WEIGHTED SUM FORMULA FOR MULTIPLE ZETA VALUES

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ABSTRACT. The sum formula is a basic identity of multiple zeta values that expresses a Riemann zeta value as a homogeneous sum of multiple zeta values of a given dimension. This formula was already known to Euler in the dimension two case, conjectured in the early 1990s for higher dimensions and then proved by Granville and Zagier independently. Recently a weighted form of Euler's formula was obtained by Ohno and Zudilin. We generalize it to a weighted sum formula for multiple zeta values of all dimensions.

1. INTRODUCTION

Multiple zeta values (MZVs) are special values of the multi-variable analytic function

$$(1) \qquad \zeta(s_1, \cdots, s_k) = \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

at integers $s_1 \geqslant 2, s_i \geqslant 1, 1 \leqslant i \leqslant k$. Their study in the two variable case went back to Goldbach and Euler [21]. The general concept was introduced in the early 1990s, with motivation from both mathematics [15, 33] and physics [5]. Since then the subject has been studied intensively with interactions to a broad range of areas in mathematics and physics, including arithmetic geometry, combinatorics, number theory, knot theory, Hopf algebra, quantum field theory and mirror symmetry [2, 3, 9, 14, 16, 18, 20, 23, 30, 32].

A principle goal in the theoretical study of MZVs is to determine all algebraic relations among them. Conjecturally all such relations come from the so-called extended double shuffle relations. But there are no definite way to exhaust all of them and new identities of MZVs are being found steadily [3, 6, 13, 19, 28, 35].

One of the first established and most well-known among these identities is the striking **sum formula**, stating that, for given positive integers k and $n \geqslant k + 1$,

$$(2) \qquad \sum_{s_i \geqslant 1, s_1 \geqslant 2, s_1 + \cdots + s_k = n} \zeta(s_1, \cdots, s_k) = \zeta(n),$$

suggesting intriguing connection between Riemann zeta values and multiple zeta values. This formula was obtained by Euler when $k = 2$, known as **Euler's sum formula** [7]

$$(3) \qquad \sum_{i=2}^{n-1} \zeta(i, n-i) = \zeta(n)$$

which includes the basic example $\zeta(2,1) = \zeta(3)$. Its general form was conjectured in [16] and proved by Granville [10] and Zagier [34]. Since then the sum formula has been generalized and extended in various directions [4, 6, 19, 22, 24, 25, 26, 27, 29]. Ohno and Zudilin [28] recently proved a weighted form of Euler's sum formula (the **weighted Euler**

sum formula)

$$\sum_{i=2}^{n-1} 2^i \zeta(i, n-i) = (n+1)\zeta(n), \quad n \geqslant 2, \tag{4}$$

and applied it to study multiple zeta star values.

In this paper we generalize the weighted Euler sum formula of Ohno and Zudilin to higher dimensions.

Theorem 1.1. (Weighted sum formula) *For positive integers $k \geqslant 2$ and $n \geqslant k+1$, we have*

$$\sum_{\substack{s_i \geqslant 1, s_1 \geqslant 2 \\ s_1 + \cdots + s_k = n}} \Big[2^{s_1 - 1} + (2^{s_1 - 1} - 1)\Big(\big(\sum_{i=2}^{k-1} 2^{S_i - s_1 - (i-1)}\big) + 2^{S_{k-1} - s_1 - (k-2)}\Big)\Big] \zeta(s_1, \cdots, s_k) = n\zeta(n),$$

where $S_i = s_1 + \cdots + s_i$ for $i = 1, \cdots, k-1$.

See Theorem 2.2 for a more concise formulation. When $k = 2$, this formula becomes

$$\sum_{i=2}^{n-1} (2^i - 1)\zeta(i, n-i) = n\zeta(n),$$

which gives Eq. (4) after applying Eq. (3).

After introducing background notations and results, we prove our weighted sum formula in Section 2 by combining sum formulas for the quasi-shuffle product, for the shuffle product, as well as for MZVs in Eq. (2). The proofs of the sum formulas for the quasi-shuffle and shuffle product are given in Section 3 and Section 4 respectively. In the two dimensional case [28], the sum formula for the shuffle product was derived from Euler's decomposition formula. We have generalized Euler's decomposition formula to multiple zeta values [13]. Instead of applying this generalization directly, we obtain the sum formula for the shuffle product by induction.

2. Double shuffle relations and the weighted sum theorem

After introducing preliminary notations and results on MZVs in Section 2.1, we prove the weighted sum formula (Theorem 1.1) in Section 2.2 by applying several shuffle and quasi-shuffle (stuffle) relations in Section 2.2. The proofs of these relations will be given in the next two sections.

2.1. **Double shuffle relations of MZVs.** As is well-known, there are two ways to express a MZV:

$$
(5) \qquad \zeta(s_1, \cdots, s_k) : \ = \sum_{n_1 > \cdots > n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}
$$

$$
(6) \qquad = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{|\vec{s}|-1}} \frac{dt_1}{f_1(t_1)} \cdots \frac{dt_{|\vec{s}|}}{f_{|\vec{s}|}(t_{|\vec{s}|})}
$$

for integers $s_i \geqslant 1$ and $s_1 > 1$. Here

$$
f_j(t) = \begin{cases} 1 - t_j, & j = s_1, s_1 + s_2, \cdots, s_1 + \cdots + s_k, \\ t_j, & \text{otherwise.} \end{cases}
$$

The product of two such sums is a $\mathbb{Z}$-linear combination of other such sums and the product of two such integrals is a $\mathbb{Z}$-linear combination of other such integrals. Thus the $\mathbb{Z}$-linear span of the MZVs forms an algebra

$$
(7) \qquad \mathbf{MZV} := \mathbb{Z}\{\zeta(s_1, \cdots, s_k) \mid s_i \geqslant 1, s_1 \geqslant 2\}.
$$

The multiplication rules of the MZVs reflected by these two representations are encoded in two algebras, the quasi-shuffle (stuffle) algebra for the sum representation and the shuffle algebra for the integral representation [16, 20].

For the sum representation, let $\mathcal{H}^*$ be the quasi-shuffle algebra whose underlying additive group is that of the noncommutative polynomial algebra

$$
\mathbb{Z}\langle z_s \mid s \geqslant 1 \rangle = \mathbb{Z}M(z_s \mid s \geqslant 1)
$$

where $M(z_s \mid i \geqslant 1)$ is the free monoid generated by the set $\{z_s \mid s \geqslant 1\}$. So an element of $M(z_s \mid s \geqslant 1)$ is either the unit 1, also called the empty word, or is of the form

$$
z_{s_1, \cdots, s_k} := z_{s_1} \cdots z_{s_k}, s_j \geqslant 1, 1 \leqslant j \leqslant k, k \geqslant 1.
$$

The product on $\mathcal{H}^*$ is the quasi-shuffle (also called harmonic shuffle or stuffle) product [2, 16, 17] defined recursively by

$$
(8) \ \ (z_{r_1} u) * (z_{s_1} v) = z_{r_1}(u * (z_{s_1} v)) + z_{s_1}((z_{r_1} u) * v) + z_{r_1 + s_1}(u * v), \quad u, v \in M(z_s \mid s \geqslant 1)
$$

with the convention that $1 * u = u * 1 = u$ for $u \in M(z_s \mid s \geqslant 1)$. In the quasi-shuffle algebra $(\mathcal{H}^*, *)$ there is a subalgebra

$$
(9) \qquad \mathcal{H}_0^* := \mathbb{Z} \oplus \Big(\bigoplus_{s_1 > 1} \mathbb{Z} z_{s_1} \cdots z_{s_k} \Big) \subseteq \mathcal{H}^*.
$$

Then the multiplication rule of MZVs according to their summation representation follows from the statement that the linear map

$$
(10) \qquad \zeta^* : \mathcal{H}_0^* \to \mathbf{MZV}, \quad z_{s_1, \cdots, s_k} \mapsto \zeta(s_1, \cdots, s_k).
$$

is an algebra homomorphism.

For the integral representation, let $\mathcal{H}^{\text{\tiny III}}$ be the shuffle algebra whose underlying additive group is that of the noncommutative polynomial algebra

$$
\mathbb{Z}\langle x_0, x_1 \rangle = \mathbb{Z}M(x_0, x_1)
$$

where $M(x_0, x_1)$ is the free monoid generated by x_0 and x_1. The product on $\mathcal{H}^{\text{Ш}}$ is the shuffle product defined recursively by

$$(11) \qquad (au) \,\text{Ш}\, (bv) = a(u \,\text{Ш}\, (bv)) + b((au) \,\text{Ш}\, v), a, b \in \{x_0, x_1\}, u, v \in M(x_0, x_1)$$

with the convention that $u \,\text{Ш}\, 1 = 1 \,\text{Ш}\, u = u$ for $u \in M(x_0, x_1)$. In the shuffle algebra $\mathcal{H}^{\text{Ш}}$, there are subalgebras

$$(12) \qquad \mathcal{H}_0^{\text{Ш}} := \mathbb{Z} \oplus x_0 \mathcal{H}^{\text{Ш}} x_1 \subseteq \mathcal{H}_1^{\text{Ш}} := \mathbb{Z} \oplus \mathcal{H}^{\text{Ш}} x_1 \subseteq \mathcal{H}^{\text{Ш}}.$$

Then the multiplication rule of MZVs according to their integral representations follows from the statement that the linear map

$$(13) \qquad \zeta^{\text{Ш}} : \mathcal{H}_0^{\text{Ш}} \to \mathbf{MZV}, \quad x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 \mapsto \zeta(s_1, \cdots, s_k)$$

is an algebra homomorphism.

There is a natural bijection of abelian groups (but *not* algebras)

$$(14) \qquad \eta : \mathcal{H}_1^{\text{Ш}} \to \mathcal{H}^*, \quad 1 \leftrightarrow 1, \; x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 \leftrightarrow z_{s_1, \cdots, s_k}.$$

that restricts to a bijection of abelian groups

$$(15) \qquad \eta : \mathcal{H}_0^{\text{Ш}} \to \mathcal{H}_0^*, \quad 1 \leftrightarrow 1, \; x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 \leftrightarrow z_{s_1, \cdots, s_k}.$$

Then the fact that MZVs can be multiplied in two ways is reflected by the commutative diagram

$$(16)$$

$$\mathcal{H}_0^{\text{Ш}} \xrightarrow{\;\;\eta\;\;} \mathcal{H}_0^*$$
$$\zeta^{\text{Ш}} \searrow \qquad \swarrow \zeta^*$$
$$\mathbf{MZV}$$

Through η, the shuffle product Ш on $\mathcal{H}_1^{\text{Ш}}$ and $\mathcal{H}_0^{\text{Ш}}$ transports to a product Ш_* on $\mathcal{H}^*$ and $\mathcal{H}_0^*$. That is, for $w_1, w_2 \in \mathcal{H}_0^*$, define

$$(17) \qquad w_1 \,\text{Ш}_*\, w_2 := \eta(\eta^{-1}(w_1) \,\text{Ш}\, \eta^{-1}(w_2)).$$

Then the **double shuffle relation** is simply the set

$$\{w_1 \,\text{Ш}_*\, w_2 - w_1 * w_2 \mid w_1, w_2 \in \mathcal{H}_0^*\}$$

and the **extended double shuffle relation** [20, 30] is the set

$$(18) \qquad \{w_1 \,\text{Ш}_*\, w_2 - w_1 * w_2, \; z_1 \,\text{Ш}_*\, w_2 - z_1 * w_2 \mid w_1, w_2 \in \mathcal{H}_0^*\}.$$

Theorem 2.1. ([16, 20, 30]) *Let $I_{\mathbf{EDS}}$ be the ideal of $\mathcal{H}_0^*$ generated by the extended double shuffle relation in Eq. (18). Then $I_{\mathbf{EDS}}$ is in the kernel of ζ^*.*

It is conjectured that $I_{\mathbf{EDS}}$ is in fact the kernel of ζ^*.

2.2. **Proof of the weighted sum theorem.** Let k be a positive integer $\geqslant 2$. For positive integers $t_1, \cdots, t_{k-1}$, denote

(19)
$$\mathcal{C}(t_1, \cdots, t_{k-1}) := \left(\sum_{j=1}^{k-1} 2^{t_1 + \cdots + t_j - j}\right) + 2^{t_1 + \cdots + t_{k-1} - (k-1)}$$

$$= \left(\sum_{j=1}^{k-2} 2^{t_1 + \cdots + t_j - j}\right) + 2^{t_1 + \cdots + t_{k-1} - (k-1) + 1}.$$

They satisfy the following simple relations that can be checked easily from the definition:

(20)
$$\mathcal{C}(t_1 + 1, t_2, \cdots, t_{k-1}) = 2\mathcal{C}(t_1, t_2, \cdots, t_{k-1}),$$

(21)
$$\mathcal{C}(1, t_2, \cdots, t_{k-1}) = \mathcal{C}(t_2, \cdots, t_{k-1}) + 1$$

with the convention that $\mathcal{C}(t_2, \cdots, t_{k-1}) = \mathcal{C}(\emptyset) = 1$ when $k = 2$.

Using the notation $\mathcal{C}(t_1, \cdots, t_{k-1})$ we can restate Theorem 1.1 as follows.

Theorem 2.2. (Second form of weighted sum formula) *For positive integers k and $n \geqslant k + 1$, we have*

(22)
$$\sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + s_k = n}} [\mathcal{C}(t_1, \cdots, t_{k-1}) - \mathcal{C}(t_2, \cdots, t_{k-1})]\zeta(t_1, \cdots, t_k) = n\zeta(n),$$

with the convention that $\mathcal{C}(t_2, \cdots, t_{k-1}) = \mathcal{C}(\emptyset) = 1$ when $k = 2$.

We will prove Theorem 2.2 and hence the weighted sum formula in Theorem 1.1 by using the following auxiliary Theorem 2.4 and Theorem 2.6. They are a type of sum formula on the products $*$ and III_* respectively and are interesting on their own right. Their proofs will be given in Section 3 and Section 4 respectively.

We first display the sum formulas on the product $*$.

Theorem 2.3. *For positive integers $k \geqslant 2$ and $n \geqslant k$, we have*

(23)
$$\sum_{\substack{r, s_i \geqslant 1 \\ r + s_1 + \cdots + s_{k-1} = n}} z_r * z_{s_1, \cdots, s_{k-1}} = k \sum_{\substack{t_i \geqslant 1 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k} + (n - k + 1) \sum_{\substack{u_i \geqslant 1 \\ u_1 + \cdots + u_{k-1} = n}} z_{u_1, \cdots, u_{k-1}}.$$

This theorem will be applied to MZVs through the following

Theorem 2.4. *For positive integers $k \geqslant 2$ and $n \geqslant k + 1$, we have*

(24)
$$\sum_{\substack{r, s_i \geqslant 1, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} z_r * z_{s_1, \cdots, s_{k-1}}$$

$$= \sum_{\substack{t_i \geqslant 1, t_1 = 1, t_2 \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, t_2, \cdots, t_k} + (k - 1) \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2, t_2 = 1 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k}$$

$$+ k \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2, t_2 \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k} + (n - k) \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1 + \cdots + u_{k-1} = n}} z_{u_1, \cdots, u_{k-1}}.$$

The proofs of these two theorems will be given in Section 3. Similarly for the product $ш_*$, we have the following sum formulas.

Theorem 2.5. *For positive integers $k \geqslant 2$ and $n \geqslant k$, we have*

$$(25) \qquad \sum_{\substack{r,s_i \geqslant 1, r+s_1+\cdots+s_{k-1}=n}} z_r \; ш_* \; z_{s_1,\cdots,s_{k-1}} = \sum_{\substack{t_i \geqslant 1, t_1+\cdots+t_k=n}} \mathcal{C}(t_1,\cdots,t_{k-1}) z_{t_1,\cdots,t_k}.$$

Theorem 2.6. *For positive integers $k \geqslant 2$ and $n \geqslant k+1$, we have*

$$
\begin{aligned}
& \sum_{\substack{r,s_i \geqslant 1, s_1 \geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n}} z_r \; ш_* \; z_{s_1,\cdots,s_{k-1}} \\
(26) \quad & = \sum_{\substack{t_i \geqslant 1 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,\cdots,t_{k-1}) - \mathcal{C}(t_2,\cdots,t_{k-1})] z_{t_1,\cdots,t_k} - \sum_{\substack{t_i \geqslant 1, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}
\end{aligned}
$$

with the convention that $\mathcal{C}(t_2,\cdots,t_{k-1}) = \mathcal{C}(\emptyset) = 1$ when $k = 2$.

The proofs of these two theorems will be given in Section 4. Now we derive Theorem 2.2 and hence Theorem 1.1 from Theorem 2.4 and Theorem 2.6.

Proof of Theorem 2.2. Regrouping the sums in Eq. (26) of Theorem 2.4 and applying the summation relation

$$\sum_{\substack{t_1=1, t_2 \geqslant 2 \\ t_1+\cdots+t_k=n}} - \sum_{\substack{t \geqslant 2, t_2=1 \\ t_1+\cdots+t_k=n}} = \sum_{\substack{t_1=1 \\ t_1+\cdots+t_k=n}} - \sum_{\substack{t_2=1 \\ t_1+\cdots+t_k=n}}$$

we obtain

$$
\begin{aligned}
& \sum_{\substack{r,s_i \geqslant 1, s_1 \geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n}} z_r * z_{s_1,\cdots,s_{k-1}} \\
& = \sum_{\substack{t_i \geqslant 1, t_1=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + k \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} - \sum_{\substack{t_i \geqslant 1, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (n-k) \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1+\cdots+u_{k-1}=n}} z_{u_1,\cdots,u_{k-1}}.
\end{aligned}
$$

From this equation and Theorem 2.6 we obtain

$$
\begin{aligned}
& \sum_{\substack{r,s_i \geqslant 1, s_1 \geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n}} \left(z_r \; ш_* \; z_{s_1,\cdots,s_{k-1}} - z_r * z_{s_1,\cdots,s_{k-1}} \right) \\
& = \sum_{\substack{t_i \geqslant 1, t_1=1 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,\cdots,t_{k-1}) - \mathcal{C}(t_2,\cdots,t_{k-1}) - 1] z_{t_1,\cdots,t_k} \\
& \quad + \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,\cdots,t_{k-1}) - \mathcal{C}(t_2,\cdots,t_{k-1}) - k] z_{t_1,\cdots,t_k} \\
& \quad - (n-k) \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1+\cdots+u_{k-1}=n}} z_{u_1,\cdots,u_{k-1}}
\end{aligned}
$$

$$= \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + t_k = n}} [\mathcal{C}(t_1, \cdots, t_{k-1}) - \mathcal{C}(t_2, \cdots, t_{k-1}) - k] z_{t_1, \cdots, t_k}$$

$$- (n-k) \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1 + \cdots + u_{k-1} = n}} z_{u_1, \cdots, u_{k-1}}.$$

by Eq. (21). Since $\zeta^*(z_r \text{ ш}_* z_{s_1, \cdots, s_{k-1}} - z_r * z_{s_1, \cdots, s_{k-1}}) = 0$, by Theorem 2.1, this gives

$$\sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + t_k = n}} [\mathcal{C}(t_1, \cdots, t_{k-1}) - \mathcal{C}(t_2, \cdots, t_{k-1}) - k] \zeta(t_1, \cdots, t_k)$$

$$= (n-k) \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1 + \cdots + u_{k-1} = n}} \zeta(u_1, \cdots, u_{k-1}).$$

The sum formula in Eq. (2) shows that

$$\sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + t_k = n}} \zeta(t_1, \cdots, t_k) = \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1 + \cdots + u_{k-1} = n}} \zeta(u_1, \cdots, u_{k-1}) = \zeta(n).$$

Therefore

$$\sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + t_k = n}} [\mathcal{C}(t_1, \cdots, t_{k-1}) - \mathcal{C}(t_2, \cdots, t_{k-1})] \zeta(t_1, \cdots, t_k) = n\zeta(n),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. PROOFS OF THEOREM 2.3 AND THEOREM 2.4

In this section we prove the two sum relations of the quasi-shuffle product.

3.1. **Proof of Theorem 2.3.** We first consider the case when $k = 2$. By the basic relation $z_r * z_{s_1} = z_{r, s_1} + z_{s_1, r} + z_{r+s_1}$ from Eq. (8), we have

$$\sum_{r, s_1 \geqslant 1, r+s_1 = n} (z_r * z_{s_1}) = \sum_{r, s_1 \geqslant 1, r+s_1 = n} z_{r, s_1} + \sum_{r, s_1 \geqslant 1, r+s_1 = n} z_{s_1, r} + \sum_{r, s_1 \geqslant 1, r+s_1 = n} z_{r+s_1}$$

$$= 2 \sum_{t_i \geqslant 1, t_1 + t_2 = n} z_{t_1, t_2} + (n-1) z_n,$$

as needed.

In the general case we prove Eq. (23) by induction on n. If $n = 2$, then $k = 2$ and we are done. For a given integer $m \geqslant 3$, assume that Eq. (23) holds when $n < m$ and consider Eq. (23) when $n = m$ and $k \geqslant 2$. Since we have proved the case when $k = 2$, we may assume that $k \geqslant 3$. By Eq. (8), we have

(27)
$$\sum_{\substack{r, s_i \geqslant 1 \\ r + s_1 + \cdots + s_{k-1} = n}} (z_r * z_{s_1, \cdots, s_{k-1}})$$

$$= \sum_{\substack{r, s_i \geqslant 1 \\ r + s_1 + \cdots + s_{k-1} = n}} z_{r, s_1, \cdots, s_{k-1}} + \sum_{\substack{r, s_i \geqslant 1 \\ r + s_1 + \cdots + s_{k-1} = n}} z_{s_1}(z_r * z_{s_2, \cdots, s_{k-1}}) + \sum_{\substack{r, s_i \geqslant 1 \\ r + s_1 + \cdots + s_{k-1} = n}} z_{r+s_1, s_2, \cdots, s_{k-1}}.$$

In the second sum on the right hand side of Eq. (27), for any fixed $s_1 \geqslant 1$, by the induction hypothesis we have

$$\sum_{r,s_i \geqslant 1, r+s_2+\cdots+s_{k-1}=n-s_1} z_r * z_{s_2,\cdots,s_{k-1}}$$

$$= (k-1) \sum_{t_i \geqslant 1, t_1+\cdots+t_{k-1}=n-s_1} z_{t_1,\cdots,t_{k-1}} + (n-s_1-k+2) \sum_{u_i \geqslant 1, u_1+\cdots+u_{k-2}=n-s_1} z_{u_1,\cdots,u_{k-2}}.$$

So the second sum becomes

$$\sum_{\substack{r,s_i \geqslant 1 \\ r+s_1+s_2+\cdots+s_{k-1}=n}} z_{s_1}(z_r * z_{s_2,\cdots,s_{k-1}})$$

$$= (k-1) \sum_{\substack{s_1 \geqslant 1, t_i \geqslant 1 \\ s_1+t_1+\cdots+t_{k-1}=n}} z_{s_1,t_1,\cdots,t_{k-1}} + (n-s_1-k+2) \sum_{\substack{s_1 \geqslant 1, u_i \geqslant 1 \\ s_1+u_1+\cdots+u_{k-2}=n}} z_{s_1,u_1,\cdots,u_{k-2}}$$

$$= (k-1) \sum_{\substack{t_i \geqslant 1, t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (n-u_1-k+2) \sum_{u_i \geqslant 1, u_1+\cdots+u_{k-1}=n} z_{u_1,\cdots,u_{k-1}}.$$

For the third sum on the right hand side of Eq. (27), we have

$$\sum_{r,s_i \geqslant 1, r+s_1+\cdots+s_{k-1}=n} z_{r+s_1,s_2,\cdots,s_{k-1}} = \sum_{u_i \geqslant 1, u_1+\cdots+u_{k-1}=n} (u_1-1)z_{u_1,\cdots,u_{k-1}}.$$

Therefore Eq. (27) becomes

$$\sum_{\substack{r,s_i \geqslant 1 \\ r+s_1+\cdots+s_{k-1}=n}} z_r * z_{s_1,\cdots,s_{k-1}}$$

$$= \sum_{t_i \geqslant 1, t_1+\cdots+t_k=n} z_{t_1,\cdots,t_k} + \sum_{t_i \geqslant 1, t_1+\cdots+t_k=n} (k-1)z_{t_1,\cdots,t_k}$$

$$+ \sum_{u_i \geqslant 1, u_1+\cdots+u_{k-1}=n} (n-u_1-k+2)z_{u_1,\cdots,u_{k-1}} + \sum_{u_i \geqslant 1, u_1+\cdots+u_{k-1}=n} (u_1-1)z_{u_1,\cdots,u_{k-1}}$$

$$= \sum_{t_i \geqslant 1, t_1+\cdots+t_k=n} k z_{t_1,\cdots,t_k} + \sum_{u_i \geqslant 1, u_1+\cdots+u_{k-1}=n} (n-k+1)z_{u_1,\cdots,u_{k-1}}.$$

This means that Eq. (23) holds when $n = m$, completing the inductive proof of Theorem 2.3.

3.2. **Proof of Theorem 2.4.**

We now prove Theorem 2.4 by applying Theorem 2.3. When $k = 2$ we can verify Eq. (24) directly by Eq. (8) as in the case of Theorem 2.3. For $k \geqslant 3$, applying Eq. (8) and Eq. (23), we have

$$\sum_{\substack{r,s_i \geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} z_r * z_{1,s_2,\cdots,s_{k-1}}$$

$$= \sum_{\substack{r,s_i \geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} z_{r,1,s_2,\cdots,s_{k-1}} + \sum_{\substack{r,s_i \geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} z_1(z_r * z_{s_2,\cdots,s_{k-1}}) + \sum_{\substack{r,s_i \geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} z_{r+1,s_2,\cdots,s_{k-1}}$$

$$= \sum_{\substack{t_i \geqslant 1, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + \left((k-1) \sum_{t_2+\cdots+t_k=n-1} z_{1,t_2,\cdots,t_k} \right.$$

$$+ [(n-1)-(k-1)+1] \sum_{u_2+\cdots+u_{k-1}=n-1} z_{1,u_2,\cdots,u_{k-1}} \left. \right) + \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1+\cdots+u_{k-1}=n}} z_{u_1,\cdots,u_{k-1}},$$

where k and n in Eq. (23) are replaced by $k-1$ and $n-1$. By regrouping, this expression can be further simplified to

$$(k-1) \sum_{\substack{t_i \geqslant 1, t_1=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + \sum_{\substack{t_i \geqslant 1, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k}$$

$$+ (n-k+1) \sum_{\substack{u_i \geqslant 1, u_1=1 \\ u_1+\cdots+u_{k-1}=n}} z_{u_1,u_2,\cdots,u_{k-1}} + \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1+\cdots+u_{k-1}=n}} z_{u_1,u_2,\cdots,u_{k-1}}$$

$$= (k-1) \sum_{\substack{t_i \geqslant 1, t_1=1, t_2 \geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k}$$

$$+ k \sum_{\substack{t_i \geqslant 1, t_1=t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (n-k+1) \sum_{\substack{u_i \geqslant 1, u_1=1 \\ u_1+\cdots+u_{k-1}=n}} z_{u_1,u_2,\cdots,u_{k-1}}$$

$$+ \sum_{\substack{u_i \geqslant 1, u_1 \geqslant 2 \\ u_1+\cdots+u_{k-1}=n}} z_{u_1,u_2,\cdots,u_{k-1}}.$$

Now Eq. (24) follows from this and Eq. (23).

4. PROOFS OF THEOREM 2.5 AND THEOREM 2.6

This sections gives the proofs of the two sum relations of the shuffle product.

4.1. A preparational lemma.
In this subsection we provide a lemma that is needed in the proofs of Theorem 2.5 and Theorem 2.6.

Let $\mathcal{H}^{*+}$ be the subring of $\mathcal{H}^*$ generated by $z_{\vec{s}}$ with $\vec{s} \in \mathbb{Z}_{\geqslant 1}^k, k \geqslant 1$. Then

$$\mathcal{H}^* = \mathbb{Z} \oplus \mathcal{H}^{*+}.$$

Define two operators

(28)
$$P : \mathcal{H}^{*+} \to \mathcal{H}^*, P(z_{s_1,s_2,\cdots,s_k}) = z_{s_1+1,s_2,\cdots,s_k},$$

$$Q : \mathcal{H}^* \to \mathcal{H}^*, Q(w) = \begin{cases} z_1 w, & w \neq 1, \\ z_1, & w = 1. \end{cases}$$

These operators are simply the transports of the operators

$$I_0 : \mathcal{H}_1^{\text{Ш}+}(\overline{G}) \to \mathcal{H}_1^{\text{Ш}}(\overline{G}), \quad I_0(u) = x_0 u,$$

$$I_1 : \mathcal{H}_1^{\text{Ш}}(\overline{G}) \to \mathcal{H}_1^{\text{Ш}}(\overline{G}), \quad I_1(u) = \begin{cases} x_1 u, & u \neq 1, \\ x_1, & u = 1. \end{cases}$$

Thus the recursive definition of $ш$ in Eq. (11) gives the following Rota-Baxter relation [1, 11, 12, 31] from [13].

Proposition 4.1. ([13]) *The multiplication $ш_*$ on $\mathcal{H}^*$ defined in Eq. (17) is the unique one such that*

$$
\begin{aligned}
(29) \qquad
& P(w_1) \, ш_* \, P(w_2) = P\big(w_1 \, ш_* \, P(w_2)\big) + P\big(P(w_1) \, ш_* \, w_2\big), && w_1, w_2 \in \mathcal{H}^{*+}, \\
& Q(w_1) \, ш_* \, Q(w_2) = Q\big(w_1 \, ш_* \, Q(w_2)\big) + Q\big(Q(w_1) \, ш_* \, w_2\big), && w_1, w_2 \in \mathcal{H}^{*}, \\
& P(w_1) \, ш_* \, Q(w_2) = Q\big(P(w_1) \, ш_* \, w_2\big) + P\big(w_1 \, ш_* \, Q(w_2)\big), && w_1 \in \mathcal{H}^{*+}, w_2 \in \mathcal{H}^{*}, \\
& Q(w_1) \, ш_* \, P(w_2) = Q\big(w_1 \, ш_* \, P(w_2)\big) + P\big(Q(w_1) \, ш_* \, w_2\big), && w_1 \in \mathcal{H}^{*}, w_2 \in \mathcal{H}^{*+}.
\end{aligned}
$$

with the initial condition that $1 \, ш_* \, w = w \, ш_* \, 1 = w$ for $w \in \mathcal{H}^*$.

Lemma 4.2. *For positive integers $k \geqslant 2$ and $n \geqslant k$, we have*

$$
\begin{aligned}
(30) \qquad
& z_1 \, ш_* \sum_{\substack{s_i \geqslant 1, s_1 \geqslant 2 \\ s_1 + \cdots + s_{k-1} = n-1}} z_{s_1, \cdots, s_{k-1}} \\
& = \sum_{\substack{t_i \geqslant 1, t_1 = 1, t_2 \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, t_2, \cdots, t_k} + k \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2, t_k = 1 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k} + (k-1) \sum_{\substack{t_i \geqslant 1, t_1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k}.
\end{aligned}
$$

and

$$
(31) \qquad z_1 \, ш_* \sum_{\substack{s_i \geqslant 1 \\ s_1 + \cdots + s_{k-1} = n-1}} z_{s_1, \cdots, s_{k-1}} = k \sum_{\substack{t_i \geqslant 1, t_k = 1 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k} + (k-1) \sum_{\substack{t_i \geqslant 1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k}.
$$

Both sides of Eq. (30) are zero when $n = k$.

Proof. We prove Eq. (30) and (31) by induction on n. If $n = 2, 3$, Eq. (30) and (31) can be verified directly. Let $m \geqslant 4$ be an integer. Assume that Eq. (30) and (31) hold when $n < m$. Now assume that $n = m$.

By Eq. (28), Eq. (29) and the induction hypothesis, the left hand side of Eq. (30) becomes

$$
\begin{aligned}
& z_1 \, ш_* \sum_{\substack{s_i \geqslant 1, s_1 \geqslant 2 \\ s_1 + \cdots + s_{k-1} = n-1}} z_{s_1, \cdots, s_{k-1}} = Q(1) \, ш_* \, P\Big(\sum_{\substack{s_i \geqslant 1, s_1 + \cdots + s_{k-1} = n-2}} z_{s_1, \cdots, s_{k-1}} \Big) \\
& = Q\Big(\sum_{\substack{s_i \geqslant 1, s_1 \geqslant 2 \\ s_1 + \cdots + s_{k-1} = n-1}} z_{s_1, \cdots, s_{k-1}} \Big) + P\Big(z_1 \, ш_* \sum_{\substack{s_i \geqslant 1 \\ s_1 + \cdots + s_{k-1} = n-2}} z_{s_1, \cdots, s_{k-1}} \Big) \\
& = \sum_{\substack{t_i \geqslant 1, t_2 \geqslant 2 \\ t_2 + \cdots + t_{k+1} = n-1}} z_{1, t_2, \cdots, t_{k+1}} + P\Big(k \sum_{\substack{t_i \geqslant 1, t_k = 1 \\ t_1 + \cdots + t_k = n-1}} z_{t_1, \cdots, t_k} + (k-1) \sum_{\substack{t_i \geqslant 1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n-1}} z_{t_1, \cdots, t_k} \Big) \\
& = \sum_{\substack{t_i \geqslant 1, t_2 \geqslant 2 \\ t_2 + \cdots + t_{k+1} = n-1}} z_{1, t_2, \cdots, t_{k+1}} + k \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2, t_k = 1 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k} + (k-1) \sum_{\substack{t_i \geqslant 1, t_1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k},
\end{aligned}
$$

which shows that Eq. (30) holds when $n = m$.

By a similar argument, we have

$$z_1 \amalg_* \sum_{s_2+\cdots+s_{k-1}=n-2} z_{1,s_2,\cdots,s_{k-1}}$$

$$= Q\Big(\sum_{s_2+\cdots+s_{k-1}=n-2} z_{1,s_2,\cdots,s_{k-1}}\Big) + Q\Big(z_1 \amalg_* \sum_{s_2+\cdots+s_{k-1}=n-2} z_{s_2,\cdots,s_{k-1}}\Big)$$

$$= \sum_{s_i\geqslant 1, s_2+\cdots+s_{k-1}=n-2} z_{1,1,s_2,\cdots,s_{k-1}}$$

$$+ Q\Big((k-1)\sum_{\substack{t_i\geqslant 1, t_{k-1}=1 \\ t_1+\cdots+t_{k-1}=n-1}} z_{t_1,\cdots,t_{k-1}} + (k-2)\sum_{\substack{t_i\geqslant 1, t_{k-1}\geqslant 2 \\ t_1+\cdots+t_{k-1}=n-1}} z_{t_1,\cdots,t_{k-1}}\Big)$$

$$= \sum_{\substack{t_i\geqslant 1, t_1=t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_1=t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (k-2)\sum_{\substack{t_i\geqslant 1, t_1=1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k}$$

Adding this with Eq. (30) we obtain

$$z_1 \amalg_* \sum_{\substack{s_i\geqslant 1 \\ s_1+\cdots+s_{k-1}=n-1}} z_{s_1,\cdots,s_{k-1}}$$

$$= \sum_{\substack{t_i\geqslant 1, t_1=1, t_2\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + k\sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}$$

$$+ \sum_{\substack{t_i\geqslant 1, t_1=t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_1=t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (k-2)\sum_{\substack{t_i\geqslant 1, t_1=1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k}$$

$$= k\sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}$$

$$+ \sum_{\substack{t_i\geqslant 1, t_1=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_1=t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (k-2)\sum_{\substack{t_i\geqslant 1, t_1=1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k}$$

$$= k\sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}$$

$$+ k\sum_{\substack{t_i\geqslant 1, t_1=t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_1=1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k}$$

$$= k\sum_{\substack{t_i\geqslant 1, t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (k-1)\sum_{\substack{t_i\geqslant 1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k},$$

which shows that Eq. (31) holds when $n = m$. This completes the induction. $\square$

4.2. **Proof of Theorem 2.5.** For the proof of Theorem 2.5, we first consider the case of $k = 2$. In this case we have Euler's decomposition formula [8, 13]

$$z_r \ш_* z_{s_1} = \sum_{t_1 \geqslant 1, t_2 \geqslant 1, t_1 + t_2 = n} \left(\binom{t_1 - 1}{r - 1} + \binom{t_1 - 1}{s_1 - 1} \right) z_{t_1, t_2}.$$

So

$$\sum_{r \geqslant 1, s_1 \geqslant 1, r + s_1 = n} z_r \ш_* z_{s_1} = \sum_{t_1 \geqslant 1, t_2 \geqslant 1, t_1 + t_2 = n} \left(\sum_{r=1}^{t_1} \binom{t_1 - 1}{r - 1} + \sum_{s_1=1}^{t_1} \binom{t_1 - 1}{s_1 - 1} \right) z_{t_1, t_2} = 2^{t_1} z_{t_1, t_2},$$

as required.

For the general case we prove Eq. (25) by induction on n. If $n = 2$, then $k = 2$ and so we are done by the above argument. Let m be a positive integer $\geqslant 3$ and assume that Eq. (25) holds for $n < m$. Now assume that $n = m$. Since we have dealt with the case of $k = 2$, we may assume that $k \geqslant 3$.

We decompose the left hands side of Eq. (25) into three disjoint parts when $r = 1, s_1 \geqslant 1$, when $r \geqslant 2, s_1 \geqslant 2$ and when $r \geqslant 2, s_1 = 1$:

(32)
$$\sum_{\substack{r, s_i \geqslant 1 \\ r + s_1 + \cdots + s_{k-1} = n}} z_r \ш_* z_{s_1, \cdots, s_{k-1}} = \sum_{\substack{s_i \geqslant 1 \\ s_1 + \cdots + s_{k-1} = n-1}} z_1 \ш_* z_{s_1, \cdots, s_{k-1}}$$
$$+ \sum_{\substack{s_i \geqslant 1, r, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} z_r \ш_* z_{s_1, \cdots, s_{k-1}}$$
$$+ \sum_{\substack{r \geqslant 2, s_i \geqslant 1 \\ r + s_2 + \cdots + s_{k-1} = n-1}} z_r \ш_* z_{1, s_2, \cdots, s_{k-1}}.$$

We denote the three sums by $\mathfrak{S}_1, \mathfrak{S}_2$ and $\mathfrak{S}_3$ respectively with the given order.

For the sum $\mathfrak{S}_2$, by Eq. (28) and (29), we have

$$\sum_{\substack{s_i \geqslant 1, r, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} z_r \ш_* z_{s_1, \cdots, s_{k-1}}$$
$$= \sum_{\substack{s_i \geqslant 1, r, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} P(z_{r-1}) \ш_* P(z_{s_1 - 1, \cdots, s_{k-1}})$$
$$= P\left(\sum_{\substack{s_i \geqslant 1, r, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} z_{r-1} \ш_* P(z_{s_1 - 1, \cdots, s_{k-1}}) \right) + P\left(\sum_{\substack{s_i \geqslant 1, r, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} P(z_{r-1}) \ш_* z_{s_1, \cdots, s_{k-1}} \right)$$
$$= P\left(\sum_{\substack{s_i \geqslant 1, r, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} z_{r-1} \ш_* z_{s_1, \cdots, s_{k-1}} \right) + P\left(\sum_{\substack{s_i \geqslant 1, r, s_1 \geqslant 2 \\ r + s_1 + \cdots + s_{k-1} = n}} z_r \ш_* z_{s_1 - 1, \cdots, s_{k-1}} \right).$$

We will denote $\mathfrak{S}_{2,1}$ and $\mathfrak{S}_{2,2}$ for the first and second sum respectively in the last expression. Similarly for the sum $\mathfrak{S}_3$, we have

$$\sum_{\substack{r \geqslant 2, s_i \geqslant 1 \\ r + s_2 + \cdots + s_{k-1} = n-1}} z_r \ш_* z_{1, s_2, \cdots, s_{k-1}}$$

$$= \sum_{\substack{r\geqslant 2,s_i\geqslant 1 \\ r+s_2+\cdots+s_k=n-1}} P(z_{r-1})\,\text{\cyr sh}_*\,Q(z_{s_2,\cdots,s_{k-1}})$$

$$= P\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} z_{r-1}\,\text{\cyr sh}_*\,Q(z_{s_2,\cdots,s_{k-1}})\Big) + Q\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} P(z_{r-1})\,\text{\cyr sh}_*\,z_{s_2,\cdots,s_{k-1}}\Big)$$

$$= P\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} z_{r-1}\,\text{\cyr sh}_*\,z_{1,s_2,\cdots,s_{k-1}}\Big) + Q\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1 \\ r+s_2+\cdots+s_{k-1}=n-1}} z_r\,\text{\cyr sh}_*\,z_{s_2,\cdots,s_{k-1}}\Big)$$

We let $\mathfrak{S}_{3,1}$ and $\mathfrak{S}_{3,2}$ to denote the first and the second sum respectively in the last expression.

By Lemma 4.2 we have

$$(33) \qquad \mathfrak{S}_1 = k\sum_{\substack{t_i\geqslant 1,t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1,t_k\geqslant 2,t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}.$$

We also have

$$\mathfrak{S}_{2,1}+\mathfrak{S}_{3,1} = P\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1 \\ r+s_1+\cdots+s_{k-1}=n}} z_{r-1}\,\text{\cyr sh}_*\,z_{s_1,s_2,\cdots,s_{k-1}}\Big)$$

$$= P\Big(\sum_{\substack{r,s_i\geqslant 1 \\ r+s_1+\cdots+s_{k-1}=n-1}} z_r\,\text{\cyr sh}_*\,z_{s_1,s_2,\cdots,s_{k-1}}\Big)$$

$$(34) \qquad = P\Big(\sum_{\substack{t_i\geqslant 1 \\ t_1+\cdots+t_k=n-1}} \mathcal{C}(t_1,\cdots,t_{k-1})z_{t_1,\cdots,t_k}\Big) \quad \text{(by the induction hypothesis)}$$

$$= \sum_{\substack{t_i\geqslant 1,t_1+\cdots+t_k=n-1}} \mathcal{C}(t_1,\cdots,t_{k-1})z_{t_1+1,\cdots,t_k}$$

$$= \sum_{\substack{t_i\geqslant 1,t_1\geqslant 2,t_1+\cdots+t_k=n}} \mathcal{C}(t_1-1,t_2,\cdots,t_{k-1})z_{t_1,\cdots,t_k}.$$

For the sum $\mathfrak{S}_{2,2}$, we have

$$\mathfrak{S}_{2,2} = P\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1,r+s_1+\cdots+s_{k-1}=n-1}} z_r\,\text{\cyr sh}_*\,z_{s_1,\cdots,s_{k-1}}\Big)$$

$$= P\Big(\sum_{\substack{r,s_i\geqslant 1,r+s_1+\cdots+s_{k-1}=n-1}} z_r\,\text{\cyr sh}_*\,z_{s_1,\cdots,s_{k-1}} - \sum_{\substack{s_i\geqslant 1,s_1+\cdots+s_{k-1}=n-2}} z_1\,\text{\cyr sh}_*\,z_{s_1,\cdots,s_{k-1}}\Big)$$

$$= P\Big(\sum_{\substack{t_i\geqslant 1,t_1+\cdots+t_k=n-1}} \mathcal{C}(t_1,\cdots,t_{k-1})z_{t_1,\cdots,t_k} - k\sum_{\substack{t_i\geqslant 1,t_k=1,t_1+\cdots+t_k=n-1}} z_{t_1,\cdots,t_k}$$

$$(35) \qquad -(k-1)\sum_{\substack{t_i\geqslant 1,t_k\geqslant 2 \\ t_1+\cdots+t_k=n-1}} z_{t_1,\cdots,t_k}\Big) \quad \text{(by the induction hypothesis and Eq. (31))}$$

$$= \sum_{\substack{t_i\geqslant 1 \\ t_1+\cdots+t_k=n-1}} \mathcal{C}(t_1,\cdots,t_{k-1})z_{t_1+1,\cdots,t_k} - k\sum_{\substack{t_i\geqslant 1 \\ t_1+\cdots+t_k=n-1}} z_{t_1+1,\cdots,t_k} + \sum_{\substack{t_i\geqslant 1,t_k\geqslant 2 \\ t_1+\cdots+t_k=n-1}} z_{t_1+1,\cdots,t_k}$$

$$= \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + t_k = n}} \left(\mathcal{C}(t_1 - 1, \cdots, t_{k-1}) - k \right) z_{t_1, \cdots, t_k} + \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k}.$$

For the sum $\mathfrak{S}_{3,2}$, we have

$$\mathfrak{S}_{3,2} = Q\Big(\sum_{r, s_i \geqslant 1, r + s_2 + \cdots + s_{k-1} = n-1} z_r \, \text{Ш}_* \, z_{s_2, \cdots, s_{k-1}} - \sum_{s_2 + \cdots + s_{k-1} = n-2} z_1 \, \text{Ш}_* \, z_{s_2, \cdots, s_{k-1}} \Big)$$

$$= Q\Big(\sum_{\substack{t_i \geqslant 1, t_2 + \cdots + t_k = n-1}} \mathcal{C}(t_2, \cdots, t_{k-1}) z_{t_2, \cdots, t_k} - \Big[\sum_{\substack{t_i \geqslant 1, t_k = 1, t_2 + \cdots + t_k = n-1}} (k-1) z_{t_2, \cdots, t_k}$$

(36)
$$+ \sum_{\substack{t_i \geqslant 1, t_k \geqslant 2 \\ t_2 + \cdots + t_k = n-1}} (k-2) z_{t_2, \cdots, t_k} \Big] \Big) \quad \text{(by the induction hypothesis and Eq. (31))}$$

$$= Q\Big(\sum_{\substack{t_i \geqslant 1, t_2 + \cdots + t_k = n-1}} [\mathcal{C}(t_2, \cdots, t_{k-1}) - (k-1)] z_{t_2, \cdots, t_k} + \sum_{\substack{t_i \geqslant 1, t_k \geqslant 2 \\ t_2 + \cdots + t_k = n-1}} z_{t_2, \cdots, t_k} \Big)$$

$$= \sum_{\substack{t_i \geqslant 1, t_1 = 1 \\ t_1 + \cdots + t_k = n}} [\mathcal{C}(t_1, \cdots, t_{k-1}) - k] z_{t_1, t_2, \cdots, t_k} + \sum_{\substack{t_i \geqslant 1, t_1 = 1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, t_2, \cdots, t_k}$$

by Eq. (21).

Adding Eq. (34) and (35) we obtain

$$\mathfrak{S}_{2,1} + \mathfrak{S}_{2,2} + \mathfrak{S}_{3,1} = \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + t_k = n}} (2\mathcal{C}(t_1 - 1, \cdots, t_{k-1}) - k) z_{t_1, \cdots, t_k} + \sum_{\substack{t_i \geqslant 1, t_1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k}$$

(37)
$$= \sum_{\substack{t_i \geqslant 1, t_1 \geqslant 2 \\ t_1 + \cdots + t_k = n}} (\mathcal{C}(t_1, \cdots, t_{k-1}) - k) z_{t_1, \cdots, t_k} + \sum_{\substack{t_i \geqslant 1, t_1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k}$$

by Eq. (20). Next adding Eq. (36) and (37) together, we get

(38)
$$\mathfrak{S}_{2,1} + \mathfrak{S}_{2,2} + \mathfrak{S}_{3,1} + \mathfrak{S}_{3,2} = \sum_{\substack{t_i \geqslant 1 \\ t_1 + \cdots + t_k = n}} \Big(\mathcal{C}(t_1, \cdots, t_{k-1}) - k \Big) z_{t_1, \cdots, t_k} + \sum_{\substack{t_i \geqslant 1, t_k \geqslant 2 \\ t_1 + \cdots + t_k = n}} z_{t_1, \cdots, t_k}.$$

Finally adding Eq. (33) and (38), we obtain Eq. (25). This completes the inductive proof of Theorem 2.5.

4.3. **Proof of Theorem 2.6.** The proof of Theorem 2.6 is similar to that of Theorem 2.5. First we prove that Eq. (26) holds when $k = 2$. This follows from

$$\sum_{\substack{r \geqslant 1, s_1 \geqslant 2 \\ r + s_1 = n}} z_r \, \text{Ш}_* \, z_{s_1} = \sum_{\substack{r \geqslant 1, s_1 \geqslant 2 \\ r + s_1 = n}} \sum_{\substack{t_1, t_2 \geqslant 1 \\ t_1 + t_2 = n}} \left(\binom{t_1 - 1}{r - 1} + \binom{t_1 - 1}{s_1 - 1} \right) z_{t_1} z_{t_2}$$

$$= \sum_{\substack{t_1, t_2 \geqslant 1, t_1 + t_2 = n}} \left(\sum_{r=1}^{\min(t_1, n-2)} \binom{t_1 - 1}{r - 1} + \sum_{s_1 = 2}^{t_1} \binom{t_1 - 1}{s_1 - 1} \right) z_{t_1} z_{t_2}$$

$$= \left(\sum_{t_1 = 1}^{n-2} (2^{t_1} - 1) z_{t_1, n - t_1} \right) + (2^{n-1} - 2) z_{n-1, 1}$$

$$= \Big(\sum_{t_1=1}^{n-1}(2^{t_1}-1)z_{t_1,n-t_1}\Big) - z_{n-1,1}.$$

For the general case we prove Eq. (26) by induction on n. If $n=3$, then $k=2$ and we are done. Let $m \geqslant 4$ be an integer. Assume that Eq. (26) holds when $n \leqslant m-1$. We will prove that it holds when $n=m$. Since we have dealt with the case of $k=2$, we may assume that $k \geqslant 3$ without loss of generality.

By Eq. (29) the left hand side of Eq. (26) is equal to

$$\sum_{\substack{r,s_i\geqslant 1,s_1\geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n}} z_r \amalg_* z_{s_1,\cdots,s_{k-1}}$$

$$= z_1 \amalg_* \Big(\sum_{\substack{s_i\geqslant 1,s_1\geqslant 2 \\ s_1+\cdots+s_{k-1}=n-1}} z_{s_1,\cdots,s_{k-1}}\Big) + \sum_{\substack{r\geqslant 2,s_i\geqslant 1,s_1\geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n}} P(z_{r-1}) \amalg_* P(z_{s_1-1,s_2,\cdots,s_{k-1}})$$

$$= z_1 \amalg_* \Big(\sum_{\substack{s_i\geqslant 1,s_1\geqslant 2 \\ s_1+\cdots+s_{k-1}=n-1}} z_{s_1,\cdots,s_{k-1}}\Big) + P\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1,s_1\geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n}} z_{r-1} \amalg_* z_{s_1,s_2,\cdots,s_{k-1}}\Big)$$

$$+ P\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1,s_1\geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n}} z_r \amalg_* z_{s_1-1,s_2,\cdots,s_{k-1}}\Big)$$

$$= z_1 \amalg_* \Big(\sum_{\substack{s_i\geqslant 1,s_1\geqslant 2 \\ s_1+\cdots+s_{k-1}=n-1}} z_{s_1,\cdots,s_{k-1}}\Big) + P\Big(\sum_{\substack{r\geqslant 1,s_i\geqslant 1,s_1\geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n-1}} z_r \amalg_* z_{s_1,s_2,\cdots,s_{k-1}}\Big)$$

$$+ P\Big(\sum_{\substack{r\geqslant 2,s_i\geqslant 1 \\ r+s_1+\cdots+s_{k-1}=n-1}} z_r \amalg_* z_{s_1,s_2,\cdots,s_{k-1}}\Big).$$

We denote the three sums in the last expression by $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_3$ respectively with the given order.

By Eq. (30) and (21), we have

$$\mathfrak{S}_1 = \sum_{\substack{t_i\geqslant 1,t_1=1,t_2\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,t_2,\cdots,t_k} + k \sum_{\substack{t_i\geqslant 1,t_1\geqslant 2,t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (k-1) \sum_{\substack{t_i\geqslant 1,t_1,t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}$$

(39)
$$= \sum_{\substack{t_i\geqslant 1,t_1=1 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,\cdots,t_k) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1,t_1=t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}$$

$$+ k \sum_{\substack{t_i\geqslant 1,t_1\geqslant 2,t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (k-1) \sum_{\substack{t_i\geqslant 1,t_1\geqslant 2,t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}.$$

By the induction hypothesis, we have

$$\mathfrak{S}_2 = P\Big(\sum_{\substack{r\geqslant 1,s_i\geqslant 1,s_1\geqslant 2 \\ r+s_1+\cdots+s_{k-1}=n-1}} z_r \amalg_* z_{s_1,s_2,\cdots,s_{k-1}}\Big)$$

$$= P\Big(\sum_{\substack{t_i\geqslant 1 \\ t_1+\cdots+t_k=n-1}} [\mathcal{C}(t_1,\cdots,t_{k-1}) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1, t_2=1 \\ t_1+\cdots+t_k=n-1}} z_{t_1,\cdots,t_k} \Big)$$

$$= \sum_{\substack{t_i\geqslant 1 \\ t_1+\cdots+t_k=n-1}} [\mathcal{C}(t_1,\cdots,t_{k-1}) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1+1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1, t_2=1 \\ t_1+\cdots+t_k=n-1}} z_{t_1+1,\cdots,t_k}$$

$$= \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1-1,\cdots,t_{k-1}) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}$$

By Theorem 2.5 and Eq. (31), we have

$$\mathfrak{S}_3 = P\Big(\sum_{\substack{r\geqslant 2, s_i\geqslant 1 \\ r+s_1+\cdots+s_{k-1}=n-1}} z_r \,{\text ш}_*\, z_{s_1,s_2,\cdots,s_{k-1}} \Big)$$

$$= P\Big(\sum_{\substack{r\geqslant 1, s_i\geqslant 1 \\ r+s_1+\cdots+s_{k-1}=n-1}} z_r \,{\text ш}_*\, z_{s_1,s_2,\cdots,s_{k-1}} - \sum_{\substack{s_i\geqslant 1 \\ s_1+\cdots+s_{k-1}=n-2}} z_1 \,{\text ш}_*\, z_{s_1,s_2,\cdots,s_{k-1}} \Big)$$

$$= P\Big(\sum_{\substack{t_i\geqslant 1 \\ t_1+\cdots+t_k=n-1}} \mathcal{C}(t_1,\cdots,t_{k-1}) - k\sum_{\substack{t_i\geqslant 1, t_k=1 \\ t_1+\cdots+t_k=n-1}} z_{t_1,\cdots,t_k} - (k-1)\sum_{\substack{t_i\geqslant 1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n-1}} z_{t_1,\cdots,t_k} \Big)$$

$$= P\Big(\sum_{\substack{t_i\geqslant 1, t_1+\cdots+t_k=n-1}} [\mathcal{C}(t_1,\cdots,t_{k-1}) - k]\, z_{t_1,\cdots,t_k} + \sum_{\substack{t_i\geqslant 1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n-1}} z_{t_1,\cdots,t_k} \Big)$$

$$= \sum_{\substack{t_i\geqslant 1, t_1+\cdots+t_k=n-1}} [\mathcal{C}(t_1,\cdots,t_{k-1}) - k]\, z_{t_1+1,\cdots,t_k} + \sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n-1}} z_{t_1+1,\cdots,t_k}$$

$$= \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_1+\cdots+t_k=n}} [\mathcal{C}(t_1-1,t_2,\cdots,t_{k-1}) - k]\, z_{t_1,\cdots,t_k} + \sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}.$$

Hence, using Eq. (20) we obtain

$$(40) \quad \begin{aligned} \mathfrak{S}_2 + \mathfrak{S}_3 &= \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2 \\ t_1+\cdots+t_k=n}} \Big(2\mathcal{C}(t_1-1,t_2,\cdots,t_{k-1}) - \mathcal{C}(t_2,\cdots,t_{k-1}) - k\Big)z_{t_1,\cdots,t_k} \\ &\quad - \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + \sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} \\ &= \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2 \\ t_1+\cdots+t_k=n}} \Big(\mathcal{C}(t_1,t_2,\cdots,t_k) - \mathcal{C}(t_2,\cdots,t_{k-1}) - k\Big)z_{t_1,\cdots,t_k} \\ &\quad - \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + \sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k}. \end{aligned}$$

Adding Eq. (39) and (40) together we obtain

$$
\begin{aligned}
\mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 \;=\; & \sum_{\substack{t_i\geqslant 1, t_1=1 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,\cdots,t_k) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1, t_1=t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} \\
& + k \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_k=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + (k-1) \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} \\
& + \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,t_2,\cdots,t_k) - \mathcal{C}(t_2,\cdots,t_{k-1}) - k]z_{t_1,\cdots,t_k} \\
& - \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} + \sum_{\substack{t_i\geqslant 1, t_1, t_k\geqslant 2 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} \\
=\; & \sum_{\substack{t_i\geqslant 1, t_1=1 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,\cdots,t_k) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1, t_1=t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} \\
& + \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,t_2,\cdots,t_k) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1, t_1\geqslant 2, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k} \\
=\; & \sum_{\substack{t_i\geqslant 1 \\ t_1+\cdots+t_k=n}} [\mathcal{C}(t_1,t_2,\cdots,t_k) - \mathcal{C}(t_2,\cdots,t_{k-1})]z_{t_1,\cdots,t_k} - \sum_{\substack{t_i\geqslant 1, t_2=1 \\ t_1+\cdots+t_k=n}} z_{t_1,\cdots,t_k},
\end{aligned}
$$

as desired. This completes the inductive proof of Theorem 2.6.

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