Derivatives of Frobenius and Derivatives of Hodge weights

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Abstract

In this paper we study the derivatives of Frobenius and the derivatives of Hodge weights for families of Galois representations with triangulations. We generalize the Fontaine-Mazur $\mathcal{L}$-invariant and use it to build a formula which is a generalization of the Greenberg-Stevens-Colmez formula. For the purpose of proving this formula we show two auxiliary results called projection vanishing property and "projection vanishing implying $\mathcal{L}$-invariants" property.

Introduction

It is well known that Galois Representation is one of the most fundamental objects in number theory. In this paper we concentrate on the p-adic representations of the absolute Galois group of $\mathbf{Q}_p$, where p is a fixed prime number. Among them semistable representations are special but important. To such representations Fontaine [18] attached linear algebra objects called filtered (φ, N)-modules. Colmez and Fontaine [14] proved that there is an equivalence of categories between the category of semistable representations and the category of admissible filtered (φ, N)-modules. Using the associated filtered (φ, N)-module we can attach to each semistable representation two kinds of invariants, i.e. Hodge weights and the eigenvalues of Frobenius φ. A famous fact is that the Newton polygon is always above the Hodge polygon, which is the main significance of admissibility.

Recently there are a lot of papers studying families of Galois representations. For example, see [6, 21, 22, 27]. A natural question on families of Galois representations is the following

Question 0.1. *For a family of p-adic representations of $G_{\mathbf{Q}_p}$, what is the relation of derivatives of Hodge weights and derivatives of eigenvalues of Frobenius?*

However, Hodge weights and eigenvalues of Frobenius are not defined for a general representation of $G_{\mathbf{Q}_p}$. Therefore, we need to specify certain conditions so that the two kinds of derivatives in Question 0.1 can be reasonably explained. A good choice is the families with triangulations. The significance of triangulations has been confirmed by many works. See [23, 11, 13, 24, 8] for example.

To explain what a triangulation is, we need the theory of (φ, Γ)-modules. The (φ, Γ)-modules are modules over various rings of power series (denoted by $\mathscr{E}$, $\mathscr{E}^\dagger$ and $\mathscr{R}$). See [16, 9, 20] for precise constructions of these rings and definitions of (φ, Γ)-modules.

Theorem 0.2. *([16, 9, 20]) There is an equivalence of categories between the category of p-adic representations of $G_{\mathbf{Q}_p}$ and the category of étale (φ, Γ)-modules over either $\mathscr{E}$, $\mathscr{E}^\dagger$ or $\mathscr{R}$.*

What we need is a version of Theorem 0.2 with p-adic representations (i.e. $\mathbf{Q}_p$-representations) replaced by E-representations where E is a finite extension of $\mathbf{Q}_p$. Such a variant version follows directly from Theorem 0.2 itself.

Let E be a finite extension of $\mathbf{Q}_p$. For a (not necessarily étale) (φ, Γ)-module M over $\mathscr{R}_E$, by a triangulation of M we mean a filtration $\mathrm{Fil}_\bullet M$ on M consisting of saturated (φ, Γ)-submodules of M with $\mathrm{rank}_{\mathscr{R}_E}\mathrm{Fil}_i M = i$ such that $\mathrm{Fil}_i M/\mathrm{Fil}_{i-1}M$ ($1 \leq i \leq \mathrm{rank}_{\mathscr{R}_E} M$) is of rank 1, i.e. of the form $\mathscr{R}_E(\delta_i)$ where δ_i is an $E^\times$-valued character of $\mathbf{Q}_p^\times$. We call $(\delta_1, \cdots, \delta_n)$ the triangulation data for M.

When M comes from a semistable representation V, $-w_{\delta_1}, \cdots, -w_{\delta_n}$ coincide with the Hodge weights of V, and $\delta_1(p)p^{w_{\delta_1}}, \cdots, \delta_n(p)p^{w_{\delta_n}}$ coincide with eigenvalues of Frobenius of V. Here for a character δ of $\mathbf{Q}_p^\times$, w_δ is the weight of δ whose definition is given in Section 3.

Hence, for a family of representations of $G_{\mathbf{Q}_p}$ with triangulation data $(\delta_1, \cdots, \delta_n)$ we can regard $\mathrm{d}w_{\delta_i}$ ($i = 1, \cdots, n$) as the derivatives of Hodge weights, and regard $\frac{\mathrm{d}\delta_i(p)}{\delta_i(p)} + \log(p)\mathrm{d}w_{\delta_i}$ ($i = 1, \cdots, n$) formally as the derivatives of "logarithmic of Frobenius eigenvalues". The value of $\log(p)$ depends on which component of the logarithmic we take.

Now specifying the families of representations of $G_{\mathbf{Q}_p}$ with triangulations, Question 0.1 becomes the following

Question 0.3. *For an S-representation of $G_{\mathbf{Q}_p}$ with triangulation date $(\delta_1, \cdots, \delta_n)$, what is the relation among $\frac{\mathrm{d}\delta_1(p)}{\delta_i(p)}, \cdots, \frac{\mathrm{d}\delta_n(p)}{\delta_n(p)}, \mathrm{d}w_{\delta_1}, \cdots, \mathrm{d}w_{\delta_n}$? We will always take S to be an affinoid E-algebra.*

When $n=2$, Question 0.3 has been researched by Greenberg-Stevens [19] (for ordinary semistable point) and Colmez [12] (for general semistable point). The precise statement of Colmez's theorem will be recalled below. Later Colmez's theorem was generalized by Zhang [28] (again for $n=2$ but the base field $\mathbf{Q}_p$ is replaced by any finite extension of $\mathbf{Q}_p$).

Let S be an affiniod E-algebra, $\mathcal{V}$ a 2-dimensional S-representation of $G_{\mathbf{Q}_p}$. Without loss of generality we may assume that $\mathcal{V}$ is free, and let $\{v_1,v_2\}$ be a basis of $\mathcal{V}$ over S. Let $\sigma\mapsto A_\sigma$ be the matrix of $\sigma\in G_{\mathbf{Q}_p}$ with respect to this basis. Then there exist $\delta,\kappa\in S$ such that

$$\log(\det A_\sigma)=\delta\psi_1(\sigma)+\kappa\psi_2(\sigma)$$

for any $\sigma\in G_{\mathbf{Q}_p}$. Here, $\psi_1: G_{\mathbf{Q}_p}\to E$ is the unramified additive character of $G_{\mathbf{Q}_p}$ such that $\psi_1(\sigma)=1$ if σ induces the Frobenius $x\mapsto x^p$ on $\overline{\mathbf{F}}_p$; $\psi_2: G_{\mathbf{Q}_p}\to E$ is the additive character that is the logarithmic of the cyclotomic character χ_{cyc}.

Theorem 0.4. *([12]) Suppose that $\mathcal{V}$ admits a fixed Hodge weight 0 and there exists $\alpha\in S$ such that $(\mathbf{B}^{\varphi=\alpha}_{\mathrm{cris},S}\widehat{\otimes}_S\mathcal{V})^{G_{\mathbf{Q}_p}}$ is locally free of rank 1 over S. Suppose z_0 is a closed point of $\mathrm{Max}(S)$ such that $\mathcal{V}_{z_0}$ is semistable with Hodge weights 0 and $k\geq 1$. Then the differential*

$$\frac{\mathrm{d}\alpha}{\alpha}-\frac{1}{2}\mathcal{L}\mathrm{d}\kappa+\frac{1}{2}\mathrm{d}\delta$$

is zero at z_0, where $\mathcal{L}$ is the Fontaine-Mazur $\mathcal{L}$-invariant of $\mathcal{V}_{z_0}$.

Theorem 0.4 hints that Question 0.3 should be closely related to the following

Question 0.5. *What is the generalization of Fontaine-Mazur $\mathcal{L}$-invariants?*

Let $(D,\varphi,N,\mathrm{Fil}^\bullet)$ be an admissible filtered E-(φ,N)-module with a refinement $\mathcal{F}$. Throughout this paper we assume that φ is semisimple on D.[1] The monodromy N induces an operator $N_{\mathcal{F}}$ on the grading module

$$\mathrm{gr}^{\mathcal{F}}_{\bullet}D=\bigoplus_{i=1}^{\dim_E D}\mathcal{F}_iD/\mathcal{F}_{i-1}D.$$

[1] This is not an essential condition. However, to include the result for the general case (φ maybe not semisimple) we need much more knowledge and technique from the theory of (φ,Γ)-modules, which does not fit with the style of the present paper. The general case will be considered in a sequel paper, where we study the families of (not necessarily étale) (φ,Γ)-modules instead of families of Galois representations.

If $s,t\in\{1,\cdots,\dim_E D\}$ satisfy $s<t$ and $N_{\mathcal{F}}(\mathrm{gr}^{\mathcal{F}}_t D)=\mathrm{gr}^{\mathcal{F}}_s D$, then we say that s is critical for $\mathcal{F}$ and write $t=t_{\mathcal{F}}(s)$. The criticality does not depend on φ and $\mathrm{Fil}^\bullet$. We will introduce another notion "strong criticality" (see Definition 4.8) which depends not only on N and $\mathcal{F}$ but also on φ and $\mathrm{Fil}^\bullet$. If s is strongly critical, we can attach to s an invariant denoted by $\mathcal{L}_{\mathcal{F},s}$. For the solution to Question 0.5 we regard the set

$$\{\mathcal{L}_{\mathcal{F},s} : s \text{ is strongly critical for } \mathcal{F}\}$$

as the generalization of the Fontaine-Mazur $\mathcal{L}$-invariant. [2]

Now we can state our main theorem as follows.

Theorem 0.6. *Let S be an affinoid E-algebra. Let $\mathcal{V}$ be an S-representation of $G_{\mathbf{Q}_p}$ with a triangulation and the associated triangulation date $(\delta_1,\cdots,\delta_n)$. Let z_0 be a closed point of $\mathrm{Max}(S)$, E_{z_0} the residue field of S at z_0. Suppose that $\mathcal{V}_{z_0}$ is semistable and φ is semisimple on D, where D is the filtered E_{z_0}-(φ,N)-module attached to $\mathcal{V}_{z_0}$. Let $\mathcal{F}$ be the refinement on D corresponding to the triangulation of $\mathcal{V}_{z_0}$. Suppose that $s\in\{1,\cdots,n-1\}$ is strongly critical for $\mathcal{F}$, $t=t_{\mathcal{F}}(s)$. Then*

$$\frac{\mathrm{d}\delta_t(p)}{\delta_t(p)}-\frac{\mathrm{d}\delta_s(p)}{\delta_s(p)}+\mathcal{L}_{\mathcal{F},s}(\mathrm{d}w_{\delta_t}-\mathrm{d}w_{\delta_s})$$

is zero at z_0.

We remark that, when s is critical for $\mathcal{F}$ and $t_{\mathcal{F}}(s)=s+1$, s is strongly critical for $\mathcal{F}$ if and only if $w_{\delta_s,z_0}>w_{\delta_{s+1},z_0}$.

An especially interesting case is when the rank of the monodromy N of D is equal to $\dim_E D-1$. Let e_n be an element not in $N(D)$ such that $\varphi(e_n)\in Ee_n$. For $i=1,\cdots,n-1$ put $e_i=N^{n-i}e_n$. Then D admits a unique triangulation $\mathcal{F}$ and $\mathcal{F}_iD=Ee_1\oplus\cdots\oplus Ee_i$ for all $i=1,\cdots,n$. Write $k_i=-w_{\delta_{i,z_0}}$. Then $k_1,\cdots,k_n$ are Hodge weights of $\mathcal{V}_{z_0}$. There always exists an upper-triangular matrix $(\ell_{j,i})_{n\times n}$ such that $\{e_i+\sum\limits_{1\leq j<i}\ell_{j,i}e_j : i=1,\cdots,n\}$ is an E-basis of D compatible with the Hodge filtration.

[2] The reader may be mystified that our $\mathcal{L}$-invariant is defined for Galois representations with triangulations instead of Galois representations themselves. On one hand, such a definition is suitable for Question 0.3, which can be seen in Theorem 0.6. On the other hands, Galois representations with triangulations play the fundamental role in many aspects, for example the definition of p-adic L-functions for modular forms [26] and the construction of eigenvarieties [1, 7]. In Fontaine and Mazur's definition of $\mathcal{L}$-invariants, the information of triangulation is hidden. Indeed, a semistable (but non-crystalline) 2-dimensional Galois representation admits a unique triangulation.

Theorem 0.7. *With the above notations suppose that $k_1 < k_2 < \cdots < k_n$. Then*

$$\frac{\mathrm{d}\delta_{s+1}(p)}{\delta_{s+1}(p)} - \frac{\mathrm{d}\delta_s(p)}{\delta_s(p)} + \ell_{s,s+1}(\mathrm{d}w_{\delta_{s+1}} - \mathrm{d}w_{\delta_s})$$

is zero at z_0.

When $n = 2$, the condition $k_1 < k_2$ automatically holds. So Theorem 0.7 covers Theorem 0.4. Indeed, under the condition of Theorem 0.4 we have $\mathrm{d}w_{\delta_1} = 0$, $\frac{\mathrm{d}\alpha}{\alpha} = \frac{\mathrm{d}\delta_1(p)}{\delta_1(p)}$, $\mathrm{d}\delta = -\frac{\mathrm{d}\delta_1(p)}{\delta_1(p)} - \frac{\mathrm{d}\delta_2(p)}{\delta_2(p)}$ and $\mathrm{d}\kappa = \mathrm{d}w_{\delta_2}$.

We sketch the proof of Theorem 0.6.

From $\mathcal{V}$ we obtain an infinitesimal deformation of $\mathcal{V}_{z_0}$ and attach to this infinitesimal deformation a 1-cocycle $c : G_{\mathbf{Q}_p} \to \mathcal{V}_{z_0}^* \otimes_{E_{z_0}} \mathcal{V}_{z_0}$. Let $\{e_1, \cdots, e_n\}$ be a basis of D that is s-perfect for $\mathcal{F}$, $\{e_1^*, \cdots e_n^*\}$ the dual basis of $\{e_1, \cdots, e_n\}$. (See Definition 4.10 for the precise meaning of s-perfect basis.) Let $\pi_{h,\ell}$ be the composition of the inclusion

$$\mathcal{V}_{z_0}^* \otimes_{E_{z_0}} \mathcal{V}_{z_0} \hookrightarrow \mathbf{B}_{\mathrm{st},E_{z_0}} \otimes_{E_{z_0}} (\mathcal{V}_{z_0}^* \otimes_{E_{z_0}} \mathcal{V}_{z_0})$$

and the projection

$$\mathbf{B}_{\mathrm{st},E_{z_0}} \otimes_{E_{z_0}} (\mathcal{V}_{z_0}^* \otimes_{E_{z_0}} \mathcal{V}_{z_0}) \to \mathbf{B}_{\mathrm{st},E_{z_0}}, \quad \sum_{i,j} b_{ij} e_j^* \otimes e_i \mapsto b_{\ell h}.$$

We have the following projection vanishing property (Theorem 0.8) and "projection vanishing implying $\mathcal{L}$-invariant" property (Theorem 0.9).

Theorem 0.8. *Suppose that φ is semisimple on D. Let $c : G_{\mathbf{Q}_p} \to \mathcal{V}_{z_0}^* \otimes_{E_{z_0}} \mathcal{V}_{z_0}$ be a 1-cocycle coming from an infinitesimal deformation of $\mathcal{V}_{z_0}$. If $h < \ell$, then $\pi_{h,\ell}([c]) = 0$ in $H^1(\mathbf{B}_{\mathrm{st},E_{z_0}})$.*

Theorem 0.9. *Suppose that φ is semisimple on D. Let c be a 1-cocycle $G_{\mathbf{Q}_p} \to \mathcal{V}_{z_0}^* \otimes_{E_{z_0}} \mathcal{V}_{z_0}$ satisfying the projection vanishing property. If s is strongly critical for $\mathcal{F}$ and $t = t_{\mathcal{F}}(s)$, then there exist $\gamma_{s,1}, \gamma_{s,2}, \gamma_{t,1}, \gamma_{t,2} \in E_{z_0}$ and $x_s, x_t \in \mathbf{B}_{\mathrm{st},E_{z_0}}^{\varphi=1}$ such that*

$$\pi_{i,i}(c_\sigma) = \gamma_{i,1}\psi_1 + \gamma_{i,2}\psi_2 + (\sigma - 1)x_i, \quad i = s, t.$$

Furthermore $\gamma_{s,1} - \gamma_{t,1} = \mathcal{L}_{\mathcal{F},s}(\gamma_{s,2} - \gamma_{t,2})$.

Theorem 0.6 follows from Theorem 0.8, Theorem 0.9 and a computation relating $\gamma_{i,1}$, $\gamma_{i,2}$ to $\frac{\mathrm{d}\delta_i(p)}{\delta_i(p)}$ and $\mathrm{d}w_{\delta_i}$.

Our paper is organized as follows. In Section 1 we provide preliminary results on Galois cohomology. The proof of the "projection vanishing implying $\mathcal{L}$-invariant" property needs the functors $\mathbf{X}_{\mathrm{st}}$ and $\mathbf{X}_{\mathrm{dR}}$ used in [14] where they are denoted by V_{st}^0 and V_{st}^1 respectively. In Section 2 we give a systematic study on these two functors. The relation between triangulations and refinements is reviewed in Section 3. In Section 4 we introduce the concepts of criticality and strong criticality and define $\mathcal{L}$-invariants. The "projection vanishing implying $\mathcal{L}$-invariant" property is proved in Section 5, and the projection vanishing property is proved in Section 6. Finally in section 7 we combine results in Section 5 and Section 6 to prove Theorem 0.6.

There are two directions to generalize Theorem 0.6. One is to consider families of (not necessarily étale) (φ, Γ)-modules instead of families of Galois representations. The other is that the base field $\mathbf{Q}_p$ is replaced by a finite extension of $\mathbf{Q}_p$. These are in progress.

There may be two possible applications of Theorem 0.6. One is to the Exceptional Zero phenomenon, and the other is to the local-global compatibility in p-adic Langlands program. In the case of $n = 2$ the former is done in [19] and the latter is done in [15].

Notation

For a $G_{\mathbf{Q}_p}$-module M we write $H^i(M)$ for the cohomology group $H^i(G_{\mathbf{Q}_p}, M)$. For a 1-cocycle $c : G_{\mathbf{Q}_p} \to M$ let $[c]$ denote the class of c in $H^1(M)$. For a $G_{\mathbf{Q}_p}$-module M let $M(i)$ denote the twist of M by χ_{cyc}^i, where χ_{cyc} is the cyclotomic character.

Let E be a finite extension of $\mathbf{Q}_p$ considered as a base field with trivial action of $G_{\mathbf{Q}_p}$. Let $\psi_1 : G_{\mathbf{Q}_p} \to E$ be the unramified additive character of $G_{\mathbf{Q}_p}$ such that $\psi_1(\sigma) = 1$ if σ induces the Frobenius $x \mapsto x^p$ on $\overline{\mathbf{F}}_p$. Let $\psi_2 : G_{\mathbf{Q}_p} \to E$ be the additive character that is the logarithmic of χ_{cyc}. Then $[\psi_1]$ and $[\psi_2]$ form a basis of $H^1(E) = \mathrm{Hom}(G_{\mathbf{Q}_p}, E)$ over E.

If δ is a multiplicative character of $\mathbf{Q}_p^\times$, the character of $\mathbf{Q}_p^\times$ whose restriction to $\mathbf{Z}_p^\times$ coincides with $\delta|_{\mathbf{Z}_p^\times}$ and whose value at p is 1, is again denoted by $\delta|_{\mathbf{Z}_p^\times}$ by abuse of notation.

For an affinoid E-algebra S and a closed point $z \in \mathrm{Max}(S)$, let E_z denote the residue field of S at z. For an S-module $\mathcal{M}$ we put $\mathcal{M}_z = \mathcal{M} \otimes_S E_z$.

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ denote the set of natural numbers, integers and rational numbers respectively.

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I would like to dedicate this paper to my teacher Professor Chunlai Zhao for his 70th birthday.

1 Fontaine period rings and Galois cohomology

Let $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$ be Fontaine's period rings [17]. Put

$$\mathbf{B}_{\mathrm{cris},E} = \mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} E, \quad \mathbf{B}_{\mathrm{st},E} = \mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} E, \quad \mathbf{B}_{\mathrm{dR},E} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E.$$

We extend the actions of $G_{\mathbf{Q}_p}$ on $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$ E-linearly to $\mathbf{B}_{\mathrm{cris},E}$, $\mathbf{B}_{\mathrm{st},E}$ and $\mathbf{B}_{\mathrm{dR},E}$. We also extend the operators φ and N on $\mathbf{B}_{\mathrm{st}}$ E-linearly to $\mathbf{B}_{\mathrm{st},E}$. Then $\mathbf{B}_{\mathrm{cris},E}$ is stable under φ and $\mathbf{B}_{\mathrm{cris},E} = \mathbf{B}_{\mathrm{st},E}^{N=0}$. Let t_{cyc} be Fontaine's p-adic "$2\pi\sqrt{-1}$" [17]. We have $\varphi(t_{\mathrm{cyc}}) = p\,t_{\mathrm{cyc}}$, $Nt_{\mathrm{cyc}} = 0$ and $g(t_{\mathrm{cyc}}) = \chi_{\mathrm{cyc}}(g)t_{\mathrm{cyc}}$ for $g \in G_{\mathbf{Q}_p}$. Let Fil be the filtration on $\mathbf{B}_{\mathrm{dR},E}$ such that $\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR},E} = \mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E$. Put $\mathbf{B}^+_{\mathrm{dR},E} = \mathrm{Fil}^0\mathbf{B}_{\mathrm{dR},E} = \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E$. Then we have the following short exact sequence, the so called fundamental exact sequence [14, Proposition 1.3 v)]

$$0 \longrightarrow E \longrightarrow \mathbf{B}_{\mathrm{cris},E}^{\varphi=1} \longrightarrow \mathbf{B}_{\mathrm{dR},E}/\mathbf{B}^+_{\mathrm{dR},E} \longrightarrow 0.$$

The following lemma is well known. See [12, Proposition 1.1].

Lemma 1.1. *Let $a \leq b$ be in $\mathbf{Z} \cup \{-\infty, +\infty\}$. If either $a > 0$ or $b \leq 0$, then*

$$H^0(\mathrm{Fil}^a\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^b\mathbf{B}_{\mathrm{dR},E}) = H^1(\mathrm{Fil}^a\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^b\mathbf{B}_{\mathrm{dR},E}) = 0$$

with the convention $\mathrm{Fil}^{-\infty}\mathbf{B}_{\mathrm{dR},E} = \mathbf{B}_{\mathrm{dR},E}$ *and* $\mathrm{Fil}^{+\infty}\mathbf{B}_{\mathrm{dR},E} = 0$.

For $i \in \mathbf{N}$ and $j \in \mathbf{Z}$ put $U_{i,j} = \mathbf{B}_{\mathrm{st},E}^{N^{i+1}=0,\varphi=p^j}$. Note that $U_{i,i-1}$ coincides with the notation U_i in [12].

Lemma 1.2. *For any $i \geq 1$ we have the following short exact sequence*

$$0 \longrightarrow \mathbf{B}_{\mathrm{cris},E}^{\varphi=p^j} \longrightarrow U_{i,j} \xrightarrow{N} U_{i-1,j-1} \longrightarrow 0.$$

Proof. We only need to prove the surjectivity of $N : U_{i,j} \to U_{i-1,j-1}$. Let u be the element in $\mathbf{B}_{\mathrm{st}}$, considered as an element in $\mathbf{B}_{\mathrm{st},E}$, that is denoted by $\log[\pi]$ in [14, §1.5]. Then $\mathbf{B}_{\mathrm{st},E} = \mathbf{B}_{\mathrm{cris},E}[u]$ and $\varphi(u) = pu$, $N(u) = -1$. For $x \in U_{i-1,j-1}$ write $x = \sum_{\ell=0}^{i-1} a_\ell u^\ell$ with $a_\ell \in \mathbf{B}_{\mathrm{cris},E}$. Then a_ℓ is in $\mathbf{B}_{\mathrm{cris},E}^{\varphi=p^{i-1-\ell}}$. So $y = -\sum_{\ell=0}^{i-1} a_\ell \frac{u^{\ell+1}}{\ell+1}$ is in $U_{i,j}$ and $N(y) = x$. □

Proposition 1.3. *If $i \geq 1$, then the inclusion $E \subset U_{i,0}$ induces an isomorphism*

$$H^1(E) \xrightarrow{\sim} \ker(H^1(U_{i,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E})).$$

Proof. We prove the assertion by induction on i. For $i = 1$, the assertion is [12, Proposition 1.2].

By definition $U_{0,-i} = \mathbf{B}_{\mathrm{cris},E}^{\varphi=p^{-i}}$. From the fundamental exact sequence we obtain the following exact sequence

$$0 \longrightarrow Et^{-i} \longrightarrow U_{0,-i} \longrightarrow \mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{-i}\mathbf{B}_{\mathrm{dR},E} \longrightarrow 0.$$

So we have an isomorphism $H^1(E(-i)) = H^1(Et^{-i}) \to H^1(U_{0,-i})$ since by Lemme 1.1

$$H^0(\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{-i}\mathbf{B}_{\mathrm{dR},E}) = H^1(\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{-i}\mathbf{B}_{\mathrm{dR},E}) = 0$$

for $i \geq 0$. When $i \geq 1$, each nontrivial extension of E by $E(-i)$ is not semistable. (This is a well known fact; it also follows from Proposition 2.6 below.) Thus $H^1(E(-i)) \to H^1(\mathbf{B}_{\mathrm{st},E})$ induced by the natural inclusion $E(-i) \subset \mathbf{B}_{\mathrm{st},E}$ is injective. As $H^1(E(-i)) \to H^1(U_{0,-i})$ is an isomorphism, it follows that $H^1(U_{0,-i}) \to H^1(\mathbf{B}_{\mathrm{st},E})$ is also injective.

Note that

$$\ker(H^1(U_{i,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E})) \subset \ker(H^1(U_{i,0}) \xrightarrow{N^i} H^1(\mathbf{B}_{\mathrm{st},E})).$$

We consider the exact sequence

$$0 \longrightarrow U_{i-1,0} \longrightarrow U_{i,0} \xrightarrow{N^i} U_{0,-i} \longrightarrow 0.$$

As $H^0(U_{0,-i}) = 0$, from this short exact sequence we derive an isomorphism

$$H^1(U_{i-1,0}) \xrightarrow{\sim} \ker(H^1(U_{i,0}) \xrightarrow{N^i} H^1(U_{0,-i})).$$

In particular the natural map $H^1(U_{i-1,0}) \to H^1(U_{i,0})$ is injective. As $H^1(U_{0,-i})$ injects into $H^1(\mathbf{B}_{\mathrm{st},E})$, we have

$$\ker(H^1(U_{i,0}) \xrightarrow{N^i} H^1(\mathbf{B}_{\mathrm{st},E})) = \ker(H^1(U_{i,0}) \xrightarrow{N^i} H^1(U_{0,-i})).$$

It follows that $\ker(H^1(U_{i,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E}))$ lies in the image of $H^1(U_{i-1,0}) \to H^1(U_{i,0})$. Since $H^1(U_{i-1,0})$ injects into $H^1(U_{i,0})$, we have an isomorphism

$$\ker(H^1(U_{i-1,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E})) \xrightarrow{\sim} \ker(H^1(U_{i,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E})).$$

This completes the inductive proof. □

Corollary 1.4. *The inclusion $E \subset \mathbf{B}_{\mathrm{st},E}^{\varphi=1}$ induces an isomorphism*

$$H^1(E) \xrightarrow{\sim} \ker(H^1(\mathbf{B}_{\mathrm{st},E}^{\varphi=1}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E})).$$

Proof. First we prove that $H^1(E) \to H^1(\mathbf{B}_{\mathrm{st},E}^{\varphi=1})$ is injective. Let c be a 1-cocycle with values in E. If the image of $[c]$ in $H^1(\mathbf{B}_{\mathrm{st},E}^{\varphi=1})$ is zero, then there exists some $y \in \mathbf{B}_{\mathrm{st},E}^{\varphi=1}$ such that $c_\sigma = (\sigma - 1)y$ for all $\sigma \in G_{\mathbf{Q}_p}$. As $\mathbf{B}_{\mathrm{st},E}^{\varphi=1} = \cup_{i\geq 1} U_{i,0}$, y is in $U_{i,0}$ for some $i \geq 1$, which implies that the image of $[c]$ in $H^1(U_{i,0})$ is zero. But by Proposition 1.3, $H^1(E)$ injects to $H^1(U_{i,0})$, so $[c] = 0$ (in $H^1(E)$).

Now, let c be a 1-cocycle with values in $\mathbf{B}_{\mathrm{st},E}^{\varphi=1}$ such that the image of $[c]$ by $N : H^1(\mathbf{B}_{\mathrm{st},E}^{\varphi=1}) \to H^1(\mathbf{B}_{\mathrm{st},E})$ is zero. Then there exists some $z \in \mathbf{B}_{\mathrm{st},E}$ such that $N(c_\sigma) = (\sigma - 1)z$ for all $\sigma \in G_{\mathbf{Q}_p}$. Let i be a positive integer such that $N^i(z) = 0$. Then $c_\sigma \in U_{i,0}$ for all $\sigma \in G_{\mathbf{Q}_p}$. In other words, $[c]$ comes from an element in $\ker(H^1(U_{i,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E}))$ by the map $H^1(U_{i,0}) \to H^1(\mathbf{B}_{\mathrm{st},E}^{\varphi=1})$. So by Proposition 1.3, $[c]$ comes from an element in $H^1(E)$ by the map $H^1(E) \to H^1(\mathbf{B}_{\mathrm{st},E}^{\varphi=1})$. □

2 Some facts on Galois representations

Throughout this section a *filtration* on an E-vector space D means an exhaustive descending $\mathbf{Z}$-indexed filtration.

2.1 $\mathbf{X}_{\mathrm{st}}$ and $\mathbf{X}_{\mathrm{dR}}$

We will use the functors $\mathbf{X}_{\mathrm{st}}$ and $\mathbf{X}_{\mathrm{dR}}$ defined in [12]. These functors were already used in [14] to show that every admissible filtered (φ, N)-module comes from a Galois representation. In [14] $\mathbf{X}_{\mathrm{st}}$ and $\mathbf{X}_{\mathrm{dR}}$ are denoted by V_{st}^0 and V_{st}^1 respectively.

We refer the reader to [12] for the notions of E-(φ, N)-modules, filtered E-modules, filtered E-(φ, N)-modules and admissible filtered E-(φ, N)-modules. Note that, if D, D_1 and D_2 are filtered E-(φ, N)-modules, then there exist natural filtered E-(φ, N)-module structures on D^* and $D_1 \otimes_E D_2$.

If V is a finite-dimensional E-representation of $G_{\mathbf{Q}_p}$, then $\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{B}_{\mathrm{st},E} \otimes_E V)^{G_{\mathbf{Q}_p}}$ is a filtered E-(φ, N)-module induced from the natural filtered E-(φ, N)-module structure on $\mathbf{B}_{\mathrm{st},E} \otimes_E V$. We always have $\dim_E \mathbf{D}_{\mathrm{st}}(V) \leq \dim_E V$, and say that V is *semistable* if $\dim_E \mathbf{D}_{\mathrm{st}}(V) = \dim_E V$.

If D is a finite-dimensional E-(φ, N)-module, let $\mathbf{X}_{\mathrm{st}}(D)$ be the $\mathbf{B}^{\varphi=1}_{\mathrm{cris},E}$-module defined by

$$\mathbf{X}_{\mathrm{st}}(D) = (\mathbf{B}_{\mathrm{st},E} \otimes_E D)^{\varphi=1,N=0}.$$

If $\mathrm{Fil} = (\mathrm{Fil}^j)_{j\in\mathbf{Z}}$ is a filtration on a finite-dimensional E-vector space D, let $\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil})$ or just $\mathbf{X}_{\mathrm{dR}}(D)$ if there is no confusion, be the $\mathbf{B}^{+}_{\mathrm{dR},E}$-module

$$\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}) = (\mathbf{B}_{\mathrm{dR},E} \otimes_E D)/\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E D).$$

By [14, Proposition 5.1, Proposition 5.2] $\mathbf{X}_{\mathrm{st}}$ and $\mathbf{X}_{\mathrm{dR}}$ are exact.

If (D, Fil) is a filtered E-(φ, N)-module, then there is a natural E-linear map $\mathbf{X}_{\mathrm{st}}(D) \to \mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil})$ induced by the inclusion $\mathbf{B}_{\mathrm{st},E} \otimes_E D \to \mathbf{B}_{\mathrm{dR},E} \otimes_E D$. Let $\mathbf{V}_{\mathrm{st}}(D, \mathrm{Fil})$ be the kernel of this map, which is an E-vector space.

By [12, Theorem 2.1] $\mathbf{V}_{\mathrm{st}}$ is an equivalence of categories from the category of admissible filtered E-(φ, N)-modules to the category of semistable E-representations of $G_{\mathbf{Q}_p}$, with quasi-inverse $\mathbf{D}_{\mathrm{st}}$. Furthermore, $\mathbf{V}_{\mathrm{st}}$ and $\mathbf{D}_{\mathrm{st}}$ respect tensor products and duals.

If (D, Fil) is an admissible filtered E-(φ, N)-module, then the sequence

$$0 \longrightarrow \mathbf{V}_{\mathrm{st}}(D, \mathrm{Fil}) \longrightarrow \mathbf{X}_{\mathrm{st}}(D) \longrightarrow \mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}) \longrightarrow 0 \tag{2.1}$$

is exact, and the natural map

$$\mathbf{B}_{\mathrm{st},E} \otimes_E \mathbf{V}_{\mathrm{st}}(D, \mathrm{Fil}) \to \mathbf{B}_{\mathrm{st},E} \otimes_E D$$

is an isomorphism respecting the actions of $G_{\mathbf{Q}_p}$, φ, N and the filtrations.

If D is an E-(φ, N)-module, and e^* is an element in the dual E-(φ, N)-module D^*, we have a $G_{\mathbf{Q}_p}$-equivariant map

$$\pi_{e^*} : \mathbf{X}_{\mathrm{st}}(D) \to \mathbf{B}_{\mathrm{st},E}, \quad x \mapsto < e^*, x > .$$

Here $< \cdot, \cdot >$ denotes the $\mathbf{B}_{\mathrm{st},E}$-bilinear pairing

$$(\mathbf{B}_{\mathrm{st},E} \otimes_E D^*) \times (\mathbf{B}_{\mathrm{st},E} \otimes_E D) \to \mathbf{B}_{\mathrm{st},E}$$

induced by the canonical E-bilinear pairing $D^* \times D \to E$.

Lemma 2.1. (a) *We have* $N \circ \pi_{e^*} = \pi_{Ne^*}$.

(b) *If* $N^{i+1}e^* = 0$, *then the image of* π_{e^*} *is in* $\mathbf{B}_{\mathrm{st},E}^{N^{i+1}=0}$. *If furthermore* $\varphi(e^*) = e^*$, *then the image of* π_{e^*} *is in* $U_{i,0}$.

Proof. For any $x \in \mathbf{X}_{\mathrm{st}}(D)$, as $Nx = 0$, we have $N < e^*, x > = < Ne^*, x >$.

If $N^{i+1}e^* = 0$, then $N^{i+1} < e^*, x > = < N^{i+1}e^*, x > = 0$. If $\varphi e^* = e^*$, then $\varphi < e^*, x > = < \varphi e^*, \varphi x > = < e^*, x >$ since $\varphi(x) = x$. So $< e^*, x >$ is in $U_{i,0}$. □

For $e^* \in D^*$, π_{e^*} induces a map $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{B}_{\mathrm{st},E})$ again denoted by π_{e^*}. If $\varphi(e^*) = e^*$, π_{e^*} induces a map $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{B}_{\mathrm{st},E}^{\varphi=1})$ which will be denoted by $\tilde{\pi}_{e*}$.

2.2 Exactness of $H^i(\mathbf{X}_{\mathrm{dR}}(-))$

Let D be a finite-dimensional E-vector space. For a basis $\{e_1, \cdots, e_n\}$ of D over E and a filtration Fil on D we say that $\{e_1, \cdots, e_n\}$ is *compatible* with Fil if for any i, $\mathrm{Fil}^i D = \oplus_{j=1}^n \mathrm{Fil}^i D \cap Ee_j$. If we write f_j ($j = 1, \cdots, n$) for the largest integer such that $\mathrm{Fil}^{f_j} D \cap Ee_j \neq 0$, then $\{e_1, \cdots, e_n\}$ is compatible with Fil, if and only if $\mathrm{Fil}^i D = \bigoplus_{j: f_j \geq i} Ee_j$ for any i. In this case we have

$$\mathrm{Fil}^i(\mathbf{B}_{\mathrm{dR},E} \otimes_E D) = \bigoplus_j \mathrm{Fil}^{i-f_j} B_{\mathrm{dR},E} \cdot e_j.$$

Lemma 2.2. *Let*

$$0 \longrightarrow D_1 \longrightarrow D \longrightarrow D_2 \longrightarrow 0$$

be a short exact sequence of filtered E*-modules. If* $\{e_1, \cdots, e_s\}$ *is a basis of* D_1 *and* $\{\bar{e}_{s+1}, \cdots, \bar{e}_n\}$ *is a basis of* D_2 *over* E *compatible with the filtration on* D_1 *and that on* D_2 *respectively, then there exist liftings* e_j *of* $\bar{e}_j$ ($j = s+1, \cdots, n$) *such that* $\{e_1, \cdots, e_n\}$ *is a basis of* D *compatible with the filtration.*

Proof. Let f_j ($j = 1, \cdots, n$) be the largest integer such that $\mathrm{Fil}^{f_j} D_1 \cap Ee_j \neq 0$ for $j = 1, \cdots, s$, and $\mathrm{Fil}^{f_j} D_2 \cap E\bar{e}_i \neq 0$ for $j = s+1, \cdots, n$. As the filtration on D_2 is induced from that on D, there exists a lifting e_j of $\bar{e}_j$ in $\mathrm{Fil}^{f_j} D$ for any $j = s+1, \cdots, n$. Then $\bigoplus_{j: f_j \geq i} Ee_j$ is contained in $\mathrm{Fil}^i D$. However, we have

$$\begin{aligned} \dim_E \mathrm{Fil}^i D &= \dim_E \mathrm{Fil}^i D_1 + \dim_E \mathrm{Fil}^i D_2 \\ &= \sharp\{j : 1 \leq j \leq s,\ f_j \geq i\} + \sharp\{j : s+1 \leq j \leq n,\ f_j \geq i\} \\ &= \sharp\{j : 1 \leq j \leq n,\ f_j \geq i\}. \end{aligned}$$

Therefore $\mathrm{Fil}^i D = \bigoplus_{j: f_j \geq i} Ee_j$. □

Proposition 2.3. *If*

$$0 \longrightarrow D_1 \longrightarrow D \longrightarrow D_2 \longrightarrow 0$$

is a short exact sequence of filtered E-modules, then

$$0 \longrightarrow \mathbf{X}_{\mathrm{dR}}(D_1) \longrightarrow \mathbf{X}_{\mathrm{dR}}(D) \longrightarrow \mathbf{X}_{\mathrm{dR}}(D_2) \longrightarrow 0$$

is a split short exact sequence of $G_{\mathbf{Q}_p}$-modules.

Proof. Let $\{e_j\}_{j=1}^n$, $\{\bar{e}_j\}_{j=s+1}^n$ and $\{f_j\}_{j=1}^n$ be as in Lemma 2.2 and its proof. By the definition of $\mathbf{X}_{\mathrm{dR}}(-)$ we have

$$\begin{aligned}
\mathbf{X}_{\mathrm{dR}}(D_1) &= \oplus_{j=1}^{s}(\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{f_j}\mathbf{B}_{\mathrm{dR},E}) \cdot e_j, \\
\mathbf{X}_{\mathrm{dR}}(D) &= \oplus_{j=1}^{n}(\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{f_j}\mathbf{B}_{\mathrm{dR},E}) \cdot e_j, \\
\mathbf{X}_{\mathrm{dR}}(D_2) &= \oplus_{j=s+1}^{n}(\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{f_j}\mathbf{B}_{\mathrm{dR},E}) \cdot \bar{e}_j.
\end{aligned}$$

As $\{e_j\}_{j=1}^n$ and $\{\bar{e}_j\}_{j=s+1}^n$ are fixed by $G_{\mathbf{Q}_p}$, our assertion is clear now. □

The following follows directly from Proposition 2.3.

Corollary 2.4. *If*

$$0 \longrightarrow D_1 \longrightarrow D \longrightarrow D_2 \longrightarrow 0$$

is a short exact sequence of filtered E-modules, then

$$0 \longrightarrow H^i(\mathbf{X}_{\mathrm{dR}}(D_1)) \longrightarrow H^i(\mathbf{X}_{\mathrm{dR}}(D)) \longrightarrow H^i(\mathbf{X}_{\mathrm{dR}}(D_2)) \longrightarrow 0$$

is exact.

2.3 The kernel of $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}))$

In this subsection we study the kernel of $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}))$. When (D, Fil) is admissible, from the short exact sequence (2.1) we see that this kernel coincides with the image of $H^1(\mathbf{V}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{st}}(D))$.

We fix a finite-dimensional E-(φ, N)-module D. For two filtrations Fil_1 and Fil_2 on D, we write $\mathrm{Fil}_1 \approx \mathrm{Fil}_2$ if $\mathrm{Fil}_1^0 D = \mathrm{Fil}_2^0 D$. Then $\approx$ is an equivalence relation on the set of filtrations on D.

Proposition 2.5. *If $\mathrm{Fil}_1 \approx \mathrm{Fil}_2$, then the kernel of $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_1))$ coincides with the kernel of $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_2))$.*

Proof. By [14, Proposition 3.1] there exists a basis $\{e_1, \cdots, e_n\}$ of D compatible with both Fil_1 and Fil_2. Write $f_{j,\ell}$ ($j = 1, \cdots, n$ and $\ell = 1, 2$) for the largest integer such that

$$\mathrm{Fil}_\ell^{f_{j,\ell}} D \cap Ee_j \neq 0.$$

Put $\bar{f}_j = \min(f_{j,1}, f_{j,2})$.

Put

$$M = (\mathbf{B}_{\mathrm{dR},E} \otimes_E D)/\mathrm{Fil}_1^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E D) \cap \mathrm{Fil}_2^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E D).$$

Note that

$$\mathrm{Fil}_\ell^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E D) = \bigoplus_{j=1}^n \mathrm{Fil}^{-f_{j,\ell}} \mathbf{B}_{\mathrm{dR},E} \cdot e_j, \quad \ell = 1, 2,$$

$$\mathrm{Fil}_1^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E D) \cap \mathrm{Fil}_2^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E D) = \bigoplus_{j=1}^n \mathrm{Fil}^{-\bar{f}_j} \mathbf{B}_{\mathrm{dR},E} \cdot e_j.$$

So, for $\ell = 1, 2$ we have an exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^n (\mathrm{Fil}^{-f_{j,\ell}} \mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{-\bar{f}_j} \mathbf{B}_{\mathrm{dR},E}) \cdot e_j \longrightarrow M \longrightarrow X_{\mathrm{dR}}(D, \mathrm{Fil}_\ell) \longrightarrow 0.$$

As $\mathrm{Fil}_1^0 D = \mathrm{Fil}_2^0 D$, $f_{j,1} \geq 0$ if and only if $f_{j,2} \geq 0$. Thus $\bar{f}_j \geq 0$ if and only if $f_{j,\ell} \geq 0$. So by Lemma 1.1 we have

$$H^1(\mathrm{Fil}^{-f_{j,\ell}} \mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{-\bar{f}_j} \mathbf{B}_{\mathrm{dR},E}) = 0.$$

As a consequence, $H^1(M) \to H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_\ell))$ is injective.

Note that $\mathbf{X}_{\mathrm{st}}(D) \to \mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_\ell)$ ($\ell = 1, 2$) factors through $\mathbf{X}_{\mathrm{st}}(D) \to M$. Thus $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_\ell))$ factors through $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(M)$. Since $H^1(M)$ injects to $H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_\ell))$, the kernel of $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_\ell))$ coincides with the kernel of $H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(M)$. □

2.4 The map $H^1(V) \to H^1(\mathbf{B}_{\mathrm{st},E} \otimes_E V)$

Proposition 2.6. *If V is a semistable E-representation of $G_{\mathbf{Q}_p}$ with Hodge weights > 0, then any nontrivial extension of the trivial representation E of $G_{\mathbf{Q}_p}$ by V is not semistable.*

Proof. The filtered E-(φ, N)-module attached to the trivial representation E is $D_0 = E \cdot e_0$ with

$$\varphi e_0 = e_0, \quad Ne_0 = 0, \quad \mathrm{Fil}^0 D_0 = D_0, \quad \mathrm{Fil}^1 D_0 = 0.$$

Let $\widetilde{V}$ be an extension of E by V that is a semistable representation of $G_{\mathbf{Q}_p}$. Let D and $\widetilde{D}$ be the filtered E-(φ, N)-module attached to V and that attached to $\widetilde{V}$ respectively. Then we have an exact sequence of filtered E-(φ, N)-modules

$$0 \longrightarrow D \longrightarrow \widetilde{D} \longrightarrow D_0 \longrightarrow 0. \tag{2.2}$$

Let $\{e_1, \cdots, e_n\}$ be a basis of D over E, and let $A = (a_{ij})$ be the matrix of φ with respect to this basis so that $\varphi(e_i) = \sum_{j=1}^n a_{ji} e_j$. As V is of Hodge weights > 0, we have $\mathrm{Fil}^1 D = D$. By the fact that the Newton polygon is above the Hodge polygon, the lowest slope of eigenvalues of A is positive. Thus $I_n - A$ is invertible, where I_n is the unit $n \times n$-matrix.

Let $\widetilde{e}$ be any lifting of e_0. Since $\varphi(e_0) = e_0$, there are $c_1, \cdots, c_n \in E$ such that $\varphi(\widetilde{e}) = \widetilde{e} + \sum\limits_{i=1}^n c_i e_i$. As $I_n - A$ is invertible, there is a unique vector $(b_1, \cdots, b_n)^t$ such that $(I_n - A) \cdot (b_1, \cdots, b_n)^t = (c_1, \cdots, c_n)^t$. Then $e = \widetilde{e} + \sum\limits_{i=1}^n b_i e_i$ satisfies $\varphi(e) = e$. From the relation $N\varphi = p\varphi N$ we obtain $Ne \in D^{\varphi = p^{-1}} = 0$. As $\mathrm{Fil}^1 D_0 = 0$, we have $e \notin \mathrm{Fil}^1 \widetilde{D}$. Hence the exact sequence (2.2) splits and so $\widetilde{V}$ is a trivial extension of E by V. $\square$

Corollary 2.7. *Let V be a semistable E-representation of $G_{\mathbf{Q}_p}$ with Hodge weights > 0. Then the following hold:*

(a) *The natural map $H^1(V) \to H^1(\mathbf{B}_{\mathrm{st},E} \otimes_E V)$ is injective.*

(b) *Let c be in $H^1(V)$. If for any $f \in \mathrm{Hom}_{G_{\mathbf{Q}_p}}(V, \mathbf{B}_{\mathrm{st},E})$, the image of c by the map $H^1(V) \to H^1(\mathbf{B}_{\mathrm{st},E})$ induced by f is zero, then $c = 0$.*

Proof. Assertion (a) follows immediately from Proposition 2.6.

Next we prove (b). Let D be the filtered E-(φ, N)-module attached to V, and let $\{e_1, \cdots, e_n\}$ be a basis of D over E. Let π_i denote the projection

$$\mathbf{B}_{\mathrm{st},E} \otimes_E D = \mathbf{B}_{\mathrm{st},E} \otimes_E V \to \mathbf{B}_{\mathrm{st},E}, \quad \sum_{j=1}^n a_j e_j \mapsto a_i.$$

As $e_1, \cdots, e_n$ are fixed by $G_{\mathbf{Q}_p}$, π_i $(i = 1, \cdots, n)$ are in $\mathrm{Hom}_{G_{\mathbf{Q}_p}}(\mathbf{B}_{\mathrm{st},E} \otimes_E V, \mathbf{B}_{\mathrm{st},E})$. So the composition of π_i and the inclusion $V \hookrightarrow \mathbf{B}_{\mathrm{st},E} \otimes_E V$, denoted by $\tilde{\pi}_i$, is in $\mathrm{Hom}_{G_{\mathbf{Q}_p}}(V, \mathbf{B}_{\mathrm{st},E})$. In fact, $\{\tilde{\pi}_1, \cdots, \tilde{\pi}_n\}$ is a basis of $\mathrm{Hom}_{G_{\mathbf{Q}_p}}(V, \mathbf{B}_{\mathrm{st},E})$. Now the condition $\tilde{\pi}_i(c) = 0$ for $i = 1, \cdots, n$ ensures that the image of c in $H^1(\mathbf{B}_{\mathrm{st},E} \otimes_E V)$ is zero. By (a) we obtain $c = 0$. $\square$

Remark 2.8. If V is semistable with Hodge weights ≥ 0, then the natural map

$$H^1(V) \to H^1(\mathbf{X}_{\mathrm{st}}(\mathbf{D}_{\mathrm{st}}(V)))$$

is an isomorphism.

Proof. By (2.1) we have a short exact sequence

$$0 \longrightarrow V \longrightarrow \mathbf{X}_{\mathrm{st}}(\mathbf{D}_{\mathrm{st}}(V)) \longrightarrow \mathbf{X}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}(V)) \longrightarrow 0,$$

from which we obtain an exact sequence

$$H^0(\mathbf{X}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}(V))) \longrightarrow H^1(V) \longrightarrow H^1(\mathbf{X}_{\mathrm{st}}(\mathbf{D}_{\mathrm{st}}(V))) \longrightarrow H^1(\mathbf{X}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}(V))).$$

As V is of Hodge weight ≥ 0, we have $H^0(\mathbf{X}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}(V))) = H^1(\mathbf{X}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}(V))) = 0$. $\square$

3 Triangulations and refinements

We recall the theory of triangulations and refinements [1, 2, 4, 10].

If S is an affinoid E-algebra, by an *S-representation* of $G_{\mathbf{Q}_p}$ we mean a locally free S-module of finite constant rank equipped with a continuous S-linear action of $G_{\mathbf{Q}_p}$. Let $\mathscr{R}_S$ be the Robba ring over S which is a topological ring equipped with continuous actions of φ and Γ [21]. By a *(locally) free (φ,Γ)-module* over $\mathscr{R}_S$ we mean a (locally) free $\mathscr{R}_S$-module $\mathcal{M}$ of finite constant rank equipped with a semilinear action of φ such that the map $\varphi^*\mathcal{M} \to \mathcal{M}$ is an isomorphism, and a semilinear action of Γ that commutes with the φ-action and is continuous for the profinite topology on Γ and the topology on $\mathscr{R}_S$. We always consider an S-representation of $G_{\mathbf{Q}_p}$ as a family of E-representations of $G_{\mathbf{Q}_p}$ over $\mathrm{Max}(S)$, and consider an étale (φ,Γ)-module over $\mathscr{R}_S$ as a family of étale (φ,Γ)-modules over $\mathrm{Max}(S)$.

Basing on Berger and Colmez's work [6], in [21] Kedlaya and Liu defined a functor $\mathbf{D}_{\mathrm{rig}}$ from the category of S-representations of $G_{\mathbf{Q}_p}$ to the category of étale (φ,Γ)-modules over $\mathscr{R}_S$. See [21, Definition 6.3] for the notion of étale (φ,Γ)-modules over $\mathscr{R}_S$.

We recall the construction of (φ,Γ)-modules over $\mathscr{R}_S$ of rank 1, which play the important role in the definition of triangulations below. If δ is a continuous $S^\times$-valued character of $\mathbf{Q}_p^\times$, we let $\mathscr{R}_S(\delta)$ denote the rank one (φ,Γ)-module over $\mathscr{R}_S$, defined by $\mathscr{R}_S(\delta) = \mathscr{R}_S e$ with $\gamma(e) = \delta(\chi_{\mathrm{cyc}}(\gamma))e$ and $\varphi(e) = \delta(p)e$. By [22, Appendix] every (φ,Γ)-module over $\mathscr{R}_S$ of rank 1 is of this form. Let $\log(\delta|_{\mathbf{Z}_p^\times})$ be the logarithmic of $\delta|_{\mathbf{Z}_p^\times}$, which is an additive character of $\mathbf{Q}_p^\times$ with values in S and whose value at p is zero. There exists $w_\delta \in S$ such that $\log(\delta|_{\mathbf{Z}_p^\times}) = w_\delta \psi_2$. We call w_δ the *weight* (*function*) of δ. For any $z \in \mathrm{Max}(S)$, if $\mathscr{R}_S(\delta_z)$ corresponds to a semistable E_z-representation V_z of $G_{\mathbf{Q}_p}$, then the Hodge weight of V_z is $-w_\delta(z)$.

Definition 3.1. ([22]) Let $\mathcal{M}$ be a free (φ, Γ)-module over $\mathscr{R}_S$ of rank n. If there are

- a strictly increasing filtration

$$\{0\} = \mathrm{Fil}_0 D \subset \mathrm{Fil}_1 D \subset \cdots \subset \mathrm{Fil}_n D = D$$

  of saturated free $\mathscr{R}_S$-submodule stable by φ and Γ, and
- n continuous characters $\delta_i : \mathbf{Q}_p^\times \to S^\times$

such that for any $i = 1, \cdots, n$,

$$\mathrm{Fil}_i \mathcal{M}/\mathrm{Fil}_{i-1}\mathcal{M} \simeq \mathscr{R}_S(\delta_i),$$

we say that $\mathcal{M}$ is *triangulable*; we call Fil a *triangulation* of $\mathcal{M}$ and

$$(\delta_1, \cdots, \delta_n)$$

the *triangulation parameters* attached to Fil.

To discuss the relation between triangulations and refinements, we restrict ourselves to the case of $S = E$.

Let $\mathcal{D}$ be a filtered E-(φ, N)-module of rank n, and we assume that all the eigenvalues of $\varphi : \mathcal{D} \to \mathcal{D}$ are in E. Following Mazur [25] we define a *refinement* of $\mathcal{D}$ to be a filtration on $\mathcal{D}$

$$0 = \mathcal{F}_0\mathcal{D} \subset \mathcal{F}_1\mathcal{D} \subset \cdots \subset \mathcal{F}_n\mathcal{D} = \mathcal{D}$$

by E-subspaces stable by φ and N, such that each factor $\mathrm{gr}_i^{\mathcal{F}}\mathcal{D} = \mathcal{F}_i\mathcal{D}/\mathcal{F}_{i-1}\mathcal{D}$ ($i = 1, \cdots, n$) is of dimension 1. Any refinement fixes an ordering $\alpha_1, \cdots, \alpha_n$ of eigenvalues of φ and an ordering $k_1, \cdots, k_n$ of Hodge weights of $\mathcal{D}$ taken with multiplicities such that the eigenvalue of φ on $\mathrm{gr}_i^{\mathcal{F}}\mathcal{D}$ is α_i and the Hodge weight of $\mathrm{gr}_i^{\mathcal{F}}\mathcal{D}$ is k_i.

Proposition 3.2. *([2, Proposition 1.3.2]) Let $\mathcal{M}$ be a (φ, Γ)-module over $\mathscr{R}_E$ coming from a filtered E-(φ, N)-module $\mathcal{D}$ of dimension n via Berger's functor [5].*

(a) *The equivalence of categories between the category of semistable (φ,Γ)-modules and the category of filtered E-(φ,N)-modules induces a bijection between the set of triangulations on $\mathcal{M}$ and the set of refinements on $\mathcal{D}$.*

(b) *If $(\mathrm{Fil}_i\mathcal{M})$ is a triangulation of $\mathcal{M}$ corresponding to a refinement $(\mathcal{F}_i\mathcal{D})$ of $\mathcal{D}$ with the ordering of the eigenvalues of φ being $\alpha_1,\cdots,\alpha_n$ and the ordering of Hodge weights being $k_1,\cdots,k_n$, then for each $i=1,\cdots,n$, $\mathrm{Fil}_i\mathcal{M}/\mathrm{Fil}_{i-1}\mathcal{M}$ is isomorphic to $\mathscr{R}_E(\delta_i)$ where δ_i is defined by $\delta_i(p)=\alpha_i p^{-k_i}$ and $\delta_i(u)=u^{-k_i}$ ($u\in\mathbf{Z}_p^\times$).*

Remark 3.3. In [27] the author gave a family version of Berger's functor from the category of filtered (φ,N)-modules to the category of (φ,Γ)-modules [5]. Using this functor we may obtain a family version of Proposition 3.2. We omit the details since we will not use it.

4 Critical indices and $\mathcal{L}$-invariants

Let D be a filtered E-(φ,N)-module of rank n. Suppose that φ is semisimple on D. Fix a refinement $\mathcal{F}$ of D. Then $\mathcal{F}$ fixes an ordering $\alpha_1,\cdots,\alpha_n$ of the eigenvalues of φ and an ordering $k_1,\cdots,k_n$ of the Hodge weights.

4.1 The operator $N_{\mathcal{F}}$ and critical indices

We define an E-linear operator $N_{\mathcal{F}}$ on $\mathrm{gr}^{\mathcal{F}}_{\bullet}D=\bigoplus_{i=1}^{n}\mathcal{F}_iD/\mathcal{F}_{i-1}D$. For any $i\in\{1,\cdots,n\}$, if $N(\mathcal{F}_iD)=N(\mathcal{F}_{i-1}D)$, we demand that $N_{\mathcal{F}}$ maps $\mathrm{gr}^{\mathcal{F}}_iD$ to zero.

Now we assume that $N(\mathcal{F}_iD)\supsetneq N(\mathcal{F}_{i-1}D)$. Let j be the minimal integer such that

$$N(\mathcal{F}_iD)\subseteq N(\mathcal{F}_{i-1}D)+\mathcal{F}_jD.$$

Lemma 4.1. *We have $N(\mathcal{F}_{i-1}D)\cap\mathcal{F}_jD=N(\mathcal{F}_{i-1}D)\cap\mathcal{F}_{j-1}D$.*

Proof. If $N(\mathcal{F}_{i-1}D)\cap\mathcal{F}_jD\supsetneq N(\mathcal{F}_{i-1}D)\cap\mathcal{F}_{j-1}D$, then there exists $x\in\mathcal{F}_{i-1}D$ such that $N(x)$ is in $\mathcal{F}_jD$ but not in $\mathcal{F}_{j-1}D$. Then $\mathcal{F}_jD=E\cdot N(x)\oplus\mathcal{F}_{j-1}D$. Thus for any $y\in\mathcal{F}_iD$ we have

$$N(y)\subseteq N(\mathcal{F}_{i-1}D)+\mathcal{F}_jD\subseteq N(\mathcal{F}_{i-1}D)+E\cdot N(x)+\mathcal{F}_{j-1}D\subseteq N(\mathcal{F}_{i-1}D)+\mathcal{F}_{j-1}D.$$

So $N(\mathcal{F}_iD)\subseteq N(\mathcal{F}_{i-1}D)+\mathcal{F}_{j-1}D$ which contradicts the minimality of j. $\square$

For any $x \in \mathcal{F}_i D$, if we write $N(x)$ in the form $N(x) = a + z$ with $a \in N(\mathcal{F}_{i-1}D)$ and $z \in \mathcal{F}_j D$, then $z \bmod \mathcal{F}_{j-1}D$ is uniquely determined. Indeed, if $N(x) = a' + z'$ is another expression with $a' \in N(\mathcal{F}_{i-1}D)$ and $z' \in \mathcal{F}_j D$, then by Lemma 4.1 we have

$$z - z' = a' - a \in N(\mathcal{F}_{i-1}D) \cap \mathcal{F}_j D = N(\mathcal{F}_{i-1}D) \cap \mathcal{F}_{j-1}D \subseteq \mathcal{F}_{j-1}D.$$

We define

$$N_{\mathcal{F}}(x + \mathcal{F}_{i-1}D) = z + \mathcal{F}_{j-1}D \in \mathrm{gr}_j^{\mathcal{F}} D.$$

It is easy to check that

$$N_{\mathcal{F}}(\lambda(x + \mathcal{F}_{i-1}D)) = \lambda N_{\mathcal{F}}(x + \mathcal{F}_{j-1}D), \quad \lambda \in E.$$

Finally we extend $N_{\mathcal{F}}$ to the whole $\mathrm{gr}_{\bullet}^{\mathcal{F}} D$ by E-linearity. By definition we have either $N(\mathrm{gr}_i^{\mathcal{F}} D) = 0$ or $N(\mathrm{gr}_i^{\mathcal{F}} D) = \mathrm{gr}_j^{\mathcal{F}} D$ for some j.

Definition 4.2. For $j \in \{1, \cdots, n-1\}$ we say that j is *critical* (or a *critical index*) for $\mathcal{F}$ if there is some $i \in \{2, \cdots, n\}$ such that $N_{\mathcal{F}}(\mathrm{gr}_i^{\mathcal{F}} D) = \mathrm{gr}_j^{\mathcal{F}} D$.

Note that i and j in the above definition are determined by each other. We write $i = t_{\mathcal{F}}(j)$ and $j = s_{\mathcal{F}}(i)$.

Remark 4.3. We can construct an oriented graph whose vertices are the numbers $1, \cdots, n$; there is an (oriented) edge with source j and terminate i if and only if j is critical and $i = t_{\mathcal{F}}(j)$. The resulting graph consists of simple vertices and disjointed chains.

Lemma 4.4. *The following are equivalent:*

(a) *s is critical and $t = t_{\mathcal{F}}(s)$.*

(b) *$N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D = N\mathcal{F}_{t-1}D \cap \mathcal{F}_{s-1}D$ and $N\mathcal{F}_t D \cap \mathcal{F}_s D \supsetneqq N\mathcal{F}_t D \cap \mathcal{F}_{s-1}D$.*

Proof. We have already seen that, if (a) holds, then (b) holds. Conversely, we assume that (b) holds. Then $N\mathcal{F}_t D \cap \mathcal{F}_s D \supsetneqq N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D$. Thus $N\mathcal{F}_t D \supsetneqq N\mathcal{F}_{t-1}D$. From $N\mathcal{F}_t D \cap \mathcal{F}_s D \supsetneqq N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D$ we see that there is $x \in \mathcal{F}_t D \backslash \mathcal{F}_{t-1}D$ such that $N(x) \in \mathcal{F}_s D$. Thus $N\mathcal{F}_t D \subseteq N\mathcal{F}_{t-1}D + \mathcal{F}_s D$. Next we show that $N\mathcal{F}_t D \subsetneqq N\mathcal{F}_{t-1}D + \mathcal{F}_{s-1}D$. If it is not true, then there exists $y \in \mathcal{F}_t D \backslash \mathcal{F}_{t-1}D$ such that $N(y) \in \mathcal{F}_{s-1}D$. For any $z \in \mathcal{F}_t D$, write $z = w + \lambda y$ with $w \in \mathcal{F}_{t-1}D$ and $\lambda \in E$. If $N(z)$ is in $\mathcal{F}_s D$, then $N(w)$ is also in $\mathcal{F}_s D$. But $N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D = N\mathcal{F}_{t-1}D \cap \mathcal{F}_{s-1}D$. Thus $N(w)$ is in $\mathcal{F}_{s-1}D$, which implies that $N(z) = N(w) + \lambda N(y)$ is also in $\mathcal{F}_{s-1}D$. So, $N\mathcal{F}_t D \cap \mathcal{F}_s D = N\mathcal{F}_t D \cap \mathcal{F}_{s-1}D$, a contradiction. $\square$

Definition 4.5. We say that an ordered basis $S = \{e_1, \cdots, e_n\}$ of D is *compatible* with $\mathcal{F}$ if $\mathcal{F}_r D = \oplus_{i=1}^r E e_i$ for all $r \in \{1, \cdots, n\}$. If $S = \{e_1, \cdots, e_n\}$ is an ordered basis compatible with $\mathcal{F}$ and $\varphi(e_i) = \alpha_i e_i$ for any $i \in \{1, \cdots, n\}$, we say that S is *perfect* for $\mathcal{F}$.

As φ is semisimple on D, there always exists a perfect ordered basis for $\mathcal{F}$.

Lemma 4.6. (a) *If s is critical for $\mathcal{F}$ and $t = t_{\mathcal{F}}(s)$, then there exists $e_t \in \mathcal{F}_t D \backslash \mathcal{F}_{t-1} D$ such that $\varphi(e_t) = \alpha_t e_t$ and $N(e_t) \in \mathcal{F}_s D \backslash \mathcal{F}_{s-1} D$.*

(b) *If t is not $t_{\mathcal{F}}(s)$ for any s, then there exists $e_t \in \mathcal{F}_t D \backslash \mathcal{F}_{t-1} D$ such that $\varphi(e_t) = \alpha_t e_t$ and $N(e_t) = 0$.*

Proof. Let $\{e'_1, \cdots, e'_n\}$ be a perfect basis for $\mathcal{F}$.

If $t = t_{\mathcal{F}}(s)$, then there exists $x \in \mathcal{F}_t D \backslash \mathcal{F}_{t-1} D$ such that $N(x) \in \mathcal{F}_s D \backslash \mathcal{F}_{s-1} D$. Write $x = \sum_{i=1}^t \lambda_i e'_i$ and put $e_t = \sum_{1 \leq i \leq t : \alpha_i = \alpha_t} \lambda_i e'_i$. Then $\varphi(e_t) = \alpha_t e_t$ and $N(e_t) \in \mathcal{F}_s D \backslash \mathcal{F}_{s-1} D$. This proves (a). The proof of (b) is similar. □

4.2 Strongly critical indices and $\mathcal{L}$-invariants

Assume that s is critical for $\mathcal{F}$ and $t = t_{\mathcal{F}}(s)$. We consider the decompositions

$$\mathcal{F}_t D / \mathcal{F}_{s-1} D = E \bar{e}_s \oplus L \oplus E \bar{e}_t$$

that satisfy the following conditions:

• $\overline{\mathcal{F}}_1(\mathcal{F}_t D / \mathcal{F}_{s-1} D) = E \bar{e}_s$ and $\overline{\mathcal{F}}_{t-s}(\mathcal{F}_t D / \mathcal{F}_{s-1} D) = E \bar{e}_s \oplus L$, where $\overline{\mathcal{F}}$ is the refinement on $\mathcal{F}_t D / \mathcal{F}_{s-1} D$ induced by $\mathcal{F}$.

• Both L and $E \bar{e}_s \oplus E \bar{e}_t$ are stable by φ and N; $\varphi(\bar{e}_t) = \alpha_t \bar{e}_t$ and $N(\bar{e}_t) = \bar{e}_s$.

Such a decomposition is called an *s-decomposition.*

Lemma 4.7. *If s is critical, then there exists at least one s-decomposition.*

Proof. By Lemma 4.6 there exists a perfect basis $\{e_1, \cdots, e_n\}$ for $\mathcal{F}$ such that $N(e_t) = e_s$. For $i = s, \cdots, t$ let $\bar{e}_i$ denote the image of e_i in $\mathcal{F}_t D / \mathcal{F}_{s-1} D$. Then $N(\bar{e}_t) = \bar{e}_s$. Write $\tilde{L} = \mathcal{F}_{t-1} D / \mathcal{F}_{s-1} D$. For any $\alpha \in E$ put $\tilde{L}^\alpha = \{x \in \tilde{L} : \varphi(x) = \alpha x\}$. As $N \mathcal{F}_{t-1} D \cap \mathcal{F}_s D = N \mathcal{F}_{t-1} D \cap \mathcal{F}_{s-1} D$, we have $N \tilde{L}^{\alpha_t} \cap E \bar{e}_s = 0$. Let L^{α_s} be any E-subspace of $\tilde{L}^{\alpha_s}$ of codimension 1 that contains $N \tilde{L}^{\alpha_t}$ and does not contain $E \bar{e}_s$. Put $L = (\bigoplus_{\alpha \neq \alpha_s} \tilde{L}^\alpha) \bigoplus L^{\alpha_s}$. It is easy to verify that L is stable by φ and N. Then $\mathcal{F}_t D / \mathcal{F}_{s-1} D = E \bar{e}_s \oplus L \oplus E \bar{e}_t$ is an s-decomposition. □

Let dec denote an s-decomposition $\mathcal{F}_tD/\mathcal{F}_{s-1}D = E\bar{e}_s \oplus L \oplus E\bar{e}_t$. There are three possibilities for the filtration on the filtered E-(φ, N)-submodule $E\bar{e}_s \oplus E\bar{e}_t$:

Case 1. There exist an integer $k'_t > k_s$ and some $\mathcal{L}_{\mathrm{dec}} \in E$ (which must be unique) such that

$$\mathrm{Fil}^i(E\bar{e}_s \oplus E\bar{e}_t) = \begin{cases} E\bar{e}_s \oplus E\bar{e}_t & \text{if } i \leq k_s, \\ E(\bar{e}_t + \mathcal{L}_{\mathrm{dec}}\bar{e}_s) & \text{if } k_s < i \leq k'_t, \\ 0 & \text{if } i > k'_t. \end{cases}$$

Case 2. There exists an integer $k'_t < k_s$ such that

$$\mathrm{Fil}^i(E\bar{e}_s \oplus E\bar{e}_t) = \begin{cases} E\bar{e}_s \oplus E\bar{e}_t & \text{if } i \leq k'_t, \\ E\bar{e}_s & \text{if } k'_t < i \leq k_s, \\ 0 & \text{if } i > k_s. \end{cases}$$

Case 3. We have

$$\mathrm{Fil}^i(E\bar{e}_s \oplus E\bar{e}_t) = \begin{cases} E\bar{e}_s \oplus E\bar{e}_t & \text{if } i \leq k_s, \\ 0 & \text{if } i > k_s. \end{cases}$$

Similarly, there are three possibilities for the filtration on the quotient of $\mathcal{F}_tD/\mathcal{F}_{s-1}D$ by L. Below we will denote the images of $\bar{e}_s$ and $\bar{e}_t$ in $(\mathcal{F}_tD/\mathcal{F}_{s-1}D)/L$ by the original notations $\bar{e}_s$ and $\bar{e}_t$.

Case 1′. There exist an integer $k'_s < k_t$ and some $\mathcal{L}'_{\mathrm{dec}} \in E$ (which must be unique) such that

$$\mathrm{Fil}^i(E\bar{e}_s \oplus E\bar{e}_t) = \begin{cases} E\bar{e}_s \oplus E\bar{e}_t & \text{if } i \leq k'_s, \\ E(\bar{e}_t + \mathcal{L}'_{\mathrm{dec}}\bar{e}_s) & \text{if } k'_s < i \leq k_t, \\ 0 & \text{if } i > k_t. \end{cases}$$

Case 2′. There exists an integer $k'_s > k_t$ such that

$$\mathrm{Fil}^i(E\bar{e}_s \oplus E\bar{e}_t) = \begin{cases} E\bar{e}_s \oplus E\bar{e}_t & \text{if } i \leq k_t, \\ E\bar{e}_s & \text{if } k_t < i \leq k'_s, \\ 0 & \text{if } i > k'_s. \end{cases}$$

Case 3′. We have

$$\mathrm{Fil}^i(E\bar{e}_s \oplus E\bar{e}_t) = \begin{cases} E\bar{e}_s \oplus E\bar{e}_t & \text{if } i \leq k_t, \\ 0 & \text{if } i > k_t. \end{cases}$$

If Case 1 and Case 1′ happen, we always have $k_s \leq k'_s$ and $k'_t \leq k_t$. If further $k'_s < k'_t$ (which happens only when $k_s < k_t$), we say that dec is a *perfect s-decomposition* (for $\mathcal{F}$). In this case we must have $\mathcal{L}_{\mathrm{dec}} = \mathcal{L}'_{\mathrm{dec}}$.

Definition 4.8. If there exists a perfect s-decomposition, we say that s is *strongly critical* (or a *strongly critical index*). In this case we attached to s an invariant $\mathcal{L}_{\mathrm{dec}}$, where dec is a perfect s-decomposition. Proposition 4.9 below tells us that $\mathcal{L}_{\mathrm{dec}}$ is independent of the choice of perfect s-decompositions. We denote it by $\mathcal{L}_{\mathcal{F},s}$ and call it the *Fontaine-Mazur $\mathcal{L}$-invariant* associated to $(\mathcal{F}, s)$.

In the case of $t = s+1$, s is strongly critical if and only if $k_s < k_t$.

Proposition 4.9. *If* dec_1 *and* dec_2 *are two perfect s-decompositions, then* $\mathcal{L}_{\mathrm{dec}_1} = \mathcal{L}_{\mathrm{dec}_2}$.

Proof. Without loss of generality we assume that $\bar{e}_s$ in the two perfect s-decompositions are same. Let $k_s^{(1)}$, $k_t^{(1)}$, $L^{(1)}$ and $\bar{e}_t^{(1)}$ (resp. $k_s^{(2)}$, $k_t^{(2)}$, $L^{(2)}$ and $\bar{e}_t^{(2)}$) denote k'_s, k'_t, L and $\bar{e}_t$ for dec_1 (resp. dec_2). We assume that $k_s^{(1)} \leq k_s^{(2)}$.

As $N(\bar{e}_t^{(2)} - \bar{e}_t^{(1)}) = 0$, we have

$$\bar{e}_t^{(2)} - \bar{e}_t^{(1)} \in (\mathcal{F}_{t-1}D/\mathcal{F}_{s-1}D)^{\varphi=\alpha_t} = (L^{(1)})^{\varphi=\alpha_t}.$$

Thus $\bar{e}_t^{(2)}$ and $\bar{e}_t^{(1)}$ have the same image in $(\mathcal{F}_tD/\mathcal{F}_{s-1}D)/L^{(1)}$. We will denote the images of e_s, $\bar{e}_t^{(1)}$ and $\bar{e}_t^{(2)}$ in $(\mathcal{F}_tD/\mathcal{F}_{s-1}D)/L^{(1)}$ by the original notations.

As $\bar{e}_t^{(2)} + \mathcal{L}_{\mathrm{dec}_2}\bar{e}_s$ is in $\mathrm{Fil}^{k_t^{(2)}}(\mathcal{F}_tD/\mathcal{F}_{s-1}D)$, and as the Hodge weights $k_s^{(1)}$ and k_t of $(\mathcal{F}_tD/\mathcal{F}_{s-1}D)/L^{(1)}$ satisfy $k_s^{(1)} \leq k_s^{(2)} < k_t^{(2)} \leq k_t$, we have

$$\begin{aligned}\mathrm{Fil}^{k_t}\Big((\mathcal{F}_tD/\mathcal{F}_{s-1}D)/L^{(1)}\Big) &= \mathrm{Fil}^{k_t^{(2)}}\Big((\mathcal{F}_tD/\mathcal{F}_{s-1}D)/L^{(1)}\Big) \\ &= E(\bar{e}_t^{(2)} + \mathcal{L}_{\mathrm{dec}_2}\bar{e}_s) = E(\bar{e}_t^{(1)} + \mathcal{L}_{\mathrm{dec}_2}\bar{e}_s),\end{aligned}$$

which implies that $\mathcal{L}_{\mathrm{dec}_1} = \mathcal{L}_{\mathrm{dec}_2}$. □

Definition 4.10. Let s be strongly critical. We say that a perfect basis $\{e_1, \cdots, e_n\}$ for $\mathcal{F}$ is *s-perfect* if it satisfies the following conditions:

- $E\bar{e}_s \bigoplus \Big(\bigoplus_{s<i<t} E\bar{e}_i\Big) \bigoplus E\bar{e}_t$ is a perfect s-decomposition where $\bar{e}_i$ is the image of e_i in $D/\mathcal{F}_{s-1}D$,
- $N(e_t) = e_s$.
- For any $i > t_{\mathcal{F}}(s)$ writing $N(e_i) = \sum_{j=1}^{i-1} \lambda_{i,j}e_j$ we have $\lambda_{i,s} = 0$.

Remark 4.11. The first condition in Definition 4.10 implies that, for any $i = s+1, \cdots, t_{\mathcal{F}}(s) - 1$ if we write $N(e_i) = \sum_{j=1}^{i-1} \lambda_{i,j}e_j$, then $\lambda_{i,s} = 0$.

Lemma 4.12. *If s is strongly critical, then there exists an s-perfect basis.*

Proof. Write $t = t_{\mathcal{F}}(s)$. Let $\{e_1, \cdots, e_{s-1}\}$ be a perfect ordered basis of $\mathcal{F}_{r-1}D$.

As s is strongly critical, there exists a perfect s-decomposition $\mathcal{F}_tD/\mathcal{F}_{s-1}D = E\bar{e}_s \oplus L \oplus E\bar{e}_t$. Choose a perfect basis $\{\bar{e}_i : s < i < t\}$ for the induced refinement on L (identified with $\mathcal{F}_{t-1}D/\mathcal{F}_sD$). For $i \in \{s+1, \cdots, t\}$ let $e_i \in \mathcal{F}_tD$ be any lifting of $\bar{e}_i$ such that $\varphi(e_i) = \alpha_i e_i$. Put $e_s = N(e_t)$.

For any $i > t$ there exists $e'_i \in \mathcal{F}_iD \backslash \mathcal{F}_{i-1}D$ such that $\varphi(e'_i) = \alpha_i e'_i$. Next we define e_i for $i > t$ recursively from $t+1$ to n. Write $N(e'_i) = \sum_{j=1}^{i-1} \mu_{i,j} e_j$. If $\mu_{i,s} = 0$, we put $e_i = e'_i$. If $\mu_{i,s} \neq 0$, then $\alpha_i = \alpha_t$. In this case we put $e_i = e'_i - \mu_{i,s} e_t$. It is easy to prove the resulting ordered basis $\{e_1, \cdots, e_n\}$ is s-perfect. $\square$

4.3 Duality and strongly criticality

Let D be a filtered E-(φ, N)-module, with $\mathcal{F}$ a refinement on D.

Let D^* the filtered E-(φ, N)-module that is the dual of D. Let $\check{\mathcal{F}}$ be the refinement on D^* such that

$$\check{\mathcal{F}}_i D^* := (\mathcal{F}_{n-i}D)^{\perp} = \{y \in D^* : \langle y, x \rangle = 0 \text{ for all } x \in \mathcal{F}_{n-i}D\}.$$

We call $\check{\mathcal{F}}$ the *dual refinement* of $\mathcal{F}$.

Proposition 4.13. *If s is critical for $\mathcal{F}$ and $t = t_{\mathcal{F}}(s)$, then $n+1-t$ is critical for $\check{\mathcal{F}}$ and $n+1-s = t_{\check{\mathcal{F}}}(n+1-t)$.*

Proof. By Lemma 4.4 we only need to prove that

$$N\check{\mathcal{F}}_{n-s} \cap \check{\mathcal{F}}_{n+1-t} = N\check{\mathcal{F}}_{n-s} \cap \check{\mathcal{F}}_{n-t} \tag{4.1}$$

and

$$N\check{\mathcal{F}}_{n+1-s} \cap \check{\mathcal{F}}_{n+1-t} \supsetneq N\check{\mathcal{F}}_{n+1-s} \cap \check{\mathcal{F}}_{n-t}. \tag{4.2}$$

For (4.1) we have

$$\begin{aligned} N\check{\mathcal{F}}_{n-s} \cap \check{\mathcal{F}}_{n+1-t} &= \{N(y^*) : y^* \in \check{\mathcal{F}}_{n-s}, \langle N(y^*), x \rangle = 0 \ \forall \ x \in \mathcal{F}_{t-1}\} \\ &= \{N(y^*) : y^* \in \check{\mathcal{F}}_{n-s}, \langle y^*, N(x) \rangle = 0 \ \forall \ x \in \mathcal{F}_{t-1}\} \\ &= N((\mathcal{F}_s + N\mathcal{F}_{t-1})^{\perp}) \end{aligned}$$

and

$$N\check{\mathcal{F}}_{n-s} \cap \check{\mathcal{F}}_{n-t} = N((\mathcal{F}_s + N\mathcal{F}_t)^{\perp}).$$

Then (4.1) follows from the relation $\mathcal{F}_s + N\mathcal{F}_{t-1} = \mathcal{F}_s + N\mathcal{F}_t$.

For (4.2) we have

$$\begin{aligned}(N\check{\mathcal{F}}_{n+1-s})^{\perp} &= \{x \in D : \langle N(y^*), x\rangle = 0 \ \forall\ y \in \check{\mathcal{F}}_{n+1-s}\} \\ &= \{x \in D : \langle y^*, N(x)\rangle = 0 \ \forall\ y \in \check{\mathcal{F}}_{n+1-s}\} \\ &= \{x \in D : N(x) \in \mathcal{F}_{s-1}\}.\end{aligned}$$

Thus

$$N\check{\mathcal{F}}_{n+1-s} \cap \check{\mathcal{F}}_{n+1-t} = (\{x \in D : N(x) \in \mathcal{F}_{s-1}\} + \mathcal{F}_{t-1})^{\perp}$$

and

$$N\check{\mathcal{F}}_{n+1-s} \cap \check{\mathcal{F}}_{n-t} = (\{x \in D : N(x) \in \mathcal{F}_{s-1}\} + \mathcal{F}_{t})^{\perp}.$$

But $\{x \in D : N(x) \in \mathcal{F}_{s-1}\} + \mathcal{F}_t \supsetneq \{x \in D : N(x) \in \mathcal{F}_{s-1}\} + \mathcal{F}_{t-1}$. Indeed,

$$\begin{aligned}&(\{x \in D : N(x) \in \mathcal{F}_{s-1}\} + \mathcal{F}_t)/(\{x \in D : N(x) \in \mathcal{F}_{s-1}\} + \mathcal{F}_{t-1}) \\ &\cong \mathcal{F}_t/(\mathcal{F}_{t-1} + \{x \in \mathcal{F}_t : N(x) \in \mathcal{F}_{s-1}\}) = \mathcal{F}_t/\mathcal{F}_{t-1}.\end{aligned}$$

Thus (4.2) follows. $\square$

If $L \subset M$ are submodules of D, then $M^{\perp} \subset L^{\perp}$. The pairing $\langle \cdot, \cdot \rangle : L^{\perp} \times M$ induces a non-degenerate pairing on $L^{\perp}/M^{\perp} \times M/L$, so that we can identify $L^{\perp}/M^{\perp}$ with the dual of M/L naturally. In particular, $\mathrm{gr}_i^{\check{\mathcal{F}}} D^*$ is naturally isomorphic to the dual of $\mathrm{gr}_{n+1-i}^{\mathcal{F}} D$. Thus $\mathrm{gr}_{\bullet}^{\check{\mathcal{F}}} D^*$ is naturally isomorphic to the dual of $\mathrm{gr}_{\bullet}^{\mathcal{F}} D$.

Proposition 4.14. *$N_{\check{\mathcal{F}}}$ is dual to $-N_{\mathcal{F}}$.*

Proof. By Proposition 4.13, $N_{\mathcal{F}}(\mathrm{gr}_t^{\mathcal{F}} D) = \mathrm{gr}_s^{\mathcal{F}} D$ if and only if $N_{\check{\mathcal{F}}}(\mathrm{gr}_{n+1-s}^{\check{\mathcal{F}}} D^*) = \mathrm{gr}_{n+1-t}^{\check{\mathcal{F}}} D^*$. We choose a perfect basis $\{e_1, \cdots e_n\}$ for $\mathcal{F}$ such that $N(e_t) = e_s$. Then $\{e_i + \mathcal{F}_{i-1}D : i = 1, \cdots, n\}$ is a basis of $\mathrm{gr}_{\bullet}^{\mathcal{F}} D$, and its dual basis is $\{e_i^* + \check{\mathcal{F}}_{n-i}D : i = 1, \cdots, n\}$.

Note that $N_{\mathcal{F}}(e_t + \mathcal{F}_{t-1}D) = e_s + \mathcal{F}_{s-1}D$. What we need to show is that $N_{\check{\mathcal{F}}}(e_s^* + \check{\mathcal{F}}_{n-s}D^*) = -e_t^* + \check{\mathcal{F}}_{n-t}D^*$. For this we only need to prove that $Ne_s^* + e_t^*$ is in $\check{\mathcal{F}}_{n-t}D^* + N\check{\mathcal{F}}_{n-s}D^*$. We have

$$\begin{aligned}(\check{\mathcal{F}}_{n-t}D^* + N\check{\mathcal{F}}_{n-s}D^*)^{\perp} &= (\check{\mathcal{F}}_{n-t}D^*)^{\perp} \cap (N\check{\mathcal{F}}_{n-s}D^*)^{\perp} \\ &= \mathcal{F}_t D \cap \{x \in D : N(x) \in \mathcal{F}_s D\} \\ &= \{x \in \mathcal{F}_t D : N(x) \in \mathcal{F}_s D\}.\end{aligned}$$

For any $x \in \mathcal{F}_t D$ such that $N(x) \in \mathcal{F}_s D$, we can write x in the form $x = ae_t + y$ with $y \in \mathcal{F}_{t-1}D$. Then $\langle e_t^*, y\rangle = 0$. As $N(y) \in \mathcal{F}_s D \cap N\mathcal{F}_{t-1}D = \mathcal{F}_{s-1}D \cap N\mathcal{F}_{t-1}D$, we have $\langle e_s^*, N(y)\rangle = 0$. Hence

$$\langle Ne_s^* + e_t^*, x\rangle = \langle e_s^*, -N(ae_t + y)\rangle + \langle e_t^*, ae_t + y\rangle = 0,$$

as expected. $\square$

Let $\{e_1, \cdots, e_n\}$ be a perfect basis for $\mathcal{F}$, and let $\{e_1^*, \cdots, e_n^*\}$ be the dual basis of $\{e_1, \cdots, e_n\}$. Then $\{e_n^*, \cdots, e_1^*\}$ is perfect for $\check{\mathcal{F}}$.

Proposition 4.15. (a) *s is strongly critical for $\mathcal{F}$ if and only if $n+1-t_{\mathcal{F}}(s)$ is strongly critical for $\check{\mathcal{F}}$.*

(b) *If s is strongly critical for $\mathcal{F}$, then $\{e_1, \cdots, e_n\}$ is s-perfect for $\mathcal{F}$ if and only if $\{e_n^*, \cdots e_{s+1}^*, -e_s^*, e_{s-1}^*, \cdots, e_1^*\}$ is $(n+1-t_{\mathcal{F}}(s))$-perfect for $\check{\mathcal{F}}$.*

Proof. Assume that s is strongly critical for $\mathcal{F}$, $t = t_{\mathcal{F}}(s)$ and $\{e_1, \cdots, e_n\}$ is s-perfect for $\mathcal{F}$. Let $\bar{e}_i$ $(s \leq i \leq t)$ be the image of e_i in $\mathcal{F}_t D/\mathcal{F}_{s-1}D$, and put $L = \bigoplus_{s<i<t} E\bar{e}_i$. By the definition of s-perfect bases, $E\bar{e}_s \bigoplus L \bigoplus E\bar{e}_t$ is a perfect s-decomposition.

Similarly, let $\bar{e}_i^*$ $(s \leq i \leq t)$ be the image of e_i^* in $\check{\mathcal{F}}_{n+1-s}D^*/\check{\mathcal{F}}_{n-t}D^*$. Note that $\check{\mathcal{F}}_{n+1-s}D^*/\check{\mathcal{F}}_{n-t}D^*$ is naturally isomorphic to the dual of $\mathcal{F}_t D/\mathcal{F}_{s-1}D$. Put $\check{L} = \bigoplus_{s<i<t} E\bar{e}_i^*$. Then $\check{L} = (E\bar{e}_s \oplus E\bar{e}_t)^\perp$ and $L = (E\bar{e}_t^* \oplus E\bar{e}_s^*)^\perp$. Note that $E\bar{e}_t^* \oplus E\bar{e}_s^*$ is isomorphic to the dual of the quotient of $\mathcal{F}_t D/\mathcal{F}_s D$ by L, and the quotient of $\check{\mathcal{F}}_{n+1-s}D^*/\check{\mathcal{F}}_{n-t}D^*$ by $\check{L}$ is isomorphic to the dual of $Ee_s \oplus Ee_t$. Hence $E\bar{e}_t \oplus \check{L} \oplus E\bar{e}_s$ is an $(n+1-t_{\mathcal{F}}(s))$-perfect decomposition for $\check{\mathcal{F}}$. This proves (a).

For $i < s$, write $N(e_i^*) = \sum_{j=i+1}^n \lambda_{i,j} e_j^*$. Then

$$\lambda_{i,t} = \langle N(e_i^*), e_t\rangle = \langle e_i^*, -N(e_t)\rangle = \langle e_i^*, -e_s\rangle = 0.$$

Write $N(-e_s^*) = \sum_{j=t}^n \lambda_{s,j} e_j^*$. Then

$$\lambda_{s,j} = \langle N(-e_s^*), e_j\rangle = \langle e_s^*, N(e_j)\rangle = \begin{cases} 1 & \text{if } j = t, \\ 0 & \text{if } j > t. \end{cases}$$

Thus $N(-e_s^*) = e_t^*$. This proves (b). □

5 Galois cohomology of $V^* \otimes_E V$

5.1 A lemma

Let $\mathcal{L}$ be an element in E. Let D be a filtered E-(φ, N)-module $D = Ef_1 \oplus Ef_2 \oplus Ef_3$ with

$$\begin{aligned}
&\varphi(f_1) = p^{-1} f_1,\ \varphi(f_2) = f_2,\ \varphi(f_3) = f_3, \\
&N(f_1) = 0,\ N(f_2) = -f_1,\ N(f_3) = f_1, \\
&\mathrm{Fil}^0 D = E(f_2 - \mathcal{L} f_1) \oplus E(f_3 + \mathcal{L} f_1).
\end{aligned}$$

Let π_i be the projection map

$$\mathbf{X}_{\mathrm{st}}(D) \to \mathbf{B}_{\mathrm{st},E}, \quad \sum_{j=1}^{3} a_j f_j \mapsto a_j.$$

Lemma 5.1. *Let $c : G_{\mathbf{Q}_p} \to \mathbf{X}_{\mathrm{st}}(D)$ be a 1-cocycle whose class in $H^1(\mathbf{X}_{\mathrm{st}}(D))$ belongs to $\ker(H^1(\mathbf{X}_{\mathrm{st}}(D)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D)))$. Then there exist $\gamma_{2,1}, \gamma_{2,2}, \gamma_{3,1}, \gamma_{3,2} \in E$ such that $\pi_2(c) = \gamma_{2,1}\psi_1 + \gamma_{2,2}\psi_2$ and $\pi_3(c) = \gamma_{3,1}\psi_1 + \gamma_{3,2}\psi_2$. Furthermore, $\gamma_{2,1} - \gamma_{3,1} = \mathcal{L}(\gamma_{2,2} - \gamma_{3,2})$.*

The proof of Lemma 5.1 needs the Tate duality pairing $H^1(\mathbf{Q}_p) \times H^1(\mathbf{Q}_p(1)) \to H^2(\mathbf{Q}_p(1))$. We give a precise description of it following [12, §4.1].

Let $v \in \mathbf{B}_{\mathrm{cris}}^{\varphi=p}$ be such that $v/t_{\mathrm{cyc}} \in \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$ and $1/t_{\mathrm{cyc}}$ have the same image in $\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}$. Let u be the element of $\mathbf{B}_{\mathrm{st}}$ such that $u \in \mathrm{Fil}^1\mathbf{B}_{\mathrm{dR}}$, $\varphi(u) = pu$, $N(u) = -1$, and $\sigma(u) - u \in \mathbf{Q}_p\, t_{\mathrm{cyc}}$. Then $\sigma \mapsto \sigma(u) - u$ and $\sigma \mapsto \sigma(v) - v$ form an E-basis of $H^1(E(1))$. Let $(b_1, b_2) \in E \times E$ denote the 1-cocycle $\sigma \mapsto (\sigma - 1)(b_1 u + b_2 v)$. The E-representation corresponding to $(1, \ell)$ is attached to the filtered E-(φ, N)-module $(D_\ell = Ee \oplus Ef, \varphi, N, \mathrm{Fil})$ with

$$\varphi(e) = p^{-1}e, \quad \varphi(f) = f, \quad N(e) = 0, \quad N(f) = e$$

and

$$\mathrm{Fil}^j D_\ell = \begin{cases} D_\ell & \text{if } j \leq -1, \\ E \cdot (f + \ell e) & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

Let $H^1(E) \times H^1(E(1)) \to E$ be the pairing induced by the Tate duality pairing

$$H^1(\mathbf{Q}_p) \times H^1(\mathbf{Q}_p(1)) \to H^2(\mathbf{Q}_p(1))$$

and the isomorphism $H^2(\mathbf{Q}_p(1)) \cong \mathbf{Q}_p$ from local class field theory. Then precisely we have

$$< a_1\psi_1 + a_2\psi_2, (b_1, b_2) >= -a_1 b_1 + a_2 b_2.$$

Proof of Lemma 5.1. Write $c_\sigma = \lambda_{1,\sigma} f_1 + \lambda_{2,\sigma} f_2 + \lambda_{3,\sigma} f_3$. As c takes values in $\mathbf{X}_{\mathrm{st}}(D)$, we have $\lambda_{2,\sigma}, \lambda_{3,\sigma} \in E$, $\lambda_{1,\sigma} \in U_{1,1}$, and $N(\lambda_{1,\sigma}) = \lambda_{3,\sigma} - \lambda_{2,\sigma}$. This ensures the existence of $\gamma_{2,1}, \gamma_{2,2}, \gamma_{3,1}, \gamma_{3,2}$.

To show that $\gamma_{2,1} - \gamma_{3,1} = \mathcal{L}(\gamma_{2,2} - \gamma_{3,2})$, we define a new filtration Fil on D by

$$Fil^i(D) = \begin{cases} D & \text{if } i \leq -1, \\ E(f_2 - \mathcal{L}f_1) \oplus E(f_3 + \mathcal{L}f_1) & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases}$$

Then $Fil \approx \mathrm{Fil}$ and (D, Fil) is admissible. Let W be the semistable representation of $G_{\mathbf{Q}_p}$ attached to $D_W = (D, Fil)$.

As $Fil \approx \mathrm{Fil}$, by proposition 2.5, $[c]$ is in the kernel of $H^1(\mathbf{X}_{\mathrm{st}}(D_W)) \to H^1(\mathbf{X}_{\mathrm{dR}}(D_W))$. By the exact sequence

$$H^1(W) \longrightarrow H^1(\mathbf{X}_{\mathrm{st}}(D_W)) \longrightarrow H^1(\mathbf{X}_{\mathrm{dR}}(D_W))$$

there exists a 1-cocycle $c^{(1)} : G_{\mathbf{Q}_p} \to W$ such that the image of $[c^{(1)}]$ by $H^1(W) \to H^1(\mathbf{X}_{\mathrm{st}}(D_W))$ is $[c]$.

Observe that the filtered E-(φ, N)-submodule of D_W generated by f_1 (resp. by $f_2 + f_3$) is admissible and thus comes from an E-subrepresentation of W, denoted by W_0 (resp. W'). Let W_1 be the quotient of W by W', π_{W,W_1} the map $W \to W_1$. Then W_0 injects to W_1. The image of W_0 in W_1 is again denoted by W_0 by abuse of notation. The quotients of W and W_1 by W_0 are denoted by T and T_1 respectively. Then we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & W_0 \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, "\pi_{W,W_1}"] & T \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & W_0 \arrow[r] & W_1 \arrow[r] & T_1 \arrow[r] & 0, \end{tikzcd} \tag{5.1}$$

where the horizontal lines are exact.

We compute the image of $[c^{(1)}]$ by the map $H^1(W) \to H^1(T)$. Note that the action of $G_{\mathbf{Q}_p}$ on T is trivial. So we may identify T with D_T, the filtered E-(φ, N)-module attached to T. We consider the commutative diagram

$$\begin{tikzcd} H^1(W) \arrow[r] \arrow[d] & H^1(\mathbf{X}_{\mathrm{st}}(D_W)) \arrow[r] \arrow[d] & H^1(\mathbf{X}_{\mathrm{dR}}(D_W)) \arrow[d] \\ H^1(T) \arrow[r] & H^1(\mathbf{X}_{\mathrm{st}}(D_T)) \arrow[r] & H^1(\mathbf{X}_{\mathrm{dR}}(D_T)) \end{tikzcd}$$

where the horizontal lines are exact. As the image of $[c^{(1)}]$ in $H^1(\mathbf{X}_{\mathrm{st}}(D_W))$ is $[c]$, the image of $[c^{(1)}]$ in $H^1(\mathbf{X}_{\mathrm{st}}(D_T))$ by the map $H^1(W) \to H^1(T) \to H^1(\mathbf{X}_{\mathrm{st}}(D_T))$ coincides with the class of the 1-cocyle $\sigma \mapsto \lambda_{2,\sigma}\bar{f}_2 + \lambda_{3,\sigma}\bar{f}_3$, where $\bar{f}_2$ and $\bar{f}_3$ are the images of f_2 and f_3 in D_T respectively. As $H^1(T) \to H^1(\mathbf{X}_{\mathrm{st}}(D_T))$ is an isomorphism by Remark 2.8, the image of $[c]$ in $H^1(T)$ coincides with the class of $\sigma \mapsto \lambda_{2,\sigma}\bar{f}_2 + \lambda_{3,\sigma}\bar{f}_3$, where $\{\bar{f}_2, \bar{f}_3\}$ is considered as an E-basis of T.

Write $c^{(2)}$ for the 1-cocycle

$$G_{\mathbf{Q}_p} \xrightarrow{c^{(1)}} W \to T \to T_1.$$

As T_1 is the quotient of T by $E(\bar{f}_2+\bar{f}_3)$ we have

$$[c^{(2)}]=[(\lambda_2-\lambda_3)\bar{\bar{f}}_2]=[\Big((\gamma_{2,1}-\gamma_{3,1})\psi_1+(\gamma_{2,2}-\gamma_{3,2})\psi_2\Big)\bar{\bar{f}}_2]$$

where $\bar{\bar{f}}_2$ is the image of $\bar{f}_2$ in T_1.

From the diagram (5.1) we obtain the following commutative diagram

$$\begin{tikzcd} H^1(W) \arrow[r] \arrow[d, "\pi_{W,W_1}"] & H^1(T) \arrow[r] \arrow[d] & H^2(W_0) \arrow[d, equal] \\ H^1(W_1) \arrow[r] & H^1(T_1) \arrow[r] & H^2(W_0), \end{tikzcd}$$

where the horizontal lines are exact. Note that T_1 is isomorphic to E, and W_0 is isomorphic to $E(1)$. Being the image of $[\pi_{W,W_1}(c^{(1)})]$ in $H^1(T_1)$, $[c^{(2)}]$ lies in the kernel of $H^1(T_1)\to H^2(W_0)$. As an extension of E by $E(1)$, W_1 corresponds to the element $(1,\mathcal{L})\in H^1(E(1))$. So the map $H^1(T_1)\to H^2(W_0)=E$ is given by

$$(a\psi_1+b\psi_2)\bar{\bar{f}}_2\mapsto(a\psi+b\psi_2)\cup(1,\mathcal{L})=-a+b\ \mathcal{L}.$$

This implies that $\gamma_{2,1}-\gamma_{3,1}=\mathcal{L}(\gamma_{2,2}-\gamma_{3,2})$. □

5.2 1-cocycles with values in $V^*\otimes_E V$ and $\mathcal{L}$-invariants

Let D be a (not necessarily admissible) filtered E-(φ,N)-module with a refinement $\mathcal{F}$. Suppose that φ is semisimple on D. Let $\alpha_1,\cdots,\alpha_n$ be the ordering of eigenvalues of φ and let $k_1,\cdots,k_n$ be the ordering of Hodge weights fixed by $\mathcal{F}$. Let $s\in\{1,\cdots,n-1\}$ be strongly critical for $\mathcal{F}$ and $t=t_{\mathcal{F}}(s)$. Let $\{e_1,\cdots,e_n\}$ be an s-perfect basis for $\mathcal{F}$.

Let D^* be the filtered E-(φ,N)-module that is the dual of D, $\{e_1^*,\cdots,e_n^*\}$ the dual basis of $\{e_1,\cdots,e_n\}$. Let $\check{\mathcal{F}}$ be the dual refinement of $\mathcal{F}$. By Proposition 4.15, $n+1-t$ is strongly critical for $\check{\mathcal{F}}$, $t_{\check{\mathcal{F}}}(n+1-t)=n+1-s$, and $\{e_n^*,\cdots e_{s+1}^*,-e_s^*,e_{s-1}^*,\cdots,e_1^*\}$ is $(n+1-t)$-perfect for $\check{\mathcal{F}}$.

As $\{e_1,\cdots,e_n\}$ is s-perfect for $\mathcal{F}$,

$$(\bigoplus_{i<s}Ee_i)\bigoplus Ee_s\bigoplus Ee_t$$

is stable by φ and N, and let D_1 denote this filtered E-(φ,N)-submodule of D;

$$(\bigoplus_{i<s}Ee_i)\bigoplus(\bigoplus_{s<i<t}Ee_i)$$

is also stable by φ and N, and let $\overline{D}_2$ be the quotient of D by this filtered E-(φ, N)-submodule. Similarly, as $\{e_n^*, \cdots e_{s+1}^*, -e_s^*, e_{s-1}^*, \cdots, e_1^*\}$ is $(n+1-t)$-perfect for $\check{\mathcal{F}}$,

$$(\bigoplus_{j>t} Ee_j^*)\bigoplus(\bigoplus_{t>j>s} Ee_j^*)$$

is stable by φ and N. The quotient of D^* by this filtered E-(φ, N)-submodule is naturally isomorphic to the dual of D_1, so we write D_1^* for this quotient.

Put $I = \{s\} \cup \{i \in \mathbf{Z} : t \leq i \leq n\}$ and $J = \{t\} \cup \{j \in \mathbf{Z} : 1 \leq j \leq s\}$. By abuse of notations, let e_i $(i \in I)$ denote its image in $\overline{D}_2$; similarly let e_j^* $(j \in J)$ denote its image in D_1^*. Let $\mathscr{D}$ be filtered E-(φ, N)-module $D_1^* \otimes_E \overline{D}_2$. The image of $e_j^* \otimes e_i \in D^* \otimes_E D$ $(i \in I, j \in J)$ in $\mathscr{D}$ will be denoted by $e_j^* \otimes e_i$ again since this makes no confusion.

For $e \otimes e^* \in D_1 \otimes_E \overline{D}_2^* = \mathscr{D}^*$, let $\pi_{e\otimes e^*}$ be the $G_{\mathbf{Q}_p}$-equivariant map

$$\mathbf{X}_{\mathrm{st}}(\mathscr{D}) \to \mathbf{B}_{\mathrm{st},E} \quad x \mapsto < e \otimes e^*, x > .$$

We write $\pi_{j,i}$ $(i \in I, j \in J)$ for $\pi_{e_j \otimes e_i^*}$. Then $\pi_{j,i}$ is induced from the ($G_{\mathbf{Q}_p}$-equivariant) projection map

$$\mathbf{B}_{\mathrm{st},E} \otimes_E \mathscr{D} \to \mathbf{B}_{\mathrm{st},E}, \quad \sum_{h\in J}\sum_{\ell\in I} b_{h,\ell} e_h^* \otimes e_\ell \mapsto b_{j,i}.$$

The morphism $\pi_{j,i}$ $(i \in I, j \in J)$ induces a morphism $H^1(\mathbf{X}_{\mathrm{st}}(\mathscr{D})) \to H^1(\mathbf{B}_{\mathrm{st},E})$ again denoted by $\pi_{j,i}$.

Let μ_s be the minimal integer such that $N^{\mu_s+1}(e_s^* \otimes e_s) = 0$. We define μ_t similarly. By Lemma 2.1 (b) the image of $\pi_{s,s}$ is in $U_{\mu_s,0}$ and the image of $\pi_{t,t}$ is in $U_{\mu_t,0}$.

Theorem 5.2. *Let $c : G_{\mathbf{Q}_p} \to \mathbf{X}_{\mathrm{st}}(\mathscr{D})$ be a 1-cocycle.*

(a) *If $\pi_{j,s}([c]) = 0$ for any $j < s$ and if $\pi_{s,i}([c]) = 0$ for any $i \geq t$, then there exist $x_s \in U_{\mu_s,0}$ and $\gamma_{s,1}$, $\gamma_{s,2} \in E$ such that*

$$\pi_{s,s}(c_\sigma) = \gamma_{s,1}\psi_1(\sigma) + \gamma_{s,2}\psi_2(\sigma) + (\sigma - 1)x_s.$$

(b) *If $\pi_{j,t}([c]) = 0$ for any $j \leq s$ and if $\pi_{t,i}([c]) = 0$ for any $i > t$, then there exist $x_t \in U_{\mu_t,0}$ and $\gamma_{t,1}$, $\gamma_{t,2} \in E$ such that*

$$\pi_{t,t}(c_\sigma) = \gamma_{t,1}\psi_1(\sigma) + \gamma_{t,2}\psi_2(\sigma) + (\sigma - 1)x_t.$$

(c) *If the conditions in (a) and (b) hold, and if* $[c]$ *belongs to* $\ker(H^1(\mathbf{X}_{\mathrm{st}}(\mathscr{D})) \to H^1(\mathbf{X}_{\mathrm{dR}}(\mathscr{D})))$*, then*

$$\gamma_{s,1} - \gamma_{t,1} = \mathcal{L}_{\mathcal{F},s}(\gamma_{s,2} - \gamma_{t,2}),$$

where $\mathcal{L}_{\mathcal{F},s}$ *is the* $\mathcal{L}$*-invariant defined in Definition 4.8.*

Proof. By Remark 4.11 if $i < t$, then $< Ne_s^*, e_i >= - < e_s^*, Ne_i >= 0$. So Ne_s^* is an E-linear combination of e_i^* $(i \geq t)$. On the other hand, Ne_s is an E-linear combination of e_j $(j < s)$. Thus $N(e_s \otimes e_s^*)$ is an E-linear combination of $e_j \otimes e_s^*$ $(j < s)$ and $e_s \otimes e_i^*$ $(i \geq t)$. By Lemma 2.1 (a), $N \circ \pi_{s,s}$ is an E-linear combination of $\pi_{s,i}$ $(i \geq t)$ and $\pi_{j,s}$ $(j < s)$. If the condition in (a) holds, then $\tilde{\pi}_{s,s}([c]) \in H^1(U_{\mu_s,0})$ is contained in $\ker(H^1(U_{\mu_s,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E}))$.

Similarly, $N(e_t \otimes e_t^*)$ is an E-linear combination of $e_j \otimes e_t^*$ $(j \leq s)$ and $e_t \otimes e_i^*$ $(i > t)$. So $N \circ \pi_{t,t}$ is an E-linear combination of $\pi_{j,t}$ $(j \in J)$ and $\pi_{t,i}$ $(i > t)$. If the condition in (b) holds, then $\tilde{\pi}_{t,t}([c]) \in H^1(U_{\mu_t,0})$ is contained in $\ker(H^1(U_{\mu_t,0}) \xrightarrow{N} H^1(\mathbf{B}_{\mathrm{st},E}))$.

Now (a) and (b) follow from Proposition 1.3 and the fact that $H^1(E)$ is generated by ψ_1 and ψ_2.

Next we prove (c).

Let $\mathscr{D}_0$ be the filtered E-(φ, N)-submodule of $\mathscr{D}$ generated by $e_t^* \otimes e_s$, $e_s^* \otimes e_s$ and $e_t^* \otimes e_t$, and let $\mathscr{D}_1$ be the filtered E-(φ, N)-submodule of $\mathscr{D}$ generated by $\mathscr{D}_0$ and $e_s^* \otimes e_t$. As s is strongly critical for $\mathcal{F}$, there exist integers k_s' and k_t' satisfying $k_s \leq k_s' < k_t' \leq k_t$ such that the filtration of the filtered E-(φ, N)-submodule of $D/\mathcal{F}_{s-1}D$ spanned by e_s and e_t is given by

$$\mathrm{Fil}^i = \begin{cases} Ee_s \oplus Ee_t & \text{if } i \leq k_s, \\ E(e_t + \mathcal{L}_{\mathcal{F},s}e_s) & \text{if } k_s < i \leq k_t', \\ 0 & \text{if } i > k_t', \end{cases}$$

and the filtration of the filtered E-(φ, N)-submodule of $\overline{D}_2$ spanned by e_s and e_t is given by

$$\mathrm{Fil}^i = \begin{cases} Ee_s \oplus Ee_t & \text{if } i \leq k_s', \\ E(e_t + \mathcal{L}_{\mathcal{F},s}e_s) & \text{if } k_s' < i \leq k_t, \\ 0 & \text{if } i > k_t. \end{cases}$$

The dual of the former coincides with the filtered E-(φ, N)-submodule of D_1^* spanned by e_s^* and e_t^*, with the filtration given by

$$\mathrm{Fil}^i = \begin{cases} Ee_s^* \oplus Ee_t^* & \text{if } i \leq -k_t' \\ E(e_s^* - \mathcal{L}_{\mathcal{F},s}e_t^*) & \text{if } -k_t' < i \leq -k_s \\ 0 & \text{if } i > -k_s. \end{cases}$$

Therefore,

$$\mathrm{Fil}^0(\mathscr{D}_1) = Ee_t^* \otimes (e_t + \mathcal{L}_{\mathcal{F},s}e_s) \oplus E(e_s^* - \mathcal{L}_{\mathcal{F},s}e_t^*) \otimes e_s \oplus E(e_s^* - \mathcal{L}_{\mathcal{F},s}e_t^*) \otimes (e_t + \mathcal{L}_{\mathcal{F},s}e_s)$$

and

$$\mathrm{Fil}^0(\mathscr{D}_0) = E(e_t^* \otimes e_t + \mathcal{L}_{\mathcal{F},s}e_t^* \otimes e_s) \oplus E(e_s^* \otimes e_s - \mathcal{L}_{\mathcal{F},s}e_t^* \otimes e_s).$$

We will construct a 1-cocycle $c' : G_{\mathbf{Q}_p} \to \mathbf{X}_{\mathrm{st}}(\mathscr{D}_0)$ with $[c'] \in \ker(H^1(\mathbf{X}_{\mathrm{st}}(\mathscr{D}_0)) \to H^1(\mathbf{X}_{\mathrm{dR}}(\mathscr{D}_0)))$ such that $\tilde{\pi}_{s,s}([c']) = \tilde{\pi}_{s,s}([c])$ and $\tilde{\pi}_{t,t}([c']) = \tilde{\pi}_{t,t}([c])$.

As c takes values in $\mathbf{X}_{\mathrm{st}}(\mathscr{D})$, we have $\varphi(c_\sigma) = c_\sigma$ and $N(c_\sigma) = 0$. From $\varphi(c_\sigma) = c_\sigma$ we obtain

$$\varphi(\pi_{j,i}(c_\sigma)) = \alpha_i^{-1}\alpha_j\pi_{j,i}(c_\sigma)$$

for any $i \in I$ and $j \in J$. In particular, we have

$$\varphi(\pi_{s,s}(c_\sigma)) = \pi_{s,s}(c_\sigma),\ \varphi(\pi_{t,t}(c_\sigma)) = \pi_{t,t}(c_\sigma),\ \varphi(\pi_{t,s}(c_\sigma)) = p\ \pi_{t,s}(c_\sigma). \tag{5.2}$$

By Lemma 2.1 if

$$N(e_j \otimes e_i^*) = \sum_{(i',j')\in I\times J} \lambda_{j',i'}e_{j'} \otimes e_{i'}^*,$$

then

$$N(\pi_{j,i}(c_\sigma)) = \sum_{(i',j')\in I\times J} \lambda_{j',i'}\pi_{j',i'}(c_\sigma).$$

Since $N(e_t \otimes e_s^*) = e_s \otimes e_s^* - e_t \otimes e_t^*$, we have

$$N(\pi_{t,s}(c_\sigma)) = \pi_{s,s}(c_\sigma) - \pi_{t,t}(c_\sigma). \tag{5.3}$$

By Lemma 1.2 there exists some $y \in \mathbf{B}_{\mathrm{st},E}^{\varphi=p}$ such that $N(y) = x_s - x_t$. As φ commutes with $G_{\mathbf{Q}_p}$, we have $\sigma(y) \in \mathbf{B}_{\mathrm{st},E}^{\varphi=p}$ for any $\sigma \in G_{\mathbf{Q}_p}$. Let c' be the 1-cocycle with values in $\mathbf{B}_{\mathrm{st},E} \otimes_E \mathscr{D}_0$ defined by

$$\begin{aligned} c' : \sigma \ \ \mapsto \ \ & (\pi_{t,s}(c_\sigma) - (\sigma-1)y)e_t^* \otimes e_s \\ & + (\pi_{s,s}(c_\sigma) - (\sigma-1)x_s)e_s^* \otimes e_s + (\pi_{t,t}(c_\sigma) - (\sigma-1)x_t)e_t^* \otimes e_t. \end{aligned}$$

We show that c' takes values in $\mathbf{X}_{\mathrm{st}}(\mathscr{D}_0)$. What we need to check is that $\varphi(c'_\sigma) = c'_\sigma$ and $N(c'_\sigma) = 0$. By (a), (b) and the definition of c'_σ we have

$$\pi_{s,s}(c'_\sigma), \pi_{t,t}(c'_\sigma) \in E \subset \mathbf{B}_{\mathrm{st},E}^{\varphi=1,N=0}. \tag{5.4}$$

By (5.2), $\pi_{t,s}(c_\sigma)$ is in $\mathbf{B}_{\mathrm{st},E}^{\varphi=p}$. From this and the fact $(\sigma-1)y \in \mathbf{B}_{\mathrm{st},E}^{\varphi=p}$ we get

$$\pi_{t,s}(c'_\sigma) \in \mathbf{B}_{\mathrm{st},E}^{\varphi=p}. \tag{5.5}$$

From (5.3) and the fact $N(y) = x_s - x_t$ we obtain

$$N(\pi_{t,s}(c'_\sigma)) = \pi_{s,s}(c'_\sigma) - \pi_{t,t}(c'_\sigma). \tag{5.6}$$

Equalities (5.4), (5.5) and (5.6) ensure that $\varphi(c'_\sigma) = c'_\sigma$ and $N(c'_\sigma) = 0$ for any $\sigma \in G_{\mathbf{Q}_p}$.

From the definition of c' we see that $\tilde{\pi}_{s,s}([c']) = \tilde{\pi}_{s,s}([c])$ and $\tilde{\pi}_{t,t}([c']) = \tilde{\pi}_{t,t}([c])$. By Lemma 5.1 to finish our proof we only need to show that the image of $[c']$ in $H^1(\mathbf{X}_{\mathrm{dR}}(\mathscr{D}_0))$ is zero.

By Lemma 2.2 there exist $a_{i_1,i_2} \in E$ ($i_1, i_2 \in I$, $i_1 > i_2$) such that $f_i := e_i + \sum_{i' \in I, i' < i} a_{i,i'} e_{i'}$ ($i \in I$) form an E-basis of $\overline{D}_2$ compatible with the filtration on $\overline{D}_2$. Similarly, there exist $b_{j_1,j_2} \in E$ ($j_1, j_2 \in J$, $j_1 < j_2$) such that $g_j := e^*_j + \sum_{j' \in J, j' > j} b_{j,j'} e^*_{j'}$ ($j \in J$) form an E-basis of D^*_1 compatible with the filtration. Then $\{g_j \otimes f_i : i \in I, j \in J\}$ is an E-basis of $\mathscr{D}$ compatible with the filtration. Note that $a_{t,s} = -b_{s,t} = \mathcal{L}_{\mathcal{F},s}$ and

$$f_s = e_s, \quad f_t = e_t + \mathcal{L}_{\mathcal{F},s} e_s, \quad g_t = e^*_t, \quad g_s = e^*_s - \mathcal{L}_{\mathcal{F},s} e^*_t.$$

As a consequence, $\{g_t \otimes f_s, g_s \otimes f_s, g_t \otimes f_t\}$ is an E-basis of $\mathscr{D}_0$ compatible with the filtration.

Conversely, there are $\tilde{a}_{i_1,i_2}$ ($i_1, i_2 \in I$, $i_1 > i_2$) and $\tilde{b}_{j_1,j_2}$ ($j_1, j_2 \in J$, $j_1 < j_2$) in E such that $e_i = f_i + \sum_{i' \in I, i' < i} \tilde{a}_{i,i'} f_{i'}$ and $e^*_j = g_j + \sum_{j' \in J, j' > j} \tilde{b}_{j,j'} g_{j'}$. Note that $-\tilde{a}_{t,s} = \tilde{b}_{s,t} = \mathcal{L}_{\mathcal{F},s}$.

Expressing c in terms of the basis $\{g_j \otimes f_i : i \in I, j \in J\}$ we obtain

$$c = \sum_{i' \in I, j' \in J} (\pi_{j',i'}(c) + \sum_{i > i'} \tilde{a}_{i,i'} \pi_{j',i}(c) + \sum_{j < j'} \tilde{b}_{j,j'} \pi_{j,i'}(c) + \sum_{i > i', j < j'} \tilde{a}_{i,i'} \tilde{b}_{j,j'} \pi_{j,i}(c)) g_{j'} \otimes f_{i'}.$$

In particular, the coefficient of $g_t \otimes f_s$ is

$$\pi_{t,s}(c) + \sum_{i \geq t} \tilde{a}_{i,s} \pi_{t,i}(c) + \sum_{j \leq s} \tilde{b}_{j,t} \pi_{j,s}(c) + \sum_{i \geq t, j \leq s} \tilde{a}_{i,s} \tilde{b}_{j,t} \pi_{j,i}(c). \tag{5.7}$$

As the image of $[c]$ in $H^1(\mathbf{B}_{\mathrm{dR},E} \otimes_E \mathscr{D} / \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E \mathscr{D}))$ is zero, the image of the 1-cocycle (5.7) in $H^1(\mathbf{B}_{\mathrm{dR},E} / \mathrm{Fil}^{k'_t - k'_s} \mathbf{B}_{\mathrm{dR},E})$ is zero. As the images of $\pi_{t,i}(c)$ ($i > t$), $\pi_{j,s}(c)$ ($j < s$) and $\pi_{j,i}(c)$ ($i \geq t, j \leq s$) in $H^1(\mathbf{B}_{\mathrm{st},E})$ are zero, their images in $H^1(\mathbf{B}_{\mathrm{dR},E} / \mathrm{Fil}^{k'_t - k'_s} \mathbf{B}_{\mathrm{dR},E})$ are also zero. This implies that the image of the 1-cocycle

$$\pi_{t,s}(c) + \tilde{a}_{t,s} \pi_{t,t}(c) + \tilde{b}_{s,t} \pi_{s,s}(c)$$

in $H^1(\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{k'_t-k'_s}\mathbf{B}_{\mathrm{dR},E})$ is zero, and so is the image of the 1-cocycle

$$\pi_{t,s}(c') + \tilde{a}_{t,s}\pi_{t,t}(c') + \tilde{b}_{s,t}\pi_{s,s}(c').$$

Now

$$c' = (\pi_{t,s}(c') + \tilde{a}_{t,s}\pi_{t,t}(c') + \tilde{b}_{s,t}\pi_{s,s}(c'))g_t \otimes f_s + \pi_{s,s}(c')g_s \otimes f_s + \pi_{t,t}(c')g_t \otimes f_t.$$

Since $g_s \otimes f_s, g_t \otimes f_t \in \mathrm{Fil}^0\mathscr{D}_0$, the image of $[c']$ in $H^1(\mathbf{B}_{\mathrm{dR},E} \otimes_E \mathscr{D}_0/\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR},E} \otimes_E \mathscr{D}_0))$ is zero if and only if the image of the 1-cocycle $\pi_{t,s}(c') + \tilde{a}_{t,s}\pi_{t,t}(c') + \tilde{b}_{s,t}\pi_{s,s}(c')$ in $H^1(\mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^{k'_t-k'_s}\mathbf{B}_{\mathrm{dR},E})$ is zero, which is observed above. $\square$

Now let V be a semistable E-representation of $G_{\mathbf{Q}_p}$, D the associated filtered E-(φ, N)-module. Suppose that φ is semisimple on D and let $\mathcal{F}$ be a refinement on D. Assume that $s \in \{1, \cdots, n-1\}$ is strongly critical for $\mathcal{F}$, and $t = t_{\mathcal{F}}(s)$. Let $\{e_1, \cdots, e_n\}$ be an s-perfect basis for $\mathcal{F}$.

The composition of $V^* \otimes_E V \to \mathbf{X}_{\mathrm{st}}(D^* \otimes_E D)$ and

$$\pi_{j,i} : \mathbf{B}_{\mathrm{st},E} \otimes_E \mathscr{D} \to \mathbf{B}_{\mathrm{st},E}, \quad \sum_{h=1}^{n}\sum_{\ell=1}^{n} b_{h,\ell}e_h^* \otimes e_\ell \mapsto b_{j,i},$$

is again denoted by $\pi_{j,i}$, which is $G_{\mathbf{Q}_p}$-equivariant.

Corollary 5.3. *Let $c : G_{\mathbf{Q}_p} \to V^* \otimes_E V$ be a 1-cocycle. If $\pi_{j,i}([c]) = 0$ when $j < i$, then there are $x_s, x_t \in \mathbf{B}_{\mathrm{st},E}^{\varphi=1}$, and $\gamma_{s,1}, \gamma_{s,2}, \gamma_{t,1}, \gamma_{t,2} \in E$ such that*

$$\pi_{s,s}(c_\sigma) = \gamma_{s,1}\psi_1(\sigma) + \gamma_{s,2}\psi_2(\sigma) + (\sigma - 1)x_s$$

and

$$\pi_{t,t}(c_\sigma) = \gamma_{t,1}\psi_1(\sigma) + \gamma_{t,2}\psi_2(\sigma) + (\sigma - 1)x_t.$$

Furthermore $\gamma_{s,1} - \gamma_{t,1} = \mathcal{L}_{\mathcal{F},s}(\gamma_{s,2} - \gamma_{t,2})$.

Proof. We form the quotient $\mathscr{D}$ of $D^* \otimes_E D$ as at the beginning of this subsection. Then we have the following commutative diagram

$$\begin{tikzcd}
H^1(V^* \otimes_E V) \arrow[r] & H^1(\mathbf{X}_{\mathrm{st}}(D^* \otimes_E D)) \arrow[r] \arrow[d] & H^1(\mathbf{X}_{\mathrm{dR}}(D^* \otimes_E D)) \arrow[d] \\
 & H^1(\mathbf{X}_{\mathrm{st}}(\mathscr{D})) \arrow[r] & H^1(\mathbf{X}_{\mathrm{dR}}(\mathscr{D}))
\end{tikzcd}$$

where the upper horizontal line is exact, which implies that the image of $[c]$ in $H^1(\mathbf{X}_{\mathrm{st}}(\mathscr{D}))$ belongs to $\ker(H^1(\mathbf{X}_{\mathrm{st}}(\mathscr{D})) \to H^1(\mathbf{X}_{\mathrm{dR}}(\mathscr{D})))$. Hence the assertion follows from Theorem 5.2. $\square$

6 Projection vanishing property

We will attach to any infinitesimal deformation of a representation of $G_{\mathbf{Q}_p}$ i.e. an S-representation of $G_{\mathbf{Q}_p}$ a 1-cocycle, and show that, when the S-representation admits a triangulation and the residue representation is semistable, the corresponding 1-cocycle has the projection vanishing property. Here, $S = E[Z]/(Z^2)$.

Let $\mathcal{V}$ be an S-representation of $G_{\mathbf{Q}_p}$, $\mathcal{M} = \mathbf{D}_{\mathrm{rig}}(\mathcal{V})$. Suppose that $\mathcal{M}$ admits a triangulation $\mathrm{Fil}_\bullet$. Let $(\delta_1, \cdots, \delta_n)$ be the corresponding triangulation data.

Let z be the closed point defined by the maximal ideal (Z) of S. Suppose that $\mathcal{V}_z$, the evaluation of $\mathcal{V}$ at z, is semistable, and let D_z be the filtered E-(φ, N)-module attached to $\mathcal{V}_z$. Suppose that φ is semisimple on D_z. Let $\mathcal{F}$ be the refinement of D_z corresponding to the induced triangulation of $\mathcal{M}_z$. Let $\{e_{1,z}, \cdots, e_{n,z}\}$ be an ordered basis of D_z perfect for $\mathcal{F}$. Write $\varphi(e_{i,z}) = \alpha_{i,z} e_{i,z}$.

For $i = 1, \cdots, n$ there exists a continuous additive character ϵ_i of $\mathbf{Q}_p^\times$ with values in E such that $\delta_i = \delta_{i,z}(1 + Z\epsilon_i)$. By identifying Γ with $\mathbf{Z}_p^\times$ via χ_{cyc} we consider $\epsilon_i|_{\mathbf{Z}_p^\times}$ as a character of Γ or a character of $G_{\mathbf{Q}_p}$ that factors through Γ, denoted by ϵ'_i.

Fix an S-basis $\{v_1, \cdots, v_n\}$ of $\mathcal{V}$, and write the matrix of $\sigma \in G_{\mathbf{Q}_p}$ for this basis, B_σ, in the form

$$B_\sigma = (I_n + ZU_\sigma)A_\sigma \tag{6.1}$$

with $A_\sigma \in \mathrm{GL}_n(E)$ and $U_\sigma \in \mathrm{M}_n(E)$. Then $\{v_{1,z}, \cdots, v_{n,z}\}$ is an E-basis of $\mathcal{V}_z$, and A_σ is the matrix of σ for this basis. For any $\sigma \in G_{\mathbf{Q}_p}$ put

$$c_\sigma = \sum_{i,j} (U_\sigma)_{ij} v^*_{j,z} \otimes v_{i,z}.$$

Lemma 6.1. *$\sigma \mapsto c_\sigma$ is a 1-cocycle of $G_{\mathbf{Q}_p}$ with values in $\mathcal{V}_z^* \otimes_E \mathcal{V}_z$.*

Proof. From (6.1) we obtain $U_{\sigma\tau} = U_\sigma + A_\sigma U_\tau A_\sigma^{-1}$. In other words, for any $i, j \in \{1, \cdots, n\}$,

$$(U_{\sigma\tau})_{ij} = (U_\sigma)_{ij} + \sum_{h,\ell} (A_\sigma)_{ih} (U_\tau)_{h\ell} (A_\sigma^{-1})_{\ell j}.$$

Hence

$$\begin{aligned}
c_{\sigma\tau} &= \sum_{i,j} (U_{\sigma\tau})_{ij} v_{j,z}^* \otimes v_{i,z} \\
&= \sum_{i,j} \Big((U_\sigma)_{ij} + \sum_{h,\ell} (A_\sigma)_{ih} (U_\tau)_{h\ell} (A_\sigma^{-1})_{\ell j} \Big) v_{j,z}^* \otimes v_{i,z} \\
&= \sum_{i,j} (U_\sigma)_{ij} v_{j,z}^* \otimes v_{i,z} + \sum_{h\ell} (U_\tau)_{h\ell} \Big(\sum_j (A_\sigma^{-1})_{\ell j} v_{j,z}^* \Big) \otimes \Big(\sum_i (A_\sigma)_{ih} v_{i,z} \Big) \\
&= c_\sigma + \sum_{h\ell} (U_\tau)_{h\ell} (v_{\ell,z}^*)^\sigma \otimes (v_{h,z})^\sigma \\
&= c_\sigma + c_\tau^\sigma,
\end{aligned}$$

as desired. □

Let $x_{ij} \in \mathbf{B}_{\mathrm{st},E}$ $(i,j = 1, \cdots, n)$ be such that

$$e_{j,z} = x_{1j} v_{1,z} + \cdots + x_{nj} v_{n,z}. \tag{6.2}$$

Then $X = (x_{ij})$ is in $\mathrm{GL}_n(\mathbf{B}_{\mathrm{st},E})$. As $e_{1,z}, \cdots, e_{n,z}$ are fixed by $G_{\mathbf{Q}_p}$, we have $X^{-1} A_\sigma \sigma(X) = I_n$ for all $\sigma \in G_{\mathbf{Q}_p}$. For $j = 1, \cdots, n$ put $e_j = x_{1j} v_1 + \cdots + x_{nj} v_n$. Then $\{e_1, \cdots, e_n\}$ is a basis of $\mathbf{B}_{\mathrm{st},E} \otimes_E \mathcal{V}$ over $\mathbf{B}_{\mathrm{st},E} \otimes_E S$. (Note that $\mathbf{B}_{\mathrm{st},E} \widehat{\otimes}_E S = \mathbf{B}_{\mathrm{st},E} \otimes_E S$ and $\mathbf{B}_{\mathrm{st},E} \widehat{\otimes}_E \mathcal{V} = \mathbf{B}_{\mathrm{st},E} \otimes_E \mathcal{V}$.)

Lemma 6.2. *For $i = 1, \cdots, n$ we have $\varphi(e_i) = \alpha_{i,z} e_i$.*

Proof. As $v_{1,z}, \cdots, v_{n,z}$ are fixed by φ, from $\varphi(e_{j,z}) = \alpha_{j,z} e_{j,z}$ $(i = 1, \cdots, n)$ we obtain $\varphi(x_{ij}) = \alpha_{j,z} x_{ij}$ for any j. Thus $\varphi(e_j) = \sum_i \varphi(x_{ij}) v_i = \sum_i \alpha_{j,z} x_{ij} v_i = \alpha_{j,z} e_j$. □

The matrix of σ for the basis $\{e_1, \cdots, e_n\}$ is

$$X^{-1} B_\sigma \sigma(X) = I_n + Z X^{-1} U_\sigma X.$$

A simple computation shows that

$$c_\sigma = \sum_{i,j} (X^{-1} U_\sigma X)_{ij} e_{j,z}^* \otimes e_{i,z}.$$

Let $\pi_{h\ell}$ be the projection

$$\mathbf{B}_{\mathrm{st},E} \otimes_E (\mathcal{V}_z \otimes_E \mathcal{V}_z^*) \to \mathbf{B}_{\mathrm{st},E}, \quad \sum_{j,i} b_{ij} e_{j,z}^* \otimes e_{i,z} \mapsto b_{h\ell}. \tag{6.3}$$

Lemma 6.3. *Let δ'_i be the character $1+Z\epsilon'_i$. Then for $h=1,\cdots,n$ there is an element in*
$$[\mathbf{B}_{\mathrm{cris},E}^{\varphi=\prod_{i=1}^h(\alpha_{i,z}(1+Z\epsilon_i(p)))}\otimes_E(\wedge^h\mathcal{V})(\delta_1'^{-1}\cdots\delta_h'^{-1})]^{G_{\mathbf{Q}_p}}$$
denoted by $g_{1,\cdots,h}$, whose image in $\mathbf{B}_{\mathrm{st},E}\otimes_E\wedge^h\mathcal{V}_z$ is exactly $e_{1,z}\wedge\cdots\wedge e_{h,z}$.

Proof. Put $f_i=w_{\delta_i}$. By Proposition 3.2 we have $\alpha_{i,z}=\delta_{i,z}(p)p^{f_{i,z}}$ and $\delta_{i,z}(x)=x^{f_{i,z}}$ for any $x\in\mathbf{Z}_p^\times$.

For $i=1,\cdots,n$ let m_i be a nonzero element in $\mathrm{Fil}_i\mathcal{M}$ such that
$$\varphi(m_i)\equiv\delta_i(p)m_i\mod\mathrm{Fil}_{i-1}\mathcal{M}$$
and
$$\gamma(m_i)\equiv\delta_i(\chi_{\mathrm{cyc}}(\gamma))m_i\mod\mathrm{Fil}_{i-1}\mathcal{M}$$
for any $\gamma\in\Gamma$. Then $m_1\wedge\cdots\wedge m_h$ is a nonzero element in
$$(\wedge^h\mathcal{M})^{\varphi=(\delta_1\cdots\delta_h)(p),\Gamma=(\delta_1\cdots\delta_h)|_{\mathbf{Z}_p^\times}}.$$
Considered as an element in $(\wedge^h\mathcal{M})(\delta_1'^{-1}\cdots\delta_h'^{-1})[\frac{1}{t_{\mathrm{cyc}}}]$, $t_{\mathrm{cyc}}^{f_{1,z}+\cdots+f_{h,z}}m_1\wedge\cdots\wedge m_h$ is in
$$\begin{aligned}&[(\wedge^h\mathcal{M})(\delta_1'^{-1}\cdots\delta_h'^{-1})[\frac{1}{t_{\mathrm{cyc}}}]]^{\varphi=\prod_{i=1}^h(\alpha_{i,z}(1+Z\epsilon_i(p))),\Gamma=1}\\ =\;&\mathbf{D}_{\mathrm{cris}}((\wedge^h\mathcal{V})(\delta_1'^{-1}\cdots\delta_h'^{-1}))^{\varphi=\prod_{i=1}^h(\alpha_{i,z}(1+Z\epsilon_i(p)))}\\ =\;&[\mathbf{B}_{\mathrm{cris},E}^{\varphi=\prod_{i=1}^h(\alpha_{i,z}(1+Z\epsilon_i(p)))}\otimes_E(\wedge^h\mathcal{V})(\delta_1'^{-1}\cdots\delta_h'^{-1})]^{G_{\mathbf{Q}_p}},\end{aligned}$$
where the first equality follows from [3, Proposition 3.7] and the second is obvious.

Let $\mathbf{B}^\dagger_{\log,\mathbf{Q}_p}$ be the ring used in [3]. As the refinement corresponding to $\mathrm{Fil}_{\bullet,z}$ is $\mathcal{F}$, we have
$$[(\mathbf{B}^\dagger_{\log,\mathbf{Q}_p}\otimes_{\mathbf{Q}_p}E)[\frac{1}{t_{\mathrm{cyc}}}]\otimes_{\mathscr{R}_E}(\mathrm{Fil}_{i,z}\mathcal{M}_z)]^{\Gamma=1}=\mathcal{F}_iD_z.$$
Since the image of $t_{\mathrm{cyc}}^{f_{i,z}}m_{i,z}$ in $\mathscr{R}_E(\delta_i)[\frac{1}{t_{\mathrm{cyc}}}]$ is fixed by Γ, we have
$$e_{i,z}\equiv t_{\mathrm{cyc}}^{f_{i,z}}m_{i,z}\mod(\mathbf{B}^\dagger_{\log,\mathbf{Q}_p}\otimes_{\mathbf{Q}_p}E)[\frac{1}{t_{\mathrm{cyc}}}]\otimes_{\mathscr{R}_E}\mathrm{Fil}_{i-1,z}\mathcal{M}_z$$
up to a nonzero scalar. This implies that $t_{\mathrm{cyc}}^{f_{1,z}+\cdots+f_{h,z}}m_1\wedge\cdots\wedge m_h$ mod Z coincides with $e_{1,z}\wedge\cdots\wedge e_{h,z}$ up to a nonzero scalar. $\square$

Theorem 6.4. (a) *For any pair of integers (h,ℓ) such that $h<\ell$ we have $\pi_{h\ell}([c])=0$.*

(b) *For any $h = 1, \cdots, n$, $\pi_{h,h}([c])$ coincides with the image of $[\epsilon'_h]$ in $H^1(\mathbf{B}_{\mathrm{st},E})$.*

We consider (a) as the projection vanishing property.

Proof. Let $g_{1,\cdots,h}$ be as in Lemma 6.3. Write

$$g_{1,\cdots,h} = e_1 \wedge \cdots \wedge e_h + Z \sum_J \lambda_J e_J, \tag{6.4}$$

where $\lambda_J \in \mathbf{B}_{\mathrm{st},E}$ and J runs over all subsets of $\{1, \cdots, n\}$ with cardinal number h. Here, if $J = \{j_1 < \cdots < j_h\}$, then $e_J = e_{j_1} \wedge \cdots \wedge e_{j_h}$.

As the matrix of $\sigma \in G_{\mathbf{Q}_p}$ for the basis $\{e_1, \cdots, e_n\}$ is $I_n + ZX^{-1}U_\sigma X$, we have

$$\sigma(e_i) = e_i + \sum_{j=1}^n Z(X^{-1}U_\sigma X)_{ji} e_j.$$

Hence

$$\begin{aligned} g_{1,\cdots,h} = \sigma(g_{1,\cdots,h}) = {} & [1 - Z\epsilon'_1(\sigma) - \cdots - Z\epsilon'_h(\sigma)] \\ & \times \Big[\Big(e_1 + Z\sum_{j=1}^n (X^{-1}U_\sigma X)_{j1} e_j\Big) \wedge \cdots \\ & \wedge \Big(e_h + Z\sum_{j=1}^n (X^{-1}U_\sigma X)_{jh} e_j\Big) + Z\sum_J \sigma(\lambda_J) e_J\Big]. \end{aligned}$$

For any $\ell = h+1, \cdots, n$, comparing the coefficient of $e_1 \wedge \cdots \wedge e_{h-1} \wedge e_\ell$ in the right hand side of the above equality and the right hand side of (6.4), we obtain

$$\lambda_{1,\cdots,h-1,\ell} = \sigma(\lambda_{1,\cdots,h-1,\ell}) + (X^{-1}U_\sigma X)_{\ell h},$$

which proves (a).

Similarly, comparing the coefficients of $e_1 \wedge \cdots \wedge e_h$ in the above two expressions for $g_{1,\cdots,h}$ we obtain

$$\lambda_{1,\cdots,h} = \sigma(\lambda_{1,\cdots,h}) + \sum_{i=1}^h (X^{-1}U_\sigma X)_{ii} - \sum_{i=1}^h \epsilon'_i(\sigma).$$

Thus we have

$$(X^{-1}U_\sigma X)_{hh} - \epsilon'_h(\sigma) = (\sigma - 1)(\lambda_{1,\cdots,h-1} - \lambda_{1,\cdots,h}), \tag{6.5}$$

which implies (b). $\square$

Corollary 6.5. *For $h = 1, \cdots, n$, there exist $\gamma_{h,1}, \gamma_{h,2} \in E$ and $\xi_h \in \mathbf{B}_{\mathrm{st},E}^{\varphi=1}$ such that for any $\sigma \in G_{\mathbf{Q}_p}$,*

$$(X^{-1}U_\sigma X)_{hh} = \gamma_{h,1}\psi_1(\sigma) + \gamma_{h,2}\psi_2(\sigma) + (\sigma - 1)\xi_h.$$

Proof. By Theorem 6.4 (a), $\pi_{j,h}([c])$ $(j < h)$ and $\pi_{h,i}([c])$ $(i > h)$ are zero. Repeating the argument in the proof of Theorem 5.2 (a) (b) yields our assertion. □

As ψ_2 is an E-basis of $\mathrm{Hom}_{\mathrm{cont}}(\Gamma, E)$, the E-vector space of continuous homomorphisms from Γ to E, there exists $\epsilon_{h,2} \in E$ such that $\epsilon'_h = \epsilon_{h,2}\psi_2$.

Lemma 6.6. *We have $\gamma_{h,1} = -\epsilon_h(p)$ and $\gamma_{h,2} = \epsilon_{h,2}$.*

Proof. We keep to use notations in the proof of Theorem 6.4. By (6.5) and Corollary 6.5 we have

$$\begin{aligned}(\sigma - 1)(\lambda_{1,\cdots,h} - \lambda_{1,\cdots,h-1}) &= \epsilon_{h,2}\psi_2(\sigma) - (X^{-1}U_\sigma X)_{hh} \\ &= -\gamma_{h,1}\psi_1(\sigma) + (\epsilon_{h,2} - \gamma_{h,2})\psi_2(\sigma) - (\sigma - 1)\xi_h,\end{aligned}$$

with the convention that $\lambda_{1,\cdots,h-1} = 0$ when $h = 1$. Note that there exists $\omega \in \mathrm{W}(\overline{\mathbf{F}}_p)$ such that $\varphi(\omega) - \omega = 1$, where $\mathrm{W}(\overline{\mathbf{F}}_p)$ is the ring of Witt vectors with coefficients in the algebraic closure of $\mathbf{F}_p$. Then $(\sigma - 1)\omega = \psi_1(\sigma)$. Hence

$$(\epsilon_{h,2} - \gamma_{h,2})\psi_2(\sigma) = (\sigma - 1)(\lambda_{1,\cdots,h} - \lambda_{1,\cdots,h-1} + \xi_h + \gamma_{h,1}\omega).$$

As the extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p$ corresponding to ψ_2 is not Hodge-Tate, we have $\gamma_{h,2} = \epsilon_{h,2}$ and $\lambda_{1,\cdots,h} - \lambda_{1,\cdots,h-1} + \xi_h + \gamma_{h,1}\omega \in E$. Then

$$(\varphi - 1)(\lambda_{1,\cdots,h} - \lambda_{1,\cdots,h-1}) = -(\varphi - 1)\xi_h - \gamma_{h,1}(\varphi - 1)\omega = -\gamma_{h,1}. \tag{6.6}$$

Note that $\oplus_I Z e_I$, where I runs over subsets of $\{1, \cdots, n\}$ with cardinal number h except $\{1, \cdots, h\}$, is stable by φ. Let Y denote this subspace. Then we have

$$\varphi(g_{1,\cdots,h}) = (1 + Z\varphi(\lambda_{1,\cdots,h}))(\prod_{i=1}^{h}\alpha_{i,z})e_1 \wedge \cdots \wedge e_h \pmod{Y}.$$

On the other hand,

$$\begin{aligned}\varphi(g_{1,\cdots,h}) &= (1 + Z\sum_{i=1}^{h}\epsilon_i(p))(\prod_{i=1}^{h}\alpha_{i,z})g_{1,\cdots,h} \\ &= (1 + Z\sum_{i=1}^{h}\epsilon_i(p))(\prod_{i=1}^{h}\alpha_{i,z})(1 + Z\lambda_{1,\cdots,h})e_1 \wedge \cdots \wedge e_h \pmod{Y}.\end{aligned}$$

Hence we obtain

$$(\varphi-1)\lambda_{1,\cdots,h}=\sum_{i=1}^{h}\epsilon_i(p). \tag{6.7}$$

By (6.6) and (6.7) we have

$$\gamma_{h,1}=-(\varphi-1)(\lambda_{1,\cdots,h}-\lambda_{1,\cdots,h-1})=-\epsilon_h(p),$$

as wanted. $\square$

7 Proof of the main theorem

Let S be an affinoid algebra over E. Let $\mathcal{V}$ be a trianguline S-representation of $G_{\mathbf{Q}_p}$, $\mathcal{M}=\mathbf{D}_{\mathrm{rig}}(\mathcal{V})$. Fix a triangulation of $\mathcal{M}$ and let $(\delta_1,\cdots,\delta_n)$ be the corresponding triangulation data.

We restate our main theorem as follows.

Theorem 7.1. *Let z be a closed point of S such that $\mathcal{V}_z$ is semistable. Let D_z be the filtered E-(φ,N)-module attached to $\mathcal{V}_z$, and suppose that φ is semisimple on D_z. Let $\mathcal{F}$ be the refinement of D_z corresponding to the triangulation of $\mathcal{M}_z$. If s is strongly critical for $\mathcal{F}$ and $t=t_{\mathcal{F}}(s)$, then*

$$\frac{\mathrm{d}\delta_t(p)}{\delta_t(p)}-\frac{\mathrm{d}\delta_s(p)}{\delta_s(p)}+\mathcal{L}_{\mathcal{F},s}(\mathrm{d}w_{\delta_t}-\mathrm{d}w_{\delta_s})$$

is zero at z. Here, $\mathcal{L}_{\mathcal{F},s}$ is the invariant defined in Definition 4.8.

Since we only need the first order derivation, we may assume that $S=E[Z]/Z^2$ and z corresponds to the maximal ideal (Z).

For $i=1,\cdots,n$ there exists a continuous additive character ϵ_i of $\mathbf{Q}_p^\times$ with values in E such that $\delta_i=\delta_{i,z}(1+Z\epsilon_i)$. By identifying Γ with $\mathbf{Z}_p^\times$ via χ_{cyc} we consider $\epsilon_i|_{\mathbf{Z}_p^\times}$ as a character of $G_{\mathbf{Q}_p}$ that factors through Γ. Then there exists $\epsilon_{i,2}\in E$ such that $\epsilon_i|_{\mathbf{Z}_p^\times}=\epsilon_{i,2}\psi_2$. Clearly $w_{\delta_i}=w_{\delta_{i,z}}+Z\epsilon_{i,2}$. Thus

$$\frac{\mathrm{d}\delta_i(p)}{\delta_i(p)}=\epsilon_i(p)\mathrm{d}Z,\quad \mathrm{d}w_{\delta_i}=\epsilon_{i,2}\mathrm{d}Z.$$

Hence Theorem 7.1 comes from the following

Proposition 7.2. $\epsilon_t(p)-\epsilon_s(p)+\mathcal{L}_{\mathcal{F},s}(\epsilon_{t,2}-\epsilon_{s,2})=0.$

Proof. Let c be the 1-cocycle attached to the infinitesimal deformation $\mathcal{V}$ of $\mathcal{V}_z$. Fix an s-perfect basis for $\mathcal{F}$, and let $\pi_{h\ell}$ ($h, \ell \in \{1, \cdots, n\}$) be the maps defined by (6.3) using this basis. By Corollary 6.5 and Lemma 6.6 there are $\xi_s, \xi_t \in \mathbf{B}_{\mathrm{st},E}^{\varphi=1}$ such that

$$\pi_{s,s}(c_\sigma) = -\epsilon_s(p)\psi_1(\sigma) + \epsilon_{s,2}\psi_2(\sigma) + (\sigma - 1)\xi_s$$

and

$$\pi_{t,t}(c_\sigma) = -\epsilon_t(p)\psi_1(\sigma) + \epsilon_{t,2}\psi_2(\sigma) + (\sigma - 1)\xi_t.$$

By Theorem 6.4 (a) we have $\pi_{h\ell}([c]) = 0$ when $h < \ell$. Thus it follows from Corollary 5.3 that $\epsilon_t(p) - \epsilon_s(p) = \mathcal{L}_{\mathcal{F},s}(\epsilon_{s,2} - \epsilon_{t,2})$. ☐

References

[1] J. Bellaïche, G. Chenevier, *Families of Galois representations and Selmer groups.* Astérisque 324 (2009), 314 page.

[2] D. Benois, *A generalization of Greenberg's $\mathscr{L}$-invariant.* Amer. J. Math. 133 (2011), 1573-1632.

[3] L. Berger, *Représentations p-adiques et équations différentielles.* Invent. Math. 148 (2002), 219-284.

[4] L. Berger, *Trianguline representations.* Bull. Lond. Math. Soc. 43 (2011), no. 4, 619-635.

[5] L. Berger, *Équations différentielles p-adiques et (φ, N)-modules filtrés.* Astérisque 319 (2008), 13-38.

[6] L. Berger, P. Colmez, *Familles des repréntations de de Rham et monodromie p-adique*, Astérisque 319 (2008), 303-337.

[7] G. Chenevier, *Families p-adiques de froms automorphes pour* GL_n. J. reine angew. Math. 570 (2004), 143-217.

[8] G. Chenevier, *Sur la densité des représentations cristallines de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Math. Ann. 355 (2013) no. 4, 1469-1525.

[9] F. Cherbonnier, P. Colmez, *Représentations p-adiques surconvergentes.* Invent. Math. 133 (1998), no.3, 581-611.

[10] P. Colmez, *Représentations triangulines de dimension* 2. Asterisque 319 (2008), 213-258.

[11] P. Colmez, *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *and* (φ,Γ)*-modules.* Astérisque 330 (2010), 281-509.

[12] P. Colmez, *Invariants* $\mathcal{L}$ *et dérivées de valeurs propres de Frobenius.* Astérisque 331 (2010), 13-28.

[13] P. Colmez, *La série principale unitaire de* $\mathrm{GL}_2(\mathbb{Q}_p)$*: Vecteurs localement analytiques,* in "Automorphic forms and Galois representations Vol 1", London Math. Soc. Lect. Note Series 415 (2014), 286-358.

[14] P. Colmez, J.-M. Fontaine, *Construction des représentations p-adiques semi-stables.* Invent. Math. 140 (2000), 1-43.

[15] Y. Ding, $\mathcal{L}$*-invariants and local-global compatibility for the group* GL_2/F, preprint, 2015.

[16] J.-M. Fontaine, *Représentations p-adiques des corps locaux. I.* The Grothendieck Festschrift, vol. II, Prog. Mth., vol 87, Birkhäuser Boston, MA, 1990, 249-309.

[17] J.-M. Fontaine, *Le corps des périodes p-adiques.* Astérisque, No. 223 (1994), 59-111.

[18] J.-M. Fontaine, *Représentaions p-adiques semi-stables.* Astérisque, No. 223 (1994), 113-184.

[19] R. Greenberg, G. Steven, *p-adic L-functions and p-adic periods of modular forms.* Invent. Math. 111 (1993), 407-447.

[20] K. Kedlaya, *Slope filtrations revisited.* Doc. Math. 10 (2005), 447-525.

[21] K. Kedlaya, R. Liu, *On families of* (φ,Γ)*-modules.* Algebra Number Theory 4 (2010) no 7, 943-967.

[22] K. Kedlaya, J. Pottharst, L. Xiao, *Cohomology of arithmetic families of* (φ,Γ)*-modules.* J. Amer. Math. Soc, 27 (2014) no. 4, 1043-1115.

[23] M. Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture.* Invent. Math. 153, 373-454.

[24] R. Liu, B. Xie, Y. Zhang, *Locally analytic vectors of unitary principal series of* $\mathrm{GL}_2(\mathbb{Q}_p)$. Ann. Sci. Éc. Norm. Supér. (4) 45 (2012), 167-190.

[25] B. Mazur, *The theme of p-adic variation*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence 2000, 433-459.

[26] B. Mazur, J. Tate, J. Teitelbaum, *On p-adic analogs of the conjectures of Birch and Swinnerton-Dyer*. Invent. Math. 84 (1986), 1-48.

[27] B. Xie, *On families of filtered (φ, N)-modules*. Math. Res. Lett. 19 (2012), 667-689.

[28] Y. Zhang, *$\mathcal{L}$-invariants and logarithmic derivatives of eigenvalues of Frobenius.* Science China Mathematics 57 (2014), 1587-1604.