarrow\mathscr{S}$. We write points of $\mathscr{S}$ (resp. $\mathscr{S}^{\mathrm{an}}$) in the form (δ_1,δ_2,c) with $c\in\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$ (resp. $c\in\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$). If $(\delta_1,\delta_2,c)\in\mathscr{S}$ is in the image of $\mathscr{S}_{\mathrm{an}}$, for our convenience we use c_{an} to denote the element in $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$ corresponding to c. For $(\delta_1,\delta_2,c)\in\mathscr{S}^{\mathrm{an}}$, since the $\mathscr{L}$-invariant induces an inclusion $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))\hookrightarrow\mathrm{P}^1(L)$, we also use $(\delta_1,\delta_2,\mathscr{L}(c))$ to denote (δ_1,δ_2,c).

If $s\in\mathscr{S}$, we assign to s the invariant $w(s)\in L$ by $w(s)=w_{\delta_1}-w_{\delta_2}$. Let $\mathscr{S}_+$ be the subset of $\mathscr{S}$ consisting of elements $s\in\mathscr{S}$ with

$$v_\pi(\delta_1(\pi))+v_\pi(\delta_2(\pi))=0,\quad v_\pi(\delta_1(\pi))\geq 0.$$

If $s\in\mathscr{S}_+$, we assign to s the invariant $u(s)\in\mathbb{Q}_+$ by

$$u(s)=v_\pi(\delta_1(\pi))=-v_\pi(\delta_2(\pi)).$$

Put $\mathscr{S}_0=\{s\in\mathscr{S}_+\mid u(s)=0\}$ and $\mathscr{S}_*=\{s\in\mathscr{S}_+\mid u(s)>0\}$. Then $\mathscr{S}_+$ is the disjoint union of $\mathscr{S}_0$ and $\mathscr{S}_*$. For $?\in\{+,0,*\}$ we put $\mathscr{S}^{\mathrm{an}}_?=\mathscr{S}^{\mathrm{an}}\cap\mathscr{S}_?$. We decompose the set $\mathscr{S}^{\mathrm{an}}_?$ as $\mathscr{S}^{\mathrm{an}}_?=\mathscr{S}^{\mathrm{ng}}_?\coprod\mathscr{S}^{\mathrm{cris}}_?\coprod\mathscr{S}^{\mathrm{st}}_?\coprod\mathscr{S}^{\mathrm{ord}}_?\coprod\mathscr{S}^{\mathrm{ncl}}_?$, where

$$\begin{aligned}
\mathscr{S}^{\mathrm{ng}}_? &= \{s\in\mathscr{S}_?\mid w(s)\text{ is not an integer }\geq 1\},\\
\mathscr{S}^{\mathrm{cris}}_? &= \{s\in\mathscr{S}_?\mid w(s)\text{ is an integer }\geq 1, u(s)<w(s),\mathscr{L}=\infty\},\\
\mathscr{S}^{\mathrm{st}}_? &= \{s\in\mathscr{S}_?\mid w(s)\text{ is an integer }\geq 1, u(s)<w(s),\mathscr{L}\neq\infty\}\\
\mathscr{S}^{\mathrm{ord}}_? &= \{s\in\mathscr{S}_?\mid w(s)\text{ is an integer }\geq 1, u(s)=w(s)\},\\
\mathscr{S}^{\mathrm{ncl}}_? &= \{s\in\mathscr{S}_?\mid w(s)\text{ is an integer }\geq 1, u(s)>w(s)\}.
\end{aligned}$$

Note that $\mathscr{S}^{\mathrm{ord}}_0$ and $\mathscr{S}^{\mathrm{ncl}}_0$ are empty.

Let D be an extension of $\mathscr{R}_L(\delta_2)$ by $\mathscr{R}_L(\delta_1)$. For any $k\in\mathbb{N}$, the preimage of $t^k_{\mathcal{F}}\mathscr{R}_L(\delta_2)$ is a (φ_q,Γ)-submodule of D, which is denoted by D'. Then D' is an extension of $\mathscr{R}_L(x^k\delta_2)$ by $\mathscr{R}_L(\delta_1)$. If D is $\mathcal{O}_F$-analytic, then so is D'.

Lemma 6.2. (a) *The class of D' in $H^1(\delta_1\delta_2^{-1}x^{-k})$ coincides with $\iota_k(c)$ up to a nonzero multiple, where c is the class of D in $H^1(\delta_1\delta_2^{-1})$.*

(b) *If D is $\mathcal{O}_F$-analytic, the class of D' in $H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}x^{-k})$ coincides with $\iota_{k,\mathrm{an}}(c)$ up to a nonzero multiple, where c is the class of D in $H^1_{\mathrm{an}}(\delta_1\delta_2^{-1})$.*

Proof. We only prove (b). The proof of (a) is similar. Let e be a basis of $\mathscr{R}_L(\delta_2)$ such that $\varphi_q(e)=\delta_2(\pi)e$ and $\sigma_a e=\delta_2(a)e$. Let $\tilde{e}$ be a lifting of e in D. The class of D, or the same, c, coincides with the class of $\Big((\delta_2(\pi)^{-1}\varphi_q-1)\tilde{e},(\nabla-w_{\delta_2})\tilde{e}\Big)$ up to a nonzero multiple. Similarly, up to a nonzero multiple, the class of D' coincides with the class of

$$\Big((\pi^{-k}\delta_2(\pi)^{-1}\varphi_q-1)(t^k_{\mathcal{F}}\tilde{e}),(\nabla-w_{\delta_2}-k)(t^k_{\mathcal{F}}\tilde{e})\Big)=\Big(t^k_{\mathcal{F}}(\delta_2(\pi)^{-1}\varphi_q-1)\tilde{e},t^k_{\mathcal{F}}(\nabla-w_{\delta_2})\tilde{e}\Big)$$

which is exactly $\iota_{k,\mathrm{an}}(c)$. □

Proposition 6.3. *Put $D=D(s)$ with $s=(\delta_1,\delta_2,c)\in\mathscr{S}$. Then the following two conditions are equivalent:*

(a) *$D(s)$ has a (φ_q,Γ)-submodule M of rank 1 such that $M\cap\mathscr{R}_L(\delta_1)=0$;*

(b) *s is in $\mathscr{S}^{\mathrm{an}}$ and satisfies $w(s)\in\mathbb{Z}_+$, $\delta_1\delta_2^{-1}\neq x^{w(s)}$ and $\mathscr{L}(c_{\mathrm{an}})=\infty$.*

Among all such M there exists a unique one, M_{sat}, that is saturated; M_{sat} is isomorphic to $\mathscr{R}_L(x^{w(s)}\delta_2)$. For any M that satisfies Condition (a), there exists some $i\in\mathbb{N}$ such that $M=t_{\mathcal{F}}^i M_{\mathrm{sat}}$.

Proof. Assume that $D(s)$ satisfies Condition (a). Since the intersection of M and $\mathscr{R}_L(\delta_1)$ is zero, the image of M in $\mathscr{R}_L(\delta_2)$ is a nonzero (φ_q,Γ)-submodule of $\mathscr{R}_L(\delta_2)$, and so must be of the form $t_{\mathcal{F}}^k\mathscr{R}_L(\delta_2)$ with $k\in\mathbb{N}$. Since $D(s)$ does not split, we have $k\geq 1$. The preimage of $t_{\mathcal{F}}^k\mathscr{R}_L(\delta_2)$ in D is exactly $M\oplus\mathscr{R}_L(\delta_1)$. Since $M\oplus\mathscr{R}_L(\delta_1)$ splits, by Lemma 6.2 we have $\iota_k(c)=0$. By Proposition 5.26 this happens only if $w(s)\in\{1,\cdots,k\}$ and $\delta_1\delta_2^{-1}\neq x^{w(s)}$. Note that, when $w(s)\in\{1,\cdots,k\}$ and $\delta_1\delta_2^{-1}\neq x^{w(s)}$, $D(s)$ is automatically $\mathcal{O}_F$-analytic. Again by Proposition 5.26 we obtain $\mathscr{L}(c_{\mathrm{an}})=\infty$. This proves (a)$\Leftrightarrow$(b).

If (a) holds, then the preimage of $t_{\mathcal{F}}^{w(s)}\mathscr{R}_L(\delta_2)$ splits as $\mathscr{R}_L(\delta_1)\oplus M_0$, where M_0 is isomorphic to $\mathscr{R}_L(x^{w(s)}\delta_2)$. We show that M_0 is saturated. Note that M_0 is not included in $t_{\mathcal{F}}D(s)$. Otherwise, the preimage of $t_{\mathcal{F}}^{w(s)-1}\mathscr{R}_L(\delta_2)$ will split, which contradicts Proposition 5.26. Let e_1 (resp. e_2, e) be a basis of $\mathscr{R}_L(\delta_1)$ (resp. $\mathscr{R}_L(\delta_2)$, M_0) such that Le_1 (resp. Le_2, Le) is stable under φ_q and Γ. Let $\tilde{e}_2$ be a lifting of e_2. Write $e=ae_1+b\tilde{e}_2$. Then $a\notin t_{\mathcal{F}}\mathscr{R}_L$ and $b\in t_{\mathcal{F}}^{w(s)}\mathscr{R}_L$. Observe that the ideal I generated by a and $t_{\mathcal{F}}^{w(s)}$ satisfies $\varphi_q(I)=I$ and $\gamma(I)=I$ for all $\gamma\in\Gamma$. Thus by Lemma 1.1, $I=\mathscr{R}_L$. It follows that M_0 is saturated. If M is another (φ_q,Γ)-submodule of $D(s)$ such that $M\cap\mathscr{R}_L(\delta_1)=0$, then the image of M in $\mathscr{R}_L(\delta_2)$ is $t_{\mathcal{F}}^k\mathscr{R}_L(\delta_2)$ for some integer $k\geq w(s)$. Then $M\subset\mathscr{R}_L(\delta_1)\oplus M_0$. Since $\delta_1\neq\delta_2 x^{w(s)}$, $\mathscr{R}_L(\delta_1)$ has no nonzero (φ_q,Γ)-submodule isomorphic to $\mathscr{R}_L(x^k\delta_2)$. It follows that $M\subset M_0$ and thus $M=t_{\mathcal{F}}^{k-w(s)}M_0$. ☐

Corollary 6.4. *Let $s=(\delta_1,\delta_2,c)$ be in $\mathscr{S}$. If s is in $\mathscr{S}^{\mathrm{an}}$ and satisfies $w(s)\in\mathbb{Z}_+$, $\delta_1\delta_2^{-1}\neq x^{w(s)}$ and $\mathscr{L}(c_{\mathrm{an}})=\infty$, then $D(s)$ has exactly two saturated (φ_q,Γ)-submodules of $D(s)$ of rank 1, one being $\mathscr{R}_L(\delta_1)$ and the other isomorphic to $\mathscr{R}_L(x^{w(s)}\delta_2)$. Otherwise, $D(s)$ has exactly one saturated (φ_q,Γ)-submodule of rank 1 which is $\mathscr{R}_L(\delta_1)$.*

Corollary 6.5. *Let $s=(\delta_1,\delta_2,c)$ and $s'=(\delta_1',\delta_2',c')$ be in $\mathscr{S}(L)$.*

(a) *If $\delta_1=\delta_1'$, then $D(s)\cong D(s')$ if and only if $s=s'$.*

(b) *If $\delta_1\neq\delta_1'$, then $D(s)\cong D(s')$ if and only if s and s' are in $\mathscr{S}^{\mathrm{an}}$ and satisfy $w(s)\in\mathbb{Z}_+$, $\delta_1'=x^{w(s)}\delta_2$, $\delta_2'=x^{-w(s)}\delta_1$ and $\mathscr{L}(c_{\mathrm{an}})=\mathscr{L}(c'_{\mathrm{an}})=\infty$.*

Proof. Assertion (a) is clear. We prove (b). Since $D(s)\cong D(s')$, there exists a (φ_q,Γ)-submodule M of $D(s)$ such that $M\cong\mathscr{R}_L(\delta_1')$ and $D(s)/M\cong\mathscr{R}_L(\delta_2')$. Since both $\mathscr{R}_L(\delta_1)$ and M are saturated (φ_q,Γ)-submodules of D, $\mathscr{R}_L(\delta_1)\cap M=0$. By Proposition 6.3 we have $w(s)\in\mathbb{Z}_+$, $\delta_1\delta_2^{-1}\neq x^{w(s)}$, $\mathscr{L}(c_{\mathrm{an}})=\infty$ and $\delta_1'=x^{w(s)}\delta_2$. Similarly, $\delta_1=x^{w(s')}\delta_2'$. As $\delta_1\delta_2=\delta_1'\delta_2'$, we have $w(s)=w(s')$. ☐

Proposition 6.6. *Let $s=(\delta_1,\delta_2,c)$ be in $\mathscr{S}$. Then $D(s)$ is of slope zero if and only if $s\in\mathscr{S}_+-\mathscr{S}_+^{\mathrm{ncl}}$; $D(s)$ is of slope zero and the Galois representation attached to $D(s)$ is irreducible if and only if s is in $\mathscr{S}_*-(\mathscr{S}_*^{\mathrm{ord}}\cup\mathscr{S}_*^{\mathrm{ncl}})$; $D(s)$ is of slope zero and $\mathcal{O}_F$-analytic if and only if $s\in\mathscr{S}_+^{\mathrm{an}}-\mathscr{S}_+^{\mathrm{ncl}}$.*

Proof. By Kedlaya's slope filtration theorem, $D(s)$ is of slope zero if and only if $v_\pi(\delta_1(\pi)\delta_2(\pi))=0$ and $D(s)$ has no (φ_q,Γ)-submodule of rank 1 that is of slope <0. In particular, if $D(s)$ is of slope zero, then $v_\pi(\delta_1(\pi))\geq 0$ and thus $s\in\mathscr{S}_+$. Hence we only need to consider the case of $s\in\mathscr{S}_+$. Assume that $D(s)$ has a (φ_q,Γ)-submodule of rank 1, say M, that is of slope <0. Then the intersection of M and $\mathscr{R}_L(\delta_1)$ is zero. By Proposition 6.3 we may suppose that M is saturated. By Corollary 6.4, this happens if and only if s is in $\mathscr{S}^{\mathrm{an}}$ and satisfies $w(s)\in\mathbb{Z}_+$, $\delta_1\delta_2^{-1}\neq x^{w(s)}$, $\mathscr{L}(c_{\mathrm{an}})=\infty$ and $w(s)<u(s)$. Note that $\delta_1\delta_2^{-1}\neq x^{w(s)}$ and $\mathscr{L}(c_{\mathrm{an}})=\infty$ automatically hold when $0<w(s)<u(s)$. The first assertion follows. Similarly, $D(s)$ has a saturated (φ_q,Γ)-submodule of rank 1 that is of slope zero, if and only if $u(s)=0$ or $u(s)=w(s)$. By Proposition 1.5 (c) and Remark 1.8, we know that the Galois representation attached to an étale (φ_q,Γ)-module D over $\mathscr{R}_L$ of rank 2 is irreducible if and only if D has no étale (φ_q,Γ)-submodule of rank 1. This shows the second assertion. The third assertion follows from the first one. ☐

Proof of Theorem 0.5. Assertion (a) follows from Proposition 6.6, and (b) follows from Corollary 6.5. □

Remark 6.7. Let $s \neq s'$ be as in Theorem 0.5 (b). Then $s \in \mathscr{S}_*^{\mathrm{cris}}$ if and only if $s' \in \mathscr{S}_*^{\mathrm{cris}}$; $s \in \mathscr{S}_+^{\mathrm{ord}}$ if and only if $s' \in \mathscr{S}_0^{\mathrm{cris}}$.

Remark 6.8. By an argument similar to that in [9] one can show that, if s is in $\mathscr{S}_+^{\mathrm{cris}}$ (resp. $\mathscr{S}_+^{\mathrm{ord}}$, $\mathscr{S}_+^{\mathrm{st}}$), then $D(s)$ comes from a crystalline (resp. ordinary, semistable but non-crystalline) L-representation twisted by a character.

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