On families of filtered (φ, N)-modules

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Abstract

In this article, as a generalization of Berger's construction, we give a functor from the category of families of filtered (φ, N)-modules (with certain conditions) to the category of families of (φ, Γ)-modules. Combining this with Kedlaya and Liu's theorem we show the stability of weak admissibility of filtered (φ, N)-modules.

Introduction

In p-adic Hodge theory one considers (φ, Γ)-modules as the category of linear algebra data describing p-adic Galois representations, and considers weakly admissible filtered (φ, N)-modules as the category of linear algebra data describing semistable Galois representations.

Recently mathematicians are interested in families of these modules.

In [3] Berger and Colmez defined a functor from the category of families of p-adic Galois representations to the category of families of overconvergent étale (φ, Γ)-modules. But the functor of Berger-Colmez fails to be an equivalence of categories, in contrast with the classical case.

However Kedlaya and Liu [8] showed that, when the base is an affinoid space, every family of overconvergent étale (φ, Γ)-modules can locally be converted into a family of p-adic Galois representations. Moreover they proved that the étale property is stable.

Theorem 0.1. *([8, Theorem 0.2]) Let $\mathcal{L}$ be an affinoid algebra over $\mathbb{Q}_p$, and let $\mathscr{M}_{\mathcal{L}}$ be a family of (φ, Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ in the sense of [8]. If $\mathscr{M}_x$ is étale for some $x \in \mathrm{Max}(\mathcal{L})$, then there exists an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a $\mathcal{B}$-linear representation $V_{\mathcal{B}}$ of G_K whose associated (φ, Γ)-module is isomorphic to $\mathcal{B}\widehat{\otimes}_{\mathcal{L}}\mathscr{M}_{\mathcal{L}}$. Moreover $V_{\mathcal{B}}$ is unique for this property.*

Berger and Colmez [3] also defined a functor from the category of families of semistable Galois representations to the category of families of weakly admissible filtered (φ, N)-modules, which also fails to be equivalent.

In this paper, we study the stability of weak admissibility of filtered (φ, N)-modules. Following an ideal mentioned in [8], we study this question by generalizing Berger's construction in [2] to families of filtered (φ, N)-modules and then applying Theorem 0.1.

Based on Schneider and Teitelbaum's notions of Fréchet-Stein algebras and coadmissible modules over a Fréchet-Stein algebra, we introduce a category of coadmissible (φ, Γ)-modules. As a generalization of Berger's functor given in [2], we construct a functor from the category of families of (φ, N)-modules (with certain technical conditions) to the category of coadmissible (φ, Γ)-modules.

When the base $\mathcal{L}$ is a reduced affinoid algebra, a coadmissible (φ, Γ)-module is essentially a family of (φ, Γ)-modules (in the sense of [8]), so that we can apply Theorem 0.1. As a result, we obtain that, when the base is an affinoid space, under certain conditions, the property of weakly admissibility is stable and every family of weakly admissible filtered (φ, N)-modules locally comes from some family of semistable Galois representations.

Our main result is the following

Theorem 0.2. *Let $\mathcal{L}$ be a reduced affinoid algebra and let D be a filtered (φ, N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$ which satisfies* (BN) *and* (Gr). *If D_x is weakly admissible for some $x\in \mathrm{Max}(\mathcal{L})$, then there exists an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a semi-stable $\mathcal{B}$-representation $V_{\mathcal{B}}$ of G_K whose associated filtered (φ, N)-module is isomorphic to $D_{\mathcal{B}}$. Moreover, $V_{\mathcal{B}}$ is unique for this property.*

The conditions (BN) and (Gr) will be introduced in Section 2.

The case of $N = 0$ is already considered by Hellmann. In [7] Hellmann considered stacks of filtered φ-modules over rigid analytic spaces and adic spaces, and showed that the weakly admissible locus in the stack is an open substack. Hellmann's approach is based on Rapoport-Zink's p-adic symmetric spaces. Our approach is different from Hellmann's.

We outline the structure of this paper. In Section 1 we recall the rings coming from p-adic Hodge theory. In Section 2 we recall the notion of families of filtered (φ, N)-modules. In Section 3.1 we recall the notions of free families and locally free families of (φ,Γ)-modules, and in Section 3.2 we recall Berger and Colmez's construction in [3]. In Section 4 we introduce the category of coadmissible (φ,Γ)-modules and give a functor from the category of filtered (φ, N)-modules (with certain conditions) to the category of coadmissible (φ,Γ)-modules. In Section 5 we prove Theorem 0.2.

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1 Rings of p-adic Hodge Theory

Throughout this paper let K be a finite extension of $\mathbb{Q}_p$, K_0 the maximal absolutely unramified subfield of K. Let φ be the $\mathbb{Q}_p$-automorphism of K_0 which reduces to the absolutely Frobenius of the residue field. Let μ_{p^n} be the set of p^n-th roots of unity in $\bar{\mathbb{Q}}_p$, $\mu_{p^\infty} = \cup_{n\geq 0}\,\mu_{p^n}$. For a finite extension K of $\mathbb{Q}_p$, let $K_n = K[\mu_{p^n}]$ and $K_\infty = K(\mu_{p^\infty}) = \bigcup_{n>0} K_n$. Write $\Gamma = \Gamma_K = \mathrm{Gal}(K_\infty/K)$ and $H_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K_\infty)$.

Let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$ with valuation subring $\mathcal{O}_{\mathbb{C}_p}$ and p-adic valuation v_p normalized such that $v_p(p) = 1$.

Let $\widetilde{\mathrm{E}} = \{(x^{(i)})_{i\geq 0} \mid x^{(i)}\in\mathbb{C}_p,\ (x^{(i+1)})^p = x^{(i)}\ \forall\ i\in\mathbb{N}\}$, $\widetilde{\mathrm{E}}^+$ the subset of $\widetilde{\mathrm{E}}$ such that $x^{(0)}\in\mathcal{O}_{\mathbb{C}_p}$. If $x, y\in\widetilde{\mathrm{E}}$, we define $x + y$ and xy by

$$(x+y)^{(i)} = \lim_{j\to\infty}(x^{(i+j)} + y^{(i+j)})^{p^j},\quad (xy)^{(i)} = x^{(i)}y^{(i)}.$$

Then $\widetilde{\mathrm{E}}$ is a field of characteristic p. Define a function $v_{\mathrm{E}} : \widetilde{\mathrm{E}}\to\mathbb{R}\cup\{+\infty\}$ by putting $v_{\mathrm{E}}((x^{(n)})) = v_p(x^{(0)})$. This is a valuation under which $\widetilde{\mathrm{E}}$ is complete and $\widetilde{\mathrm{E}}^+$ is the ring of integers in $\widetilde{\mathrm{E}}$. If we let $\epsilon = (\epsilon^{(n)})$ be an element of $\widetilde{\mathrm{E}}^+$ with $\epsilon^{(0)} = 1$ and $\epsilon^{(1)}\neq 1$, then $\widetilde{\mathrm{E}}$ is a completed algebraic closure of $\mathbb{F}_p((\epsilon-1))$.

Let $\widetilde{\mathrm{A}}^+$ be the ring $\mathrm{W}(\widetilde{\mathrm{E}}^+)$ of Witt vectors with coefficients in $\widetilde{\mathrm{E}}^+$, $\widetilde{\mathrm{A}}$ the ring of Witt vectors $\mathrm{W}(\tilde{\mathrm{E}})$ and $\widetilde{\mathrm{B}} = \widetilde{\mathrm{A}}[1/p]$. Let $\pi = [\epsilon]-1\in\widetilde{\mathrm{A}}^+$, where $[\epsilon]$ denotes the Teichmüller lifting of ϵ. Let A be the completion of the maximal unramified extension of $\mathbb{Z}((\pi))$ in $\widetilde{\mathrm{A}}$, $\mathrm{B} = \mathrm{A}[1/p]$.

If r, s are two elements in $\mathbb{N}[1/p]\cup\{+\infty\}$, we put $\widetilde{\mathrm{A}}^{[r,s]} = \widetilde{\mathrm{A}}^+\{\frac{p}{[\bar{\pi}^r]}, \frac{[\bar{\pi}^s]}{p}\}$ and $\widetilde{\mathrm{B}}^{[r,s]} = \widetilde{\mathrm{A}}^{[r,s]}[1/p]$ with the convention that $p/[\bar{\pi}^{+\infty}] = 1/[\bar{\pi}]$ and $[\bar{\pi}^{+\infty}]/p = 0$. If I is an interval of $\mathrm{R}\cup\{+\infty\}$, we put $\widetilde{\mathrm{B}}_I = \cap_{[r,s]\subset I}\widetilde{\mathrm{B}}^{[r,s]}$. If $I\subset J$ are two closed intervals so that $\widetilde{\mathrm{B}}^J\subset\widetilde{\mathrm{B}}^I$, we define a valuation v_I on $\widetilde{\mathrm{B}}^J$ by demanding $v_I(x) = 0$ if and only if $x\in\widetilde{\mathrm{A}}^I - p\widetilde{\mathrm{A}}^I$. Then $\widetilde{\mathrm{B}}^I$ is a Banach space for the valuation v_I and the completion of $\widetilde{\mathrm{B}}^J$ for the valuation $\widetilde{\mathrm{B}}^I$ is identified with $\widetilde{\mathrm{B}}^I$. Write

$$\widetilde{\mathrm{B}}^{\dagger,r} = \widetilde{\mathrm{B}}^{[r,+\infty]},\quad \widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{[r,+\infty[}\quad\text{and}\quad \widetilde{\mathrm{B}}^{+}_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{\dagger,0}_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{[0,+\infty[}.$$

Note that $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$ is a Fréchet space for the valuations $v^{[r,s]}$ with $s\in[r,+\infty[$, and $\widetilde{\mathrm{B}}^{\dagger,r}$ is dense in $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$. Put $\widetilde{\mathrm{B}}^{\dagger} = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}$ and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}} = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$. Equip $\widetilde{\mathrm{B}}^{\dagger}$ and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$ with the inductive limit topology. Let $\widetilde{\mathrm{B}}^{+}_{\log} = \widetilde{\mathrm{B}}^{+}_{\mathrm{rig}}[\ell_X]$ and $\widetilde{\mathrm{B}}^{\dagger}_{\log} = \widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}[\ell_X]$, where $\ell_X = \log(\pi)$.

All of the above rings admit actions of G_K. Write $\mathrm{B}_K = \mathrm{B}^{H_K}$, $\widetilde{\mathrm{B}}_K = \widetilde{\mathrm{B}}^{H_K}$, $\widetilde{\mathrm{B}}^I_K = (\widetilde{\mathrm{B}}^I)^{H_K}$, $\widetilde{\mathrm{B}}^\dagger_K = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}_K$ and $\widetilde{\mathrm{B}}^\dagger_{\mathrm{rig},K} = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig},K}$. Put $\mathrm{B}^{\dagger,r}_K = \widetilde{\mathrm{B}}^{\dagger,r}_K \cap \mathrm{B}$. We equip with $\mathrm{B}^{\dagger,r}_K$ the weak topology (see [3]). Let $\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ be the Fréchet completion of $\mathrm{B}^{\dagger,r}_K$ for the topology induced from that on $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig},K}$. Put $\mathrm{B}^\dagger_K = \cup_{r\geq 0}\mathrm{B}^{\dagger,r}_K$ and $\mathrm{B}^\dagger_{\mathrm{rig},K} = \cup_{r\geq 0}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$. The G-actions on $\mathrm{B}^\dagger_K$ and $\mathrm{B}^\dagger_{\mathrm{rig},K}$ factor through Γ. For $s \geq r$ let $\mathrm{B}^{[r,s]}_K$ be the completion of $\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ for the valuation $v^{[r,s]}$.

All of $\mathrm{B}^\dagger$, $\widetilde{\mathrm{B}}^\dagger$, $\widetilde{\mathrm{B}}^\dagger_{\mathrm{rig}}$, $\mathrm{B}^\dagger_K$ and $\mathrm{B}^\dagger_{\mathrm{rig},K}$ admit actions of φ.

There exists a sufficient large $r(K)$ such that, if $s \geq r \geq r(K)$, then $\mathrm{B}^{[r,s]}_K$ is isomorphic to the ring consisting of $f = \sum_{i=-\infty}^{+\infty} a_i T^i$, $a_i \in K_0$, convergent on the domain $p^{-1/e_K r} \leq |T| \leq p^{-1/e_K s}$, where e_K is the absolute ramification index. In fact, we may take $T = \pi_K$, where π_K is as in [1, §1.1].

If $\mathcal{L}$ is a Banach space over $\mathbb{Q}_p$ and B is a locally convex space over $\mathbb{Q}_p$, let $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ be the completion of $\mathcal{L}\otimes_{\mathbb{Q}_p} B$ for the projective tensor product topology [9, §17]. Note that, if $\mathcal{L}$ or B is finite over $\mathbb{Q}_p$, then $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B = \mathcal{L}\otimes_{\mathbb{Q}_p}B$.

Lemma 1.1. *If $\mathcal{L}$ is a Banach space over $\mathbb{Q}_p$ and B is a locally convex spaces over $\mathbb{Q}_p$ which admits an action of a group G, then the G-action can be extended $\mathcal{L}$-linearly and continuously to $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ in a unique way, and $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B)^G = \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B^G$.*

Proof. By [9, Proposition 10.1] there exists a set X such that $\mathcal{L}$ is topologically isomorphic to the Banach space $c_0(X)$ defined by

$$c_0(X) := \{f : X \to \mathbb{Q}_p \text{ such that for any } \varepsilon > 0 \text{ the set } \{x \in X : |f(x)| > \varepsilon\} \text{ is finite}\}.$$

Therefore $\mathcal{L}$ has a topological basis $\{e_x\}_{x\in X}$ if we identify $\mathcal{L}$ with $c_0(X)$ via the above isomorphism. From the definition of completion topological tensor product, we see that $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ consists of $\sum_{x\in X} a_x e_x$ with $a_x \in B$, such that for any open neighborhood U of 0 in B, the set $\{x \in X : a_x \notin U\}$ is finite. We can extend the G-action to $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ by letting $g(\sum_{x\in X} a_x e_x) = \sum_{x\in X} g(a_x)e_x$. The uniqueness of such an extension follows from the continuity. It is clear that $g(\sum_{x\in X} a_x e_x) = \sum_{x\in X} a_x e_x$ if and only if a_x is in B^G. In other words $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B)^G = \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B^G$. □

Definition 1.2. A *coefficient algebra* means a commutative Banach algebra $\mathcal{L}$ over $\mathbb{Q}_p$ satisfying the following conditions:

(a) The norm on $\mathcal{L}$ restricts to the norm on $\mathbb{Q}_p$;

(b) For each maximal ideal $\mathfrak{m}$ of $\mathcal{L}$, the residue field $L_{\mathfrak{m}} := \mathcal{L}/\mathfrak{m}$ is finite over $\mathbb{Q}_p$;

(c) The Jacobson radical of $\mathcal{L}$ is zero; in particular, $\mathcal{L}$ is reduced.

For example, any reduced affinoid algebra over $\mathbb{Q}_p$ is a coefficient algebra. In particular, any finite extension of $\mathbb{Q}_p$ is a coefficient algebra.

As $\widetilde{\mathrm{B}}^I$ and B^I are Fréchet algebras and thus are locally convex, for any coefficient algebra $\mathcal{L}$ we can form $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^I$ and $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^I$. Then we define $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$ to be $\cup_{r\geq 0}\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ and equip it the inductive limit topology. We define $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^\dagger_{\mathrm{rig}}$, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^\dagger$ and $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_K$ similarly. Then we put $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\log} := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\mathrm{rig}})[\ell_X]$ and $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^\dagger_{\log} := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^\dagger_{\mathrm{rig}})[\ell_X]$. From the proof of Lemma 1.1 we see that, if $B = \widetilde{\mathrm{B}}^I, \mathrm{B}^I$, etc, and if ξ is an endomorphism on B, $1\otimes\xi : \mathcal{L}\otimes_{\mathbb{Q}_p} B \to \mathcal{L}\otimes_{\mathbb{Q}_p} B$ can be uniquely extended to a continuous endomorphism on $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$. By abuse of notations, we always denote the resulting endomorphism by the same notation ξ.

Definition 1.3. For $\mathcal{L}$ a coefficient algebra over $\mathbb{Q}_p$ and I a subinterval of $\mathbb{R}$, let $\mathcal{R}^I_{\mathcal{L}}$ be the ring of Laurent series over $\mathcal{L}$ in the variable T which is convergent if $v(T)^{-1} \in I$. Let $v_{\mathcal{L}}$ be the valuation on $\mathcal{L}$.

When $r \geq r(K)$, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ is isomorphic to $\mathcal{R}^r_{\mathcal{L}}\otimes_{\mathbb{Q}_p} K_0$ via $\pi_K \mapsto T$, and so $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$ is isomorphic to $\mathcal{R}_{\mathcal{L}}\otimes_{\mathbb{Q}_p} K_0$.

2 Filtered (φ, N)-modules

Definition 2.1. Let $\mathcal{L}$ be a coefficient algebra. A *filtered* (φ, N)*-module* over $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$ is a locally free $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$-module D of finite rank together with the following structures:

(a) a φ-semilinear automorphism on D which is again denoted by φ;

(b) a linear endomorphism N on D satisfying $N\varphi = p\varphi N$;

(c) a descending, separated and exhaustive $\mathbb{Z}$-filtration $\mathrm{Fil}^\bullet D_K$ on $D_K := K \otimes_{K_0} D$ by $\mathcal{L} \otimes_{\mathbb{Q}_p} K$-submodules.

Let $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$ be the category of filtered (φ, N)-modules over $\mathcal{L} \otimes_{\mathbb{Q}_p} K_0$. When $\mathcal{L} = \mathbb{Q}_p$ we write $\mathrm{FilM}^{\varphi,N}_K$ for $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$.

If $\mathcal{L}'$ is another coefficient algebra and there is a continuous map $\mathcal{L} \to \mathcal{L}'$, then we have a functor

$$\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}} \to \mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}'}, \quad D \mapsto D_{\mathcal{L}'} := \mathcal{L}' \otimes_{\mathcal{L}} D.$$

In particular, if $\mathfrak{m}$ is a maximal ideal of $\mathcal{L}$, then $D_\mathfrak{m} = L_\mathfrak{m} \otimes_{\mathcal{L}} D$ is a filtered (φ, N)-module over $L_\mathfrak{m} \otimes_{\mathbb{Q}_p} K_0$. Hence a filtered (φ, N)-module over $\mathcal{L} \otimes_{\mathbb{Q}_p} K_0$ can be considered as a family of filtered (φ, N)-modules on $\mathrm{Max}(\mathcal{L})$, the maximal spectrum of $\mathcal{L}$.

Lemma 2.2. *Suppose that L is a finite extension of $\mathbb{Q}_p$. If D is an object in $\mathrm{FilM}^{\varphi,N}_{K;L}$, then D is free over $L \otimes_{\mathbb{Q}_p} K_0$.*

Proof. We write $L \otimes_{\mathbb{Q}_p} K_0 = \prod_i L_i$. Put $D_i = L_i \otimes_{L\otimes_{\mathbb{Q}_p} K_0} D$. Then $D = \bigoplus_i D_i$. As φ acts transitively on the set $\{L_i\}$, it also acts transitively on the set $\{D_i\}$. This implies that for any two indices i, j we have $\dim_{L_i} D_i = \dim_{L_j} D_j$ which ensures the freeness of D. □

Proposition 2.3. *Suppose that $\mathcal{L}$ is a reduced affinoid algebra over $\mathbb{Q}_p$. Let D be an object in $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$. Then for any $x \in \mathrm{Max}(\mathcal{L})$ there exists a neighborhood $\mathrm{Max}(\mathcal{B})$ of x such that $D_\mathcal{B} = \mathcal{B} \otimes_{\mathcal{L}} D$ is free over $\mathcal{B} \otimes_{\mathbb{Q}_p} K_0$.*

Proof. By Lemma 2.2, D_x is free over $L_x \otimes_{\mathbb{Q}_p} K_0$. Let $\{v_i\}$ be a basis of D_x over $L_x \otimes_{\mathbb{Q}_p} K_0$, and let $\{e_j\}$ be a basis of K_0 over $\mathbb{Q}_p$. Then $\{e_j v_i\}_{i,j}$ is a basis of D_x over L_x. For any i let $\tilde{v}_i$ be a lifting of v_i in D. Then there exists a neighborhood $\mathrm{Max}(\mathcal{B})$ of x such that $\mathcal{B} \otimes_{\mathcal{L}} D$, as a $\mathcal{B}$-module, is generated by $\{e_j \tilde{v}_i\}_{i,j}$, which implies that $\mathcal{B} \otimes_{\mathcal{L}} D$ is free over $\mathcal{B} \otimes_{\mathbb{Q}_p} K_0$. □

Let B_{dR} be Fontaine's de Rham period ring. Put

$$\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^+_{\mathrm{dR}} := \varprojlim_i \mathcal{L} \otimes_{\mathbb{Q}_p} (\mathrm{B}^+_{\mathrm{dR}}/t^i\mathrm{B}^+_{\mathrm{dR}}) \quad \text{and} \quad \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}} := \cup_{i\geq 0} t^{-i}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^+_{\mathrm{dR}}).$$

Let G_K act continuously on $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}}$ in the way such that the action on $\mathcal{L}$ is trivial.

Recall that

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\mathrm{rig}}[1/t])^{G_K} = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\log}[1/t])^{G_K} = \mathcal{L} \otimes_{\mathbb{Q}_p} K_0, \ \ (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}})^{G_K} = \mathcal{L} \otimes_{\mathbb{Q}_p} K.$$

Let V be an $\mathcal{L}$-representation of G_K, which means a finite locally free $\mathcal{L}$-module (of constant rank) together with a continuous action of G_K.

Definition 2.4. We say that V is *semi-stable* (resp. *crystalline*) if

$$\mathrm{D}_{\mathrm{st},\mathcal{L}}(V) = ((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\log}[1/t]) \otimes_{\mathcal{L}} V)^{G_K}$$

$$\big(\text{resp. } \mathrm{D}_{\mathrm{cris},\mathcal{L}}(V) = ((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\mathrm{rig}}[1/t]) \otimes_{\mathcal{L}} V)^{G_K}\big)$$

is a locally free $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$-module of rank $d = \mathrm{rank}_{\mathcal{L}} V$. Similarly we say that V is *de Rham* if

$$\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V) := ((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}})\otimes_{\mathcal{L}} V)^{G_K}$$

is a locally free $\mathcal{L}\otimes_{\mathbb{Q}_p} K$-module of rank d. Let $\mathrm{Rep}^{\mathrm{cris}}_{\mathcal{L}}(G_K)$, $\mathrm{Rep}^{\mathrm{st}}_{\mathcal{L}}(G_K)$ and $\mathrm{Rep}^{\mathrm{dR}}_{\mathcal{L}}(G_K)$ be the category of crystalline $\mathcal{L}$-representations of G_K, the category of semi-stable $\mathcal{L}$-representations and the category of de Rham representations, respectively.

Now we suppose that $\mathcal{L}$ is a reduced affinoid algebra till the end of this section.

In this case, by a result of Berger and Colmez [3, Corollary 6.3.3], V is crystalline (resp. semi-stable) if and only if so are $V_{\mathfrak{m}} = L_{\mathfrak{m}}\otimes_{\mathcal{L}} V$ for all $\mathfrak{m}\in\mathrm{Max}(\mathcal{L})$. Furthermore

$$\mathrm{D}_{\mathrm{cris},L_{\mathfrak{m}}}(V_{\mathfrak{m}}) = L_{\mathfrak{m}}\otimes_{\mathcal{L}}\mathrm{D}_{\mathrm{cris},\mathcal{L}}(V) \quad (\text{resp. } \mathrm{D}_{\mathrm{st},L_{\mathfrak{m}}}(V_{\mathfrak{m}}) = L_{\mathfrak{m}}\otimes_{\mathcal{L}}\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)).$$

If V is semi-stable, then $\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)$ is an object of $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$ with $\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)_K = \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)$. So $\mathrm{D}_{\mathrm{st},\mathcal{L}}$ is a functor from the category $\mathrm{Rep}^{\mathrm{st}}_{\mathcal{L}}(G_K)$ to the category $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$.

In the case when $\mathcal{L}$ is a finite extension of $\mathbb{Q}_p$, the image of the functor $\mathrm{D}_{\mathrm{st},\mathcal{L}}$ can be determined explicitly. In this case, an object in $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$ can also be considered as an object in $\mathrm{FilM}^{\varphi,N}_{K}$ by forgetting the $\mathcal{L}$-module structure. We say that D is *weakly admissible* if it is so as an object in $\mathrm{FilM}^{\varphi,N}_{K}$. Let $\mathrm{FilM}^{\varphi,N,\mathrm{wa}}_{K;\mathcal{L}}$ be the full subcategory of $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$ consisting of weakly admissible objects.

Proposition 2.5. *If L is a finite extension of $\mathbb{Q}_p$, then the functor $\mathrm{D}_{\mathrm{st},L}$ is an equivalence of categories between the category $\mathrm{Rep}^{\mathrm{st}}_{L}(G_K)$ and the category $\mathrm{FilM}^{\varphi,N,\mathrm{wa}}_{K;L}$; the quasi-inverse is the functor $\mathrm{V}_{\mathrm{st},L}$ defined by*

$$\mathrm{V}_{\mathrm{st},L}(D) = (\widetilde{\mathrm{B}}^{+}_{\log}[1/t]\otimes_{K_0} D)^{\varphi=1,N=0}\cap\mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}\otimes_K D_K).$$

Proof. By Colmez–Fontaine theorem [6], $\mathrm{D}_{\mathrm{st},\mathbb{Q}_p}$ is an equivalence of categories between $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ and $\mathrm{FilM}^{\varphi,N,\mathrm{wa}}_{K;\mathbb{Q}_p}$. But an object V of $\mathrm{Rep}^{\mathrm{st}}_{L}(G_K)$ is equivalent to an object $\tilde{V}$ of $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ together with a ring of endomorphisms of $\tilde{V}$ isomorphic to L, while an object D of $\mathrm{FilM}^{\varphi,N,\mathrm{wa}}_{K;L}$ is equivalent to an object $\tilde{D}$ of $\mathrm{FilM}^{\varphi,N,\mathrm{wa}}_{K;\mathbb{Q}_p}$ together with a ring of endomorphisms isomorphic to L. □

In the general case, it is difficult to determine the image of the functor $\mathrm{D}_{\mathrm{st},\mathcal{L}}$. In Proposition 5.5 we will give a property for these D which are in this image.

We consider the following two conditions:

(BN). Locally on $\mathcal{L}$ there exists a basis compatible with N. Explicitly, for any $x\in\mathrm{Max}(\mathcal{L})$ there exists a neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a $\mathcal{B}\otimes_{\mathbb{Q}_p} K_0$-base $\{v_1,\cdots,v_d\}$ of $D_{\mathcal{B}}$ such that $N(v_1) = 0$ and $N(v_i)\in\mathcal{L}\otimes_{\mathbb{Q}_p} K_0\cdot v_1\oplus\cdots\oplus\mathcal{L}\otimes_{\mathbb{Q}_p} K_0\cdot v_{i-1}$ for $i\geq 2$.

(Gr). For every $i\in\mathbb{Z}$, $\mathrm{Gr}^i D_K = \mathrm{Fil}^i D_K/\mathrm{Fil}^{i+1}D_K$ is locally free over $\mathcal{L}\otimes_{\mathbb{Q}_p} K$ of constant rank.

The main result of this paper is the following

Theorem 2.6. (= Theorem 0.2) *Let $\mathcal{L}$ be a reduced affinoid algebra and let D be a filtered (φ,N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$ which satisfies* (BN) *and* (Gr)*. If D_x is weakly admissible for some $x\in\mathrm{Max}(\mathcal{L})$, then there exists an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a semi-stable $\mathcal{B}$-representation $V_{\mathcal{B}}$ of G_K whose associated filtered (φ,N)-module is isomorphic to $D_{\mathcal{B}}$. Moreover, $V_{\mathcal{B}}$ is unique for this property.*

The proof of Theorem 2.6 will be given in Section 5.

3 (φ,Γ)-modules

3.1 Free and locally free (φ,Γ)-modules

By a *(locally) free* (φ,Γ)*-module* over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K$ (resp. $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$) we mean a (locally) free $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K$(resp. $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$)-module D of finite rank equipped with a semilinear (φ,Γ)-action such that the map $\varphi^*\mathrm{D}\to\mathrm{D}$ is an isomorphism.

Definition 3.1. We say that a locally free (φ,Γ)-module D over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K$ is *étale* if it admits a finite (φ,Γ)-stable $(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger}_K)$-submodule N such that $\varphi^*N\to N$ is an isomorphism and the induced map $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K)\otimes_{\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger}_K} N\to\mathrm{D}$ is an isomorphism. We say that a locally free (φ,Γ)-module D over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ is *étale* if it arise by base change extension from an étale (φ,Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K$.

By [8, Proposition 6.5] the natural base change functor from the category of étale (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K$ to the category of étale (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ is fully faithful.

The following property of free (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ is very useful.

Proposition 3.2. *Let* D *be a free* φ*-module over* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. *Then there exists* $r(\mathrm{D})>r(K)$ *sufficient large such that for any* $r\geq r(\mathrm{D})$ *there exists a unique free* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$*-submodule* D_r *of* D *satisfying the following conditions*

(a) $\mathrm{D}=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r$;

(b) *The* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pr}_{\mathrm{rig},K}$*-module* $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pr}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r$ *has a base contained in* $\varphi(\mathrm{D}_r)$.

In particular, we have $\mathrm{D}_s=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,s}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r$ *for any* $s>r$, *and if* D *is a* (φ,Γ)*-module, then* $\gamma(\mathrm{D}_r)=\mathrm{D}_r$ *for all* $\gamma\in\Gamma$.

In the case when $\mathcal{L}=\mathbb{Q}_p$, this is exactly [2, Theorem I.3.3].
Let $F(T)$ be a formal series such that $F(\pi_K)=\varphi(\pi_K)$. Write $F(T)=\varphi(T)=T^p+pf(T)$.

Lemma 3.3. *When* $r>1$, *the map* $z\mapsto F(z)$ *induces a surjection from* $\{z\in\mathbb{C}_p\mid p^{-1/pr}\leq|z|<1\}$ *to* $\{z\in\mathbb{C}_p\mid p^{-1/r}\leq|z|<1\}$.

Proof. By Weierstrass Preparation Theorem, for any $w\in\mathbb{C}_p$ with $|w|<1$, $F(z)=w$ has a solution with $|z|<1$. As $|pf(z)|\leq p^{-1}$, in the case when $|w|>p^{-1}$, we have $|z^p|=|w|$ and so $|z|=|w|^{1/p}$. □

Proposition 3.4. *Let* $\mathcal{L}$ *be a coefficient algebra. When* $r\gg 0$, *we have*

$$\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}\cap\varphi(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})=\varphi(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r/p}_{\mathrm{rig},K})$$

Proof. We choose a $\mathbb{Q}_p$-base $\{e_1,\cdots,e_d\}$ of K_0. Then $\varphi(e_1),\cdots\varphi(e_d)$ is again a $\mathbb{Q}_p$-base of K_0. When $r\gg 0$, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ is isomorphic to $\mathcal{R}^r_{\mathcal{L}}\otimes_{\mathbb{Q}_p}K_0$. Thus, if G is in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$, we may write G in the form

$$G=\sum_{j=1}^{d}(\sum_i x_{ij}T^i)\otimes e_j,$$

where $\sum_i x_{ij}z^i$ is convergent on the domain $p^{-1/e_K r}\leq|z|<1$ for any $j\in\{1,\cdots,r\}$. Let

$$H=\varphi(G)=\sum_{j=1}^{r}(\sum_i x_{ij}\varphi(T)^i)\otimes\varphi(e_j)=\sum_{j=1}^{r}(\sum_i x_{ij}F(T)^i)\otimes\varphi(e_j).$$

If H is again in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$, then by Lemma 3.3, $\sum_i x_{ij}w^i$ is convergent on the domain $p^{-1/e_K pr}\leq|w|<1$ for any $j\in\{1,\cdots,r\}$, which implies that G is in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r/p}_{\mathrm{rig},K}$. □

Proof of Proposition 3.2. The argument is similar to the proof of [2, Theorem I.3.3]. We give the details for completion.

Since D is a free $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$-module, it has a $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$-base $\{e_1,\cdots,e_d\}$. As $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}=\bigcup_{r>0}\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$, there exists $r_0=r(\mathrm{D})$ such that the matrix of φ with respect to this base is in $\mathrm{GL}_d(\mathrm{B}^{\dagger,r_0}_{\mathrm{rig},K})$. For any $r\geq r_0$ put $\mathrm{D}_r=\oplus_{i=1}^d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})e_i$. Obviously $\mathrm{D}=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r$. Further $\mathrm{D}_{pr}=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pr}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r$ has a base in $\varphi(\mathrm{D}_r)$. Indeed, $\{\varphi(e_i)\mid i=1\cdots,d\}$ is such a base. This proves the existence of D_r.

Let $\mathrm{D}_r^{(1)}$ and $\mathrm{D}_r^{(2)}$ be two $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$-submodules of D satisfying Conditions (a) and (b). We choose bases for these two submodules. Let P_1 and P_2 be respectively the matrices of φ with respect to these two bases, so P_1,P_2 are in $\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pr}_{\mathrm{rig},K})$. Let $M\in\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})$ be the transfer matrix from the base of $\mathrm{D}_r^{(1)}$ to that of $\mathrm{D}_r^{(2)}$. Then $\varphi(M)=P_1^{-1}MP_2$. We show that M is in $\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. If M is $\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,s}_{\mathrm{rig},K})$ with $s\geq pr$, then $\varphi(M)=P_1^{-1}MP_2$ is also in $\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,s}_{\mathrm{rig},K})$. By Proposition 3.4, M is in $\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,s/d}_{\mathrm{rig},K})$. Repeating this several times we see that $M\in\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. By the same reason we have $M^{-1}\in\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. So M is in $\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$, which implies that $\mathrm{D}_r^{(1)}=\mathrm{D}_r^{(2)}$. This proves the uniqueness of D_r.

If $s>r$, the module $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,s}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r$ satisfies (a) and (b) with r there replaced by s. Thus by the uniqueness of D_s we have

$$\mathrm{D}_s=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,s}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r.$$

If D is a (φ,Γ)-module, from the uniqueness of D_r we obtain $\gamma(\mathrm{D}_r)=\mathrm{D}_r$ for any $\gamma\in\Gamma$. □

3.2 Locally free (φ,Γ)-modules associated to $\mathcal{L}$-linear representations

We recall the functor of Berger and Colmez from the category of $\mathcal{L}$-representations of G_K to the category of étale (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{K}$ and the functor of Kedlaya and Liu from the the category of $\mathcal{L}$-representations of G_K to the category of étale (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$.

For any finite extension L of K we write $\mathrm{A}^{\dagger,r}_{L,n}=\varphi^{-n}(\mathrm{A}^{\dagger,p^nr}_{L})$.

Proposition 3.5. *([3, Proposition 4.2.8]) Let $\mathcal{L}$ be a coefficient algebra over $\mathbb{Q}_p$. Let $T_{\mathcal{L}}$ be a free $\mathcal{O}_{\mathcal{L}}$-linear representation of rank d. Let L be a finite Galois extension of K such that G_L acts trivially on $T_{\mathcal{L}}/12pT_{\mathcal{L}}$. Then there exists $n(L,T_{\mathcal{L}})\geq 0$ such that for $n\geq n(L,T_{\mathcal{L}})$, $\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\widetilde{\mathrm{A}}^{\dagger,(p-1)/p}\otimes_{\mathcal{O}_{\mathcal{L}}}T_{\mathcal{L}}$ has a unique $(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger,(p-1)/p}_{L,n})$-module $\mathrm{D}^{\dagger,(p-1)/p}_{\mathcal{L};L,n}(T_{\mathcal{L}})$ which is free of rank d, is fixed by H_L, has a basis which is almost invariant under Γ_L (i.e. for any $\gamma\in\Gamma_L$ the matrix of action of $\gamma-1$ on this basis has positive valuation) and satisfies*

$$(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\widetilde{\mathrm{A}}^{\dagger,(p-1)/p})\otimes_{\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger,(p-1)/p}_{L,n}}\mathrm{D}^{\dagger,(p-1)/p}_{\mathcal{L};L,n}(T_{\mathcal{L}})=(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{A}}^{\dagger,(p-1)/p})\otimes_{\mathcal{O}_{\mathcal{L}}}T_{\mathcal{L}}.$$

Theorem 3.6. *([3, Théorème 4.2.9]) Let $\mathcal{L}$ be a coefficient algebra over $\mathbb{Q}_p$. Let V be an $\mathcal{L}$-reresentation admitting a free Galois stable $\mathcal{O}_{\mathcal{L}}$-lattice T. Then there exists some $r(V)=(p-1)p^{n-1}$ such that for any $r\geq r(V)$ we may define*

$$\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V):=((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{L})\otimes_{\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger,r(V)}_{L}}\varphi^n(\mathrm{D}^{\dagger,(p-1)/p}_{\mathcal{L};L,n}(T))^{H_K}$$

for some L, n, so that the construction does not depend on the choices of T, L, n, and the following statements hold.

(a) *The $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{K})$-module $\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)$ is locally free of rank d.*

(b) *The natural map $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger,r})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{K}}\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)\to(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger,r})\otimes_{\mathcal{L}}V$ is an isomorphism.*

(c) *For any maximal ideal $\mathfrak{m}$ of $\mathcal{L}$, writing $V_{\mathfrak{m}}:=L_{\mathfrak{m}}\otimes_{\mathcal{L}}V$, the natural map $L_{\mathfrak{m}}\otimes_{\mathcal{L}}\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)\to\mathrm{D}^{\dagger,r}_{L_{\mathfrak{m}}}(V_{\mathfrak{m}})$ is an isomorphism.*

Put

$$\mathrm{D}^{\dagger}_{\mathcal{L}}(V) := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{K}} \mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)$$

and

$$\mathrm{D}^{\dagger}_{\mathrm{rig},\mathcal{L}}(V) := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{K}} \mathrm{D}^{\dagger}_{\mathcal{L}}(V).$$

Then $\mathrm{D}^{\dagger}_{\mathcal{L}}(V)$ (resp. $\mathrm{D}^{\dagger}_{\mathrm{rig},\mathcal{L}}(V)$) is an étale (φ,Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{K}$ (resp. $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$).

Proposition 3.7. *([8]) We have*

$$V = \left((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{K}} \mathrm{D}^{\dagger}_{\mathcal{L}}(V)\right)^{\varphi=1} = \left((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D}^{\dagger}_{\mathrm{rig},\mathcal{L}}(V)\right)^{\varphi=1}.$$

4 Coadmissible (φ,Γ)-modules and filtered (φ,N)-modules

In this section we introduce a notion of coadmissible (φ,Γ)-modules and construct a functor, a family version of Berger's functor [2], from the category of filtered (φ,N)-modules satisfying certain conditions (GBN) and (GFF) to the category of coadmissible (φ,Γ)-modules. In this section we always suppose that $\mathcal{L}$ is noetherian and satisfies the condition (FL) given in Section 4.1.

4.1 Coadmissible (φ,Γ)-modules

First we recall the notions of Fréchet-Stein algebras and coadmissible modules defined by Schneider and Teitelbaum [10, §3].

Definition 4.1. A (commutative) Fréchet-algebra A over K is called a *Fréchet-Stein algebra* if there is a sequence $q_1 \le \cdots \le q_n \le \cdots$ of continuous algebra seminorms on A which defines the Fréchet topology on A such that

- $A_{q_n} := A/\{x \in A \,|\, q_n(x) = 0\}$ is a noetherian Banach algebra,
- A_{q_n} is a flat $A_{q_{n+1}}$-module for any $n \in \mathbb{N}$.

For $(A,(q_n))$ as above we have $A \xrightarrow{\simeq} \varprojlim_n A_{q_n}$.

Definition 4.2. A *coherent sheaf* for $(A,(q_n))$ is a sequence $\{M_n\}_{n\in\mathbb{N}}$, where M_n is a finite A_{q_n}-module, together with isomorphisms $A_{q_n} \otimes_{A_{q_{n+1}}} M_{n+1} \xrightarrow{\simeq} M_n$.

If $\{M_n\}$ is a coherent sheaf for $(A,(q_n))$, its A-module of "global sections" is defined by $\mathit{\Gamma}(\{M_n\}) := \varprojlim_n M_n$. If $\{M_n\}$ is a coherent sheaf for $(A,(q_n))$ and if $M = \mathit{\Gamma}(\{M_n\})$, then the natural map $A_{q_n} \otimes_A M \to M_n$ is isomorphic for any $n \in \mathbb{N}$.

Definition 4.3. An A-module is called *coadmissible* if it is isomorphic to the module of global sections of some coherent sheaf for $(A,(q_n))$.

The "global sections" functor $\mathit{\Gamma}$ is an equivalence of categories between the category of coherent sheaves for $(A,(q_n))$ and the category of coadmissible A-modules.

If M is a coadmissible A-module associated to a coherent sheaf $\{M_n\}$, equip each M_n its canonical Banach space topology and then equip M the projective limit topology of these canonical topologies. The resulting topology on M is called the *canonical topology* of M.

Let $(A',(q'_m))$ be another Fréchet-Stein algebra and assume that there is a continuous map $A \to A'$. For a coadmissible A-module M, in general $A' \otimes_A M$ is not a coadmissible A'-module. But $\{A'_{q'_m} \otimes_A M\}$ is a coherent sheaf. Let $(A' \otimes_A M)^{\mathrm{ad}}$ denote the corresponding coadmissible A'-module. Then the natural map $A' \otimes_A M \to (A' \otimes_A M)^{\mathrm{ad}}$ has a dense image.

Throughout this section we assume that the coefficient algebra $\mathcal{L}$ is noetherian and satisfies the following condition:

(FL) For any two intervals $I \subset I'$ of $[0, +\infty[$, $\mathcal{R}_{\mathcal{L}}^{I}$ is flat over $\mathcal{R}_{\mathcal{L}}^{I'}$.

If $\mathcal{L}$ is a reduced affinoid algebra over $\mathbb{Q}_p$, then $\mathcal{L}$ satisfies (FL).

The condition (FL) ensures that $\mathcal{R}_{\mathcal{L}}^{r}$ is a Fréchet-Stein algebra and is isomorphic to the projective limit $\varprojlim_{s} \mathcal{R}_{\mathcal{L}}^{[r,s]}$ of Banach algebras. When r is sufficiently large, $\mathcal{L}\widehat{\otimes}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}$ is isomorphic to $\mathcal{R}_{\mathcal{L}}^{r} \otimes_{\mathbb{Q}_p} K_0$ and thus is a Fréchet-Stein algebra.

Definition 4.4. For a *coadmissible φ-module* (resp. *(φ, Γ)-module*) M over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}$, we shall mean a direct system $\{\mathrm{M}_r\}_{r \geq u}$ with u a positive integer, where M_r is a coadmissible $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}$-module for $r \geq u$, which satisfies the following properties:

(a) $\mathrm{M}^{[r,s]} := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{K}^{[r,s]}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} \mathrm{M}_r$ with $r \leq s$ is locally free over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{K}^{[r,s]}$ of constant rank;

(b) the natural map $\mathrm{M}_r \to \mathrm{M}_{r'}$ induces an isomorphism $\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r'}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} M\Big)^{\mathrm{ad}} \xrightarrow{\sim} \mathrm{M}_{r'}$;

(c) there exists a semilinear map $\varphi : \mathrm{M}_r \to \mathrm{M}_{pr}$ such that $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}) \cdot \varphi(\mathrm{M}_r)$ is dense in M_{pr} for the canonical topology and such that the following diagram

$$\begin{array}{ccc} \mathrm{M}_r & \longrightarrow & \mathrm{M}_{r'} \\ \downarrow \varphi & & \downarrow \varphi \\ \mathrm{M}_{pr} & \longrightarrow & \mathrm{M}_{pr'} \end{array}$$

is commutative for any pair $r < r'$ with $r \geq u$, where the horizontal arrows are natural maps.

(d) in the case of (φ, Γ)-module there exist semilinear Γ-actions on M_r, $r \geq u$, which commute with the natural maps $\mathrm{M}_r \to \mathrm{M}_{r'}$ $(r' \geq r)$ and the maps $\varphi : \mathrm{M}_r \to \mathrm{M}_{pr}$.

Condition (b) is equivalent to the following condition:

If $r \leq r' \leq s' \leq s$, then the map $\mathrm{M}_r \to \mathrm{M}_{r'}$ induces an isomorphism

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{K}^{[r',s']}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{K}^{[r,s]}} \mathrm{M}^{[r,s]} \xrightarrow{\sim} \mathrm{M}^{[r',s']}.$$

Condition (c) is equivalent to the following condition:

For any pair $r \leq s$ with $r \geq u$, there exists a semilinear map $\varphi : \mathrm{M}^{[r,s]} \to \mathrm{M}^{[pr,ps]}$ such that $\varphi(\mathrm{M}^{[r,s]})$ generates $\mathrm{M}^{[pr,ps]}$, and such that if $r \leq r' \leq s' \leq s$, then the following diagram

$$\begin{array}{ccc} \mathrm{M}^{[r,s]} & \longrightarrow & \mathrm{M}^{[r',s']} \\ \downarrow \varphi & & \downarrow \varphi \\ \mathrm{M}^{[pr,ps]} & \longrightarrow & \mathrm{M}^{[pr',ps']} \end{array}$$

is commutative where the horizontal arrows are natural maps.

Proposition 4.5. *A free φ-module (resp. (φ, Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}$ is a coadmissible φ-module (resp. (φ, Γ)-module).*

Proof. This follows from Proposition 3.2. □

If $\mathcal{L} \to \mathcal{L}'$ is a continuous map of coefficient algebras, and if M is a coadmissible φ-module (resp. (φ, Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$, then there exists a unique coadmissible φ-module (resp. (φ, Γ)-module) over $\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$, denoted by $\mathrm{M}_{\mathcal{L}'}$, such that for any pair $r < s$ as in Definition 4.4,

$$(\mathrm{M}_{\mathcal{L}'})^{[r,s]} = (\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{[r,s]}_K) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{[r,s]}_K} \mathrm{M}^{[r,s]}.$$

To end this subsection, we apply Kedlaya and Liu's result [8] to coadmissible φ-modules and (φ, Γ)-modules.

Definition 4.6. Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{L}$ be an affinoid algebra over K. Recall that $\mathcal{R}^r_K$ denotes the ring of Laurent series with coefficients in K in a variable T convergent on the annulus $0 < v_p(T) \leq 1/r$. By a *vector bundle* over $\mathcal{L}\widehat{\otimes}_K \mathcal{R}^r_K$ we will mean a coherent locally free sheaf over the product of this annulus with $\mathrm{Max}(\mathcal{L})$ in the category of rigid analytic spaces over K. (In the case when $\mathcal{L}$ is disconnected, we insist that the rank be constant.) By a vector bundle over $\mathcal{L}\widehat{\otimes}_K \mathcal{R}_K$ we will mean an object in the direct limit as $r \to +\infty$ of the categories of vector bundles over $\mathcal{L}\widehat{\otimes}_K \mathcal{R}^r_K$.

When $r \gg 0$ we have an isomorphism $\mathrm{B}^{\dagger,r}_{\mathrm{rig},K} \cong \mathcal{R}^r_{K_0}$. We thus obtain the notion of a vector bundle over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ dependent on the choice of the isomorphism. However, the notion of a vector bundle over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$ does not depend on any choices.

Definition 4.7. Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{L}$ be an affinoid algebra over $\mathbb{Q}_p$. By a *family of φ-module (resp. (φ, Γ)-modules)* over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$ we mean a vector bundle $\mathscr{M}$ over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$ equipped with an isomorphism $\varphi^*\mathscr{M} \to \mathscr{M}$ viewed as a semilinear φ-action (and a semilinear Γ-action commuting with the φ-action).

Now let $(\mathrm{M}; \{\mathrm{M}_r\}_{r\geq u})$ be a coadmissible φ-module (resp. (φ, Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$. For any $r \geq u$ let $\mathscr{M}_r$ be the coherent sheaf over $\mathrm{Max}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$ associated to M_r. Then $\mathscr{M}_r$ is a vector bundle over the affinoid space $\mathrm{Max}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. Let $\mathscr{M}$ be the direct limit of the system $\{\mathscr{M}_r \mid r \geq u\}$. Conditions (c) and (d) in Definition 4.4 ensure that $\mathscr{M}$ is a family of φ-modules (resp. (φ, Γ)-modules) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$. In this way we associate to any coadmissible φ-module (resp. (φ, Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$ a family of φ-modules (resp. (φ, Γ)-modules) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$.

By Theorem 0.1 i.e. [8, Theorem 0.2] we have the following

Corollary 4.8. *Let $\mathcal{L}$ be an affinoid algebra over $\mathbb{Q}_p$, M a coadmissible (φ, Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$. If M_x is étale for some $x \in \mathrm{Max}(\mathcal{L})$, then there exist an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a $\mathcal{B}$-linear representation $V_{\mathcal{B}}$ of G_K whose associated (φ, Γ)-module is $\mathcal{B}\widehat{\otimes}_{\mathcal{L}}\mathrm{M}$. Furthermore, $V_{\mathcal{B}}$ is unique for this property.*

4.2 Coadmissible φ-modules associated to φ-compatible sequences

Write $r_n = (p-1)p^{n-1}$. For any $r \geq (p-1)/p$, let $n(r)$ be the smallest integer n such that $r_n \geq r$.

For any $n \geq n(r)$, there exists a natural map $\varphi^{-n} : \mathrm{B}^{\dagger,r}_{\mathrm{rig},K} \hookrightarrow K_n[[t]]$. We extend it continuously to an $\mathcal{L}$-linear map $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K} \to \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]$ denoted by ι_n. This map endows an $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$-module structure on $K_n[[t]]$.

If D is a free φ-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$, the formula $\iota_n(\lambda) \cdot x = \lambda\, x$ gives an $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$-module structure on D_r, which is denoted as $\iota_n(\mathrm{D}_r)$. By abuse of notation, we write

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes_{\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})} \iota_n(\mathrm{D}_r).$$

There is a natural map

$$\begin{aligned} \varphi_n : \quad & \Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_{n+1}((t))\Big) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n((t))} \Big[\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n((t))\Big) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r\Big] \\ & \to \Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_{n+1}((t))\Big) \otimes^{\iota_{n+1}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r \end{aligned}$$

defined by $\varphi_n(f\otimes(g\otimes\iota_n(x)))=fg\otimes\iota_{n+1}(\varphi(x))$.

Definition 4.9. Let D be a free φ-module over $\mathrm{B}^{\dagger}_{\mathrm{rig},K}$, $u\geq r(\mathrm{D})$. Let $\{M_n\}_{n\geq n(u)}$ be a sequence, where M_n is an $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$-submodule of $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u$. We say that $\{M_n\}_{n\geq n(u)}$ is *φ-compatible* if

$$\varphi_n\big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_{n+1}[[t]])\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]}M_n\big)=M_{n+1}.$$

Let D be a free (φ,Γ)-module, h a positive integer and u a sufficient large number. Let M_u be a closed flat $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}$-submodule of $t^{-h}\mathrm{D}_u$ which satisfies the following conditions:

(a) $t^h\mathrm{D}_u\subset\mathrm{M}_u\subset t^{-h}\mathrm{D}_u$;

(b) M_u is Γ-invariant;

(c) $\varphi(\mathrm{M}_u)$ is contained in $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pu}_{\mathrm{rig},K})\cdot\mathrm{M}_u$;

(d) $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pu}_{\mathrm{rig},K})\cdot\varphi(\mathrm{M}_u)$ is dense in $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pu}_{\mathrm{rig},K})\cdot\mathrm{M}_u$ for the canonical topology of $t^{-h}\mathrm{D}_{pu}$.

For any $n\geq n(u)$, put

$$M_n=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{M}_u.$$

Then $\{M_n\}_{n\geq n(u)}$ is φ-compatible and satisfies

$$t^h(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u\subset M_n\subset t^{-h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u$$

for all $n\geq n(u)$.

For the converse we have the following theorem.

Theorem 4.10. *Let D be a free φ-module (resp. (φ,Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ of rank d, $u\geq r(\mathrm{D})$. If*

$$\{M_n\mid M_n\subset\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\Big)\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u\}_{n\geq n(u)}$$

is a φ-compatible sequence such that M_n is a free $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$-module of rank d and there exists a positive integer h such that

$$t^h(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u\subset M_n\subset t^{-h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u,\tag{4.1}$$

then there exists a coadmissible φ-submodule (resp. (φ,Γ)-module) M of $\mathrm{D}[1/t]$ such that

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{M}_u=M_n$$

for any $n\geq n(u)$.

For any $r\geq u$, we put

$$\mathrm{M}_r=\{x\in t^{-h}\mathrm{D}_r\mid\iota_n(x)\in M_n\text{ for any }n\geq n(r)\}.$$

Lemma 4.11. *M_r is a coadmissible $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$-submodule of $t^{-h}\mathrm{D}_r$.*

Proof. As the maps ι_n, $n\geq n(r)$, are all continuous, M_r is closed in $t^{-h}\mathrm{D}_r$. But a submodule of a coadmissible $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$-module is itself coadmissible if and only if it is closed. □

Lemma 4.12. *We have $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r=M_n$ for any $n\geq n(r)$.*

Proof. Note that $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is isomorphic to $(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)[[t]]$ and thus $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is a noetherian ring. Let $q=\varphi(\pi)/\pi$. Then $K_n[[t]]$ is the $\varphi^{n-1}(q)$-adic completion of $\varphi^{-n}(\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. It follows that $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]\cong(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)[[t]]$ is the $\varphi^{n-1}(q)$-adic completion of $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. Thus $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is flat over $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. It follows that the map

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\to(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}t^{-h}\mathrm{D}_r$$

is injective, and by the definition of M_r the image of this map is contained in M_n.

As $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r$ and M_n are finite over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$, they are complete for the t-adic topology. So we only need to show that the natural map

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\to M_n/t^hM_n$$

is surjective. By (4.1), for any $x\in M_n$, there exists $y\in t^{-h}\mathrm{D}_r$ such that $\iota_n(y)-x\in t^hM_n$. By [2, Lemma I.2.1] there exists $t_{n,3h}\in\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ such that $\iota_n(t_{n,3h})=1\mod t^{3h}K_n[[t]]$ and $\iota_m(t_{n,3h})\in t^{3h}K_m[[t]]$ if $m\geq n(r)$ and $m\neq n$. Put $z=t_{n,3h}y$. Then

$$\iota_n(z)-\iota_n(y)\in t^{2h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r\subset t^hM_n$$

and

$$\iota_m(z)\in t^{2h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_m[[t]])\otimes^{\iota_m}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r\subset t^hM_m$$

if $m\neq n$. Thus z is in M_r and the map $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\to M_n/t^hM_n$ is surjective. □

Let $s\geq r$ be two real number $\geq u$. If $n\in\mathbb{Z}$ satisfies $n(s)\geq n\geq n(r)$, then the map $\varphi^{-n}:\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}\to K_n[[t]]$ extends to a map $\varphi^{-n}:\mathrm{B}^{[r,s]}_K\to K_n[[t]]$. We also let ι_n denote the map

$$\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_{\mathrm{rig},K}\to\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]].$$

The maps ι_n with $n\in[n(r),n(s)]$ induces inclusions

$$\iota^{[r,s]}:\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K\to\prod_{n(s)\geq n\geq n(r)}\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]],$$

and

$$\bar{\iota}^{[r,s]}:\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}^{[r,s]}_K/(t))\to\prod_{n(s)\geq n\geq n(r)}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n.$$

Lemma 4.13. *The target of $\bar{\iota}^{[r,s]}$ is faithfully flat over the source.*

Proof. As $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is the $\varphi^{n-1}(q)$-adic completion of $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K)$, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is flat over $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K)$. Since the direct product of a family of flat modules over a noetherian ring is again flat [5, p.122 Exercise 4], the target of $\iota^{[r,s]}$ is flat over the source. So the target of $\bar{\iota}^{[r,s]}$ is flat over the source.

To show that the target is faithfully flat over the source, we only need to show that, for any maximal ideal I of the source, there exists a maximal ideal of target which restricts to I. Note that every prime ideal of $\mathrm{B}^{[r,s]}_K$ which contains t in fact contains $\varphi^{m-1}(q)$ for some $m\in[n(r),n(s)]$. Then I contains the image of $\varphi^{m-1}(q)$ in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}^{[r,s]}_K/(t))$ which is denoted by the same notation $\varphi^{m-1}(q)$. As the map ι_m induces an isomorphism $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K/(\varphi^{m-1}(q))\to\mathcal{L}\otimes_{\mathbb{Q}_p}K_m$, the image of I under the map $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K/(t)\to\mathcal{L}\otimes_{\mathbb{Q}_p}K_m$ is contained in a maximal ideal J_m of $\mathcal{L}\otimes_{\mathbb{Q}_p}K_m$. Then

$$J:=\left\{(x_n)_{n(s)\geq n\geq n(r)}:x_m\in J_m\text{ and }\ x_n\in\mathcal{L}\otimes_{\mathbb{Q}_p}K_n\text{ if }n\neq m\right\}$$

is a maximal ideal of $\prod_{n(s)\geq n\geq n(r)}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n$ and $\bar{\iota}(I)$ is contained in J. □

Proposition 4.14. *If $s \geq r \geq u$, then $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} \mathrm{M}_r$ is locally free over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}$ of rank $d = \mathrm{rank}(\mathrm{D})$.*

Proof. Write $\mathrm{M}^{[r,s]} = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} \mathrm{M}_r$. Since $\mathrm{M}^{[r,s]}$ is contained in a free module $t^{-h}\mathrm{D}^{[r,s]}$ of rank d and contains a free submodule $t^h\mathrm{D}^{[r,s]}$ of rank d, it suffices to show that $\mathrm{M}^{[r,s]}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}$. By Gabber's criterion [4, §2.6 Lemma 1], we only need to show that $\mathrm{M}^{[r,s]}[1/t]$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}[1/t]$, that $\mathrm{M}^{[r,s]}$ is t-torsion free and that $\mathrm{M}^{[r,s]}/t\mathrm{M}^{[r,s]}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}/(t) = \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r,s]}/(t))$. The former two are trivial.

As $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathrm{M}^{[r,s]} = M_n$ is free of rank d over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ for any $n \in [n(r), n(s)]$,

$$\Big(\prod_{n(s)\geq n\geq n(r)} \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]\Big) \otimes^{\iota^{[r,s]}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathrm{M}^{[r,s]} = \prod_{n(s)\geq n\geq n(r)} (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]) \otimes^{\iota^{[r,s]}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathrm{M}^{[r,s]}$$

is free of rank d over $\prod_{n(s)\geq n\geq n(r)} \big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]\big)$. Consequently,

$$\Big(\prod_{n(s)\geq n\geq n(r)} \mathcal{L}\otimes_{\mathbb{Q}_p} K_n\Big) \otimes^{\bar\iota^{[r,s]}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r,s]}/(t))} (\mathrm{M}^{[r,s]}/t\mathrm{M}^{[r,s]})$$

is free of rank d over $\prod_{n(s)\geq n\geq n(r)} \mathcal{L}\otimes_{\mathbb{Q}_p} K_n$. As $\prod_{n(s)\geq n\geq n(r)} \mathcal{L}\otimes_{\mathbb{Q}_p} K_n$ is faithfully flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r,s]}/(t))$, $\mathrm{M}^{[r,s]}/t\mathrm{M}^{[r,s]}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r,s]}/(t))$. □

Proposition 4.15. (a) *For any $s \geq s' \geq r' \geq r \geq u$ we have a natural isomorphism*

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathrm{M}^{[r,s]} \xrightarrow{\sim} \mathrm{M}^{[r',s']}.$$

(b) *For any pair $r' \geq r$ with $r \geq u$, $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r'}) \cdot \mathrm{M}_r$ is contained in in $\mathrm{M}_{r'}$ and is dense in the latter.*

(c) *$\varphi(\mathrm{M}_r)$ is contained in M_{pr} and $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,pr}) \cdot \varphi(\mathrm{M}_r)$ is dense in M_{pr}.*

Proof. We prove It. (a). As $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}$, the natural map

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathrm{M}^{[r,s]} = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} \mathrm{M}_r \to (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} t^{-h}\mathrm{D}_r$$

is injective. By definition, $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r'}) \cdot \mathrm{M}_r$ is contained in $\mathrm{M}_{r'}$, so the image of the above injection is contained in $\mathrm{M}^{[r',s']}$. Let N_1 denote this image and let N_2 denote $\mathrm{M}^{[r',s']}$. For any $n \in [n(r'), n(s')]$, by Lemma 4.12 we have

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}} N_1 = M_n = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}} N_2.$$

It follows that

$$\left(\prod_{n(s')\geq n\geq n(r')} \mathcal{L}\otimes_{\mathbb{Q}_p} K_n\right) \otimes^{\bar\iota^{[r',s']}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r',s']}/(t))} N_1/tN_1 = \left(\prod_{n(s')\geq n\geq n(r')} \mathcal{L}\otimes_{\mathbb{Q}_p} K_n\right) \otimes^{\bar\iota^{[r',s']}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r',s']}/(t))} N_2/tN_2.$$

Combining this with the fact that $\prod_{n(s')\geq n\geq n(r')}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n$ is faithfully flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r',s']}/(t))$, we obtain $N_1/tN_1=N_2/tN_2$. In other words, we have $N_2=N_1+tN_2$. By induction we obtain that $N_2=N_1+t^\ell N_2$ for any integer $\ell\geq 1$. In particular, $N_2=N_1+t^{2h}N_1$. As $t^{2h}N_2\subset t^h\mathrm{D}^{[r',s']}$ is contained in N_1, we have $N_1=N_2$.

Next we prove It. (b). We have already seen that $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K})\cdot\mathrm{M}_r$ is contained in in $\mathrm{M}_{r'}$. The closure of $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K})\cdot\mathrm{M}_r$ is exactly the coadmissible $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K}$-module associated to the coherent sheaf $\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\Big)_{s'\geq r'}$. Thus by It. (a), it coincides with $\mathrm{M}_{r'}$.

It. (c) can be proved similarly. We omit the details. □

Proof of Theorem 4.10. Let M be the inductive system $\{\mathrm{M}_r\}_{r\geq u}$. By Lemma 4.11, Proposition 4.14 and Proposition 4.15, $(\mathrm{M};\{\mathrm{M}_r\}_{r\geq u})$ is a coadmissible φ-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. If D is a (φ,Γ)-module, then by definition M_r is stable under Γ. In this case, $(\mathrm{M};\{\mathrm{M}_r\}_{r\geq u})$ is a coadmissible (φ,Γ)-module. □

4.3 Coadmissible (φ,Γ)-modules associated to filtered (φ,N)-modules

Recall that $\varphi(\ell_X)=p\ell_X+\log(\varphi(\pi)/\pi^p)$ and $\gamma(\ell_X)=\ell_X+\log(\gamma(\pi)/\pi)$ for any $\gamma\in\Gamma$. Let N be the $\mathrm{B}^{\dagger}_{\mathrm{rig},K}$-derivation on $\mathrm{B}^{\dagger}_{\mathrm{rig},K}[\ell_X]$ defined by $N(\ell_X)=-p/(p-1)$. We extend these operators $\mathcal{L}$-linearly and continuously to $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]$. Then we extend the inclusion $\iota_n:\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}\to\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ to $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})[\ell_X]$ by letting $\iota_n(\ell_X)=\log(\varepsilon^{(n)}\exp(t/p^n)-1)\in\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$.

Let D be a filtered (φ,N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. We suppose that D is free over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$ and satisfies the following two conditions:

(GBN). There exists a basis compatible with N. Explicitly there exists a base $\{v_1,\cdots,v_d\}$ of D over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$ such that $N(v_1)=0$ and $N(v_i)\in\mathcal{L}\otimes_{\mathbb{Q}_p}K_0\cdot v_1\oplus\cdots\oplus\mathcal{L}\otimes_{\mathbb{Q}_p}K_0\cdot v_{i-1}$ for $i\geq 2$.

(GFF). If $n\in\mathbb{N}$ is sufficient large, then for any i, $\mathrm{Fil}^i\Big((\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)((t))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0}D\Big)$ is free of rank $d=\mathrm{rank}_{\mathcal{L}}V$ over $(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)[[t]]$, where

$$\mathrm{Fil}^i\Big((\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)((t))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0}D\Big):=\sum_{j+\ell\geq i}\big(t^j(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)[[t]]\big)\cdot\mathrm{Fil}^\ell D_K.$$

Note that (GBN) is a stronger version of (BN).

Put

$$\mathrm{D}=\big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0}D\big)^{N=0}.$$

Proposition 4.16. *Suppose that D satisfies the condition* (GBN). *Then the following hold:*

(a) D *is a free (φ,Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ of rank d.*

(b) *We have*

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathrm{D}=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0}D.$$

Proof. As D satisfies (GBN), there exists an $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$-base $\{v_1,\cdots,v_d\}$ of D such that $N(v_1)=0$ and $N(v_i)\in\mathcal{L}\otimes_{\mathbb{Q}_p}K_0\cdot v_1\oplus\cdots\oplus\mathcal{L}\otimes_{\mathbb{Q}_p}K_0\cdot v_{i-1}$ for $i\geq 2$.

We show that, there exist elements $v'_1,\cdots,v'_d$ of D such that for any $i\in\{1,\cdots,d\}$ the $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]$-submodule $\langle v'_1,\cdots,v'_i\rangle$ generated by $\{v'_1,\cdots,v'_i\}$ and the submodule $\langle v_1,\cdots,v_i\rangle$ generated by $\{v'_1,\cdots,v'_i\}$ are same. We process it iteratively. For $i=1$ we put $v'_1=v_1$. Assume that $i\geq 2$ and $\langle v'_1,\cdots,v'_{i-1}\rangle=\langle v_1,\cdots,v_{i-1}\rangle$. Then there exist $a_1,\cdots,a_{i-1}\in(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]$ such that $N(v_i)=a_1v'_1+\cdots+a_{i-1}v'_{i-1}$. Since the operator $N:(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]\to(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]$ is surjective, there exist $b_1,\cdots,b_{i-1}\in(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]$ such that $N(b_1)=a_1$, $\cdots$, $N(b_{i-1})=a_{i-1}$. Put $v'_i=v_i-b_1v'_1-\cdots-b_{i-1}v'_{i-1}$, which is in D. Then $\langle v'_1,\cdots,v'_i\rangle=\langle v_1,\cdots,v_i\rangle$, as wanted. By construction $v'_1,\cdots,v'_d$ are linearly independent over $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]$.

By definition D is a (φ,Γ)-module. So, to prove It. (a), we only need show that $\mathrm{D}=\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}\cdot v_1'\oplus\cdots\oplus\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}\cdot v_d'$. For any $v\in(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[\ell_X]\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0}D$, writing $v=a_1v_1'+\cdots a_dv_d'$, we have $N(v)=N(a_1)v_1'+\cdots+N(a_d)v_d'$, and thus $N(v)=0$ if and only if $N(a_1)=\cdots=N(a_d)=0$ or equivalently $a_1,\cdots,a_d$ are in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$.

It. (b) follows from the facts that $\langle v_1',\cdots,v_d'\rangle=\langle v_1,\cdots,v_d\rangle$ and that $\mathrm{D}=\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}\cdot v_1'\oplus\cdots\oplus\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}\cdot v_d'$. □

For any $n\geq 0$ we have $\varphi^{-n}(K_0)\subset K$. Thus there are $\varphi^{-n}(K_0)$-module structures on K and on D. The latter is denoted by $\iota_n(D)$. We write $K\otimes^{\iota_n}_{K_0}D$ for $K\otimes_{\varphi^{-n}(K_0)}\iota_n(D)$. There is a map $\xi_n:K\otimes_{K_0}D\to K\otimes_{\varphi^{-n}(K_0)}\iota_n(D)$ sending $\mu\otimes x$ to $\mu\otimes\iota_n(\varphi^n(x))$. Then we obtain a filtration on the target $D^n_K=K\otimes^{\iota_n}_{K_0}D$ via the map ξ_n. Define a filtration on $(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)((t))$ by the formulas

$$\mathrm{Fil}^i\big((\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)((t))\big)=t^i(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)[[t]].$$

Then we obtain a filtration on $(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)((t))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}D^n_K$.

Put

$$M_n(D)=\mathrm{Fil}^0\big((\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)((t))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}D^n_K\big).$$

Since D satisfies (GFF), there exists a sufficient large n_0 such that, if $n\geq n_0$, then $M_n(D)$ is a free $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$-submodule of rank d.

Let $u\geq\{r(\mathrm{D}),r(n_0)\}$. If $n\geq n(u)$, we may consider M_n as an $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$-submodule of $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u$.

Proposition 4.17. *The family $\{M_n(D)\}_{n\geq n(u)}$ is φ-compatible.*

Proof. This follows from the formulas $\xi_{n+1}=\varphi_n\circ\xi_n$ on D_K. □

Let h be a positive integer such that the filtration on D_K satisfies $\mathrm{Fil}^{-h}D_K=D_K$ and $\mathrm{Fil}^hD_K=0$. Then for any $n\geq n(r)$, $M_n(D)$ satisfies

$$t^h(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u\subset M_n(D)\subset t^{-h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u.$$

Applying Theorem 4.10 we get a coadmissible (φ,Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ which is denoted by $\mathcal{M}(D)$.

Hence we obtain a functor, denoted by $\mathcal{M}$, from the category of filtered (φ,N)-modules over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$ which satisfy the conditions (GBN) and (GFF) to the category of coadmissible (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$.

The functor $\mathcal{M}$ is functorial by the following

Proposition 4.18. *If $\mathcal{L}'$ is another coefficient algebra and $\mathcal{L}\to\mathcal{L}'$ is a continuous map, then $\mathcal{M}(D_{\mathcal{L}'})=\mathcal{M}(D)_{\mathcal{L}'}$.*

Proof. We have

$$M_n(D_{\mathcal{L}'})=(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]}M_n(D).$$

Thus by definition of $\mathcal{M}(D)_r$ and $\mathcal{M}(D_{\mathcal{L}'})_r$ we have a natural map

$$(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathcal{M}(D)_r\to\mathcal{M}(D_{\mathcal{L}'})_r.$$

What we need to show is that, for any $s\geq r\geq u$ it induces an isomorphism

$$(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K)\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K}\mathcal{M}(D)^{[r,s]}\to\mathcal{M}(D_{\mathcal{L}'})^{[r,s]}.$$

Let N_1 and N_2 be respectively the source and the target of this map. Then for any $n\in[n(r),n(s)]$ we have

$$(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K}N_1$$

$$
\begin{aligned}
&= (\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathcal{M}(D)^{[r,s]} \\
&= (\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]} \left((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathcal{M}(D)^{[r,s]} \right) \\
&= (\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]} M_n(D) \\
&= M_n(D_{\mathcal{L}'}) \\
&= (\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} N_2
\end{aligned}
$$

Now repeating the argument in the proof of Proposition 4.15 It. (a), we obtain $N_1 = N_2$, as desired. □

Corollary 4.19. *If $\mathfrak{m}$ is a maximal ideal of $\mathcal{L}$, then $\mathcal{M}(D)_{\mathfrak{m}}$ is the (φ,Γ)-module over $L_{\mathfrak{m}} \otimes_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$ associated to the filtered (φ, N)-module $D_{\mathfrak{m}}$ over $L_{\mathfrak{m}} \otimes_{\mathbb{Q}_p} K_0$.*

The following proposition tells us that the functor $\mathcal{M}$ is faithful.

Proposition 4.20. *If D is a filtered (φ, N)-module over $\mathcal{L} \otimes_{\mathbb{Q}_p} K_0$ which is free over $\mathcal{L} \otimes_{\mathbb{Q}_p} K_0$ and satisfies* (GBN) *and* (GFF)*, then*

$$D = \left((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t, \ell_X] \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}} \mathcal{M}(D) \right)^{\Gamma}.$$

Lemma 4.21. *We have*

$$((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t, \ell_X])^{\Gamma} = \mathcal{L} \otimes_{\mathbb{Q}_p} K_0.$$

Proof. We define the operators $\nabla = \frac{\log(\gamma)}{\log \chi_{\mathrm{cyc}}(\gamma)}$ (γ sufficiently close to 1) and $\partial = [\varepsilon]\frac{d}{d\pi}$ on $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$ in a way similar to that in [1], and then extend them to $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t, \ell_X]$. Note that $\nabla = t\partial$. If $x \in ((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t, \ell_X])^{\Gamma}$, then $\nabla x = \partial x = 0$ and so x is in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$. As

$$(\mathrm{B}_K^{[r,s]})^{\Gamma} = (\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})^{\Gamma} = K_0,$$

by Lemma 1.1 we obtain $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})^{\Gamma} = \mathcal{L} \otimes_{\mathbb{Q}_p} K_0$. So, $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})^{\Gamma} = \mathcal{L} \otimes_{\mathbb{Q}_p} K_0$. □

Proof of Proposition 4.20. From Proposition 4.16 It. (b) and the relation $\mathrm{D}[1/t] = \mathcal{M}(D)[1/t]$ we obtain

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t, \ell_X] \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}} \mathcal{M}(D) = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t, \ell_X] \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p} K_0} D.$$

Now our conclusion follows from Lemma 4.21. □

5 Proof of Theorem 2.6

From now on we suppose that $\mathcal{L}$ is a reduced affinoid algebra. Observe that Condition (Gr) implies the following condition

(FF). For any $x \in \mathrm{Max}(\mathcal{L})$ there exists a neighborhood $\mathrm{Max}(\mathcal{B})$ of x such that, if $n \in \mathbb{N}$ is sufficient large, then for any i, $\mathrm{Fil}^i\left((\mathcal{B} \otimes_{\mathbb{Q}_p} K_n)((t)) \otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p} K_0} D_{\mathcal{B}} \right)$ is free of rank $d = \mathrm{rank}_{\mathcal{L}} V$ over $(\mathcal{B} \otimes_{\mathbb{Q}_p} K_n)[[t]]$, where

$$\mathrm{Fil}^i\left((\mathcal{L} \otimes_{\mathbb{Q}_p} K_n)((t)) \otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p} K_0} D_{\mathcal{B}} \right) := \sum_{j+\ell \geq i} \left(t^j (\mathcal{L} \otimes_{\mathbb{Q}_p} K_n)[[t]] \right) \cdot \mathrm{Fil}^{\ell}(D_{\mathcal{B}})_K.$$

So, Theorem 2.6 is a consequence of the following

Theorem 5.1. *Let $\mathcal{L}$ be a reduced affinoid algebra and let D be a filtered (φ, N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$ which satisfies* (BN) *and* (FF)*. If D_x is weakly admissible for some $x\in \mathrm{Max}(\mathcal{L})$, then there exists an affinoid neighborhood* $\mathrm{Max}(\mathcal{B})$ *of x and a semi-stable $\mathcal{B}$-representation $V_{\mathcal{B}}$ of G_K whose associated filtered (φ, N)-module is isomorphic to $D_{\mathcal{B}}$. Moreover, $V_{\mathcal{B}}$ is unique for this property.*

Proposition 5.2 and Corollary 5.3 below are useful for the proof of Theorem 5.1.
Put

$$\mathrm{D}^+_{\mathrm{st},\mathcal{L}}(V) := ((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\log})\otimes_{\mathcal{L}} V)^{G_K}.$$

Proposition 5.2. *If V is a de Rham $\mathcal{L}$-representation of G_K, then the map $\widetilde{\mathrm{B}}^+_{\log}\to\widetilde{\mathrm{B}}^\dagger_{\log}$ induces an isomorphism*

$$\mathrm{D}^+_{\mathrm{st},\mathcal{L}}(V)\to\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^\dagger_{\log})\otimes_{\mathcal{L}} V\Big)^{G_K}. \tag{5.1}$$

Proof. For any $n\in\mathbb{N}$, put $D_n=(\widetilde{\mathrm{B}}^{\dagger,r_n}_{\log}\widehat{\otimes}_{\mathbb{Q}_p}V)^{G_K}$ which is a $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$-module. Note that ι_n induces an inclusion $D_n\hookrightarrow \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)$. As V is de Rham, $\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)$ is finite over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. Thus D_n is finite over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. There is a sufficient large n_0 such that the image of $\mathrm{D}^+_{\mathrm{st},\mathcal{L}}(V)$ is contained in D_{n_0}. For any $n\geq n_0$ and any maximal ideal $\mathfrak{m}$ of $\mathcal{L}$, by [1, Proposition 3.4], the map $\mathrm{D}^+_{\mathrm{st},\mathcal{L}}(V)/\mathfrak{m}\mathrm{D}^+_{\mathrm{st},\mathcal{L}}(V)\to D_n/\mathfrak{m}D_n$ is surjective. Combining this with the fact that D_n is finite over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$, we see that the map $\mathrm{D}^+_{\mathrm{st},\mathcal{L}}(V)\to D_n$ is surjective. It follows that (5.1) is surjective. □

Corollary 5.3. *If V is a de Rham $\mathcal{L}$-representation of G_K, then the map $\widetilde{\mathrm{B}}^+_{\log}\to\widetilde{\mathrm{B}}^\dagger_{\log}$ induces an isomorphism*

$$\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)\to\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^\dagger_{\log}[1/t])\otimes_{\mathcal{L}} V\Big)^{G_K}. \tag{5.2}$$

Proposition 5.4. *If V is a semi-table $\mathcal{L}$-representation and $\mathrm{D}=\mathrm{D}^\dagger_{\mathrm{rig}}(V)$, then there exists a sufficient large $n\geq n(r(\mathrm{D}))$ such that*

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r=\mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big).$$

Proof. By [3, Lemma 4.3.1, Theorem 5.3.2] if $n\in\mathbb{N}$ is sufficient large, then

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^+_{\mathrm{dR}})\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r=\mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}})\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big)$$

and

$$\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\Big)\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r=\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\Big)\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V).$$

Combining these two facts and the fact that

$$\begin{aligned}&\mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big)\\ =&\mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}})\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big)\cap(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V),\end{aligned}$$

we obtain

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r\subset\mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big).$$

By [2] this inclusion is isomorphic after modulo $\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathcal{L}$. Therefore it is itself isomorphic. □

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. By Proposition 2.3 without loss of generality we may assume that D is free over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. As D satisfies (BN) and (FF), there is a neighborhood $\mathrm{Max}(\mathcal{L}')$ of x in $\mathrm{Max}(\mathcal{L})$ such that $D_{\mathcal{L}'}$ satisfies (GBN) and (GFF).

As $D_{\mathcal{L}'}$ is free over $\mathcal{L}'\otimes_{\mathbb{Q}_p}K_0$, and satisfies the conditions (GBN) and (GFF), $\mathcal{M}(D_{\mathcal{L}'})$ is a coadmissible (φ,Γ)-module over $\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. As the functor $\mathcal{M}$ is functorial, we have $\mathcal{M}(D)_x=\mathcal{M}(D_x)$. Because D_x is weakly admissible, by [2], $\mathcal{M}(D)_x$ is étale. Thus by Corollary 4.8 there exist an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x in $\mathrm{Max}(\mathcal{L}')$ and a $\mathcal{B}$-linear representation $V_{\mathcal{B}}$ whose associated (φ,Γ)-module is $\mathcal{B}\widehat{\otimes}_{\mathcal{L}'}\mathcal{M}(D_{\mathcal{L}'})=\mathcal{M}(\mathcal{B}\otimes_{\mathcal{L}'}D_{\mathcal{L}'})=\mathcal{M}(\mathcal{B}\otimes_{\mathcal{L}}D)$.

By Proposition 4.20

$$\begin{aligned}\mathcal{B}\otimes_{\mathcal{L}}D&=\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\log,K}[1/t])\otimes_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathcal{M}(\mathcal{B}\otimes_{\mathcal{L}}D)\Big)^{\Gamma}\\&=\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\log,K}[1/t])\otimes_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathrm{D}^{\dagger}_{\mathrm{rig}}(V_{\mathcal{B}})\Big)^{\Gamma}.\end{aligned}$$

It follows that, for any rigid point y in $\mathrm{Max}(\mathcal{B})$,

$$(\mathcal{B}\otimes_{\mathcal{L}}D)_y=\Big((L_y\otimes_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\log,K}[1/t])\otimes_{L_y\otimes_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathrm{D}^{\dagger}_{\mathrm{rig}}(V_{\mathcal{B}}\otimes_{\mathcal{B}}L_y)\Big)^{\Gamma},$$

where $L_y=\mathcal{B}/\mathfrak{m}_y$. Thus $V_{\mathcal{B}}\otimes_{\mathcal{B}}L_y$ is semistable for any $y\in\mathrm{Max}(\mathcal{B})$. Then by [3] $V_{\mathcal{B}}$ is semistable.

Note that

$$\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\log,K}[1/t])\otimes_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathrm{D}^{\dagger}_{\mathrm{rig}}(V_{\mathcal{B}})\Big)^{\Gamma}\subset\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log}[1/t])\otimes_{\mathcal{B}}V_{\mathcal{B}}\Big)^{G_K}.$$

So, by Corollary 5.3, $\mathcal{B}\otimes_{\mathcal{L}}D$ is contained in $\mathrm{D}_{\mathrm{st},\mathcal{B}}(V_{\mathcal{B}})$. The inclusion $\mathcal{B}\otimes_{\mathcal{L}}D\to\mathrm{D}_{\mathrm{st},\mathcal{B}}(V_{\mathcal{B}})$ is in fact isomorphic, since it induces isomorphisms $D_y\xrightarrow{\sim}\mathrm{D}_{\mathrm{st},L_y}(V_y)$ at all rigid points $y\in\mathrm{Max}(\mathcal{B})$.

By Lemma 4.12 there exists a sufficient large r such that for any $n\geq n(r)$,

$$(\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}^{\dagger}_{\mathrm{rig},K}(V_{\mathcal{B}})_r=\mathrm{Fil}^0\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K}(\mathcal{B}\otimes_{\mathcal{L}}D)_K\Big)$$

But by Proposition 5.4 we have

$$(\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}^{\dagger}_{\mathrm{rig},K}(V_{\mathcal{B}})_r=\mathrm{Fil}^0\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{B}}(V_{\mathcal{B}})\Big).$$

Hence

$$\mathrm{Fil}^0\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K}(\mathcal{B}\otimes_{\mathcal{L}}D)_K\Big)=\mathrm{Fil}^0\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{B}}(V_{\mathcal{B}})\Big).$$

It follows that the filtration of $\mathrm{D}_{\mathrm{dR},\mathcal{B}}(V_{\mathcal{B}})$ and the filtration of $(\mathcal{B}\widehat{\otimes}_{\mathcal{L}}D)_K$ agree. Therefore the filtered (φ,N)-module associated to $V_{\mathcal{B}}$ is $\mathcal{B}\otimes_{\mathcal{L}}D$.

The uniqueness of $V_{\mathcal{B}}$ follows from Corollary 4.8. □

Condition (FF) in Theorem 5.1 is necessary. Indeed we have the following

Proposition 5.5. *If V is a semi-stable $\mathcal{L}$-representation of G_K of rank d, then $D=\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)$ satisfies* (FF).

Proof. This follows from Proposition 5.4. □

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