BOUNDARY BLOW-UP SOLUTIONS WITH INTERIOR LAYERS AND SPIKES IN A BISTABLE PROBLEM

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Abstract. We show that for small $\epsilon > 0$, the boundary blow-up problem

$$-\epsilon^2 \Delta u = u(u - a(x))(1 - u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \infty,$$

has solutions with sharp interior layers and spikes, apart from boundary layers. We also determine the location of these layers and spikes.

1. Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with $C^2$ boundary $\partial\Omega$. We study the following boundary blow-up problem:

$$-\epsilon^2 \Delta u = u(u - a(x))(1 - u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \infty,$$  \hspace{1cm} (P_{\epsilon})

where $\epsilon > 0$ is a small parameter, $a(x)$ is a continuous function satisfying $0 < a(x) < 1$ for $x \in \overline{\Omega}$. For convenience we will denote $f(x, t) = t(t - a(x))(1 - t)$.

We say $u_\epsilon$ is a positive solution of $(P_{\epsilon})$ if $u_\epsilon \in C^1(\Omega)$, $u_\epsilon(x) > 0$ in $\Omega$,

$$\epsilon^2 \int_{\Omega} Du_\epsilon(x) D\phi(x) dx = \int_{\Omega} f(x, u_\epsilon(x)) \phi(x) dx, \quad \forall \phi \in C_0^\infty(\Omega),$$

and $u_\epsilon(x) \to \infty$ as $d(x, \partial\Omega) \to 0$.

Problem $(P_{\epsilon})$ is a well-known bistable problem. If the boundary condition is replaced by $u|_{\partial\Omega} = 1$, then it can be easily transformed (by letting $v = 1 - u$) into a similar problem but with homogeneous Dirichlet boundary conditions; in such a form, it was studied in a recent paper by Dancer and Yan [DaY1], where solutions with interior layers and spikes, as well as with boundary layers, were obtained for small $\epsilon$. The main purpose of this paper is to show that similar results hold for the boundary blow-up problem $(P_{\epsilon})$. This seems to suggest that, for small $\epsilon$, the interior layers and spikes of the solutions to the differential equation in $(P_{\epsilon})$ are not affected in any substantial way by the boundary conditions imposed on the solutions.

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Problem $(P_\epsilon)$ with homogeneous Neumann boundary conditions and with $\Omega$ an interval in $\mathbb{R}^1$ was studied by many people, see, for example, [ACH, AMPP, CP, UNY] and the references therein; for $\Omega$ in higher dimensions, see [DaY2, dN].

A special case of $(P_\epsilon)$, namely when $a(x)$ is a constant $a_0 \in (1/2, 1)$, has been considered recently in [APL] and [DuY]. It was shown in [APL] that, for $\epsilon$ sufficiently small, the problem

$$-\epsilon^2 \Delta u = u(u - a_0)(1 - u) := \rho(u) \text{ in } \Omega, \quad u|_{\partial \Omega} = \infty$$  

(1.1)

has at least three positive solutions:

$$u_\epsilon < u_\epsilon < \bar{u}_\epsilon \text{ in } \Omega,$$

and $u_\epsilon \to 0$, $\bar{u}_\epsilon \to 1$ locally uniformly in $\Omega$ as $\epsilon \to 0$. The authors in [DuY] constructed intermediate positive solutions of (1.1) with sharp interior spikes by adapting the well-known reduction method. From [GW], we know that $u_\epsilon$ and $\bar{u}_\epsilon$ are the unique "small" solution and the unique "large" solution of (1.1) respectively. In fact $u_\epsilon$ and $\bar{u}_\epsilon$ are the minimal and the maximal solutions of (1.1), and clearly they have boundary layers.

We would like to point out that for problem (1.1) with $0 < a_0 < 1/2$, in sharp contrast to the case $a_0 > 1/2$, there exists only one solution. Indeed, the existence of a solution $\bar{u}_\epsilon > 1$ follows from well-known results, and due to the monotonicity of $\rho(u)/u$ in $[1, \infty)$, it is the only solution satisfying $u > 1$; see, e.g., [DG2]. Suppose that (1.1) with $a_0 \in (0, 1/2)$ has another positive solution $u'_\epsilon$ which is different from $u_\epsilon$. Then necessarily $\min_{\Omega} u_\epsilon < 1$. Since $\rho(s) > 0$ for $s \in (a_0, 1)$, we see from the maximum principle that $\min_{\Omega} u_\epsilon < a_0$.

Using large positive constant as a super-solution and 0 as a sub-solution, we find that, for any $\beta > 1$, the problem

$$-\epsilon^2 \Delta v = \tilde{\rho}(v) \text{ in } \Omega, \quad v|_{\partial \Omega} = 0$$  

(1.2)

has a minimal solution $v_\epsilon^\beta$ satisfying $0 \leq v_\epsilon^\beta$. Since $u_\epsilon^\beta$ increases to the minimal boundary blow-up solution of the same equation as $\beta$ increases to $\infty$, we have $u_\epsilon^\beta \leq u_\epsilon$. By the maximum principle we easily deduce that $u_\epsilon^\beta \leq \beta$ in $\Omega$. Setting $v = \beta - u$, we see that the Dirichlet problem

$$-\epsilon^2 \Delta v = \tilde{\rho}(v) \text{ in } \Omega, \quad v|_{\partial \Omega} = 0$$  

(1.3)

with $\tilde{\rho}(s) := -\rho(\beta - s)$, has a positive solution $v_\epsilon := \beta - u_\epsilon^\beta$. Since $\max_{\Omega} v_\epsilon \in (\beta - a_0, \beta)$, we obtain from the necessary condition given in [CS] that $\int_{t}^{\beta} \tilde{\rho}(s)ds > 0$ for all $t \in (0, \beta)$. But this is a contradiction since $\int_{\beta - 1}^{\beta} \tilde{\rho}(s)ds < 0$.

More recently, in [DG3], the case $1/2 < a(x) < 1$ was discussed and the boundary condition in $(P_\epsilon)$ was allowed to have the more general form $u|_{\partial \Omega} = \phi$, where $1 \leq \phi \leq \infty$, and $\phi$ is a continuous function over $\{x \in \partial \Omega : \phi(x) < \infty\}$. It was shown in [DG3] that the known results for the special case $\phi \equiv 1$ continue to hold for general $1 \leq \phi \leq \infty$. 
For the general case $0 < a(x) < 1$, it was shown in [DaY1] that $(P_\epsilon)$ with boundary condition $u|_{\partial \Omega} = 1$ has solutions with sharp interior layers near the set $\{x \in \Omega : a(x) = 1/2\}$ for small $\epsilon > 0$; furthermore, there exist solutions with sharp interior spikes located near certain local extremum points of $a(x)$. However, none of the previous work covers the boundary blow-up case $u|_{\partial \Omega} = \infty$ for such general $a(x)$.

We will show in this paper that these results of [DaY1] with boundary condition $u|_{\partial \Omega} = 1$ continue to hold if the boundary condition is changed to $u|_{\partial \Omega} = \infty$. We need to overcome several difficulties. Firstly the interior layered solutions in [DaY1] were obtained by looking for minimizers of a suitable functional. Such a functional is usually undefined when the solution has infinite boundary values. To overcome this difficulty, we introduce the following definition.

We say $u_\epsilon$ is a minimizer solution to $(P_\epsilon)$ if it is a solution to $(P_\epsilon)$ and there is a sequence $\{\beta_n\}$ with $\beta_n \to \infty$ as $n \to \infty$ such that $u^{\beta_n}_\epsilon \to u_\epsilon$ in $C^1_{\text{loc}}(\Omega)$, where $u^{\beta_n}_\epsilon$ is a minimizer of

$$\inf \left\{ \frac{\epsilon^2}{2} \int_\Omega |Dw(x)|^2 - \int_\Omega F(x,w)dx, \quad w - \beta_n \in H^1_0(\Omega) \right\},$$

where $F(x,t) = \int_0^t f(x,s)ds$. Clearly $u^{\beta_n}_\epsilon$ is a solution of the problem

$$-\epsilon^2 \Delta u = f(x,u) \quad \text{in } \Omega, \quad u|_{\partial \Omega} = \beta_n. \quad (1.4)$$

We will see that for each fixed $n$, $u^{\beta_n}_\epsilon$ develops certain interior layers as $\epsilon \to 0$, and the formation of these layers is uniform in $n$ as $\epsilon \to 0$.

Secondly, in order to use the reduction method to obtain solutions with spikes located near local extremum points of $a(x)$, we need to show that the minimal (in order) solution of $(P_\epsilon)$, denoted by $u_\epsilon$, is uniformly stable in the sense that there exists $\kappa^* > 0$ independent of $\epsilon$ such that for all $\epsilon > 0$ small,

$$\int_\Omega [\epsilon^2 |Dv|^2 - f_\epsilon(x,u_\epsilon)v^2]dx \geq \kappa^* \int_\Omega v^2dx, \quad \forall v \in C^\infty_0(\Omega).$$

In the case considered in [DaY1], the corresponding result follows from earlier techniques in [CS]; in our situation here, this requires new techniques and is much more difficult to prove.

We would like to remark that all the results in this paper remain true if the boundary condition in $(P_\epsilon)$ is replaced by $u|_{\partial \Omega} = \beta$, where $\beta$ is a constant in the interval $[1, \infty)$. This case is much easier to handle, as the boundary value of the solution is finite and most of the techniques in [DaY1] can be easily adapted.

Let us now be more precise about the main results in this paper. Denote $A = \{x \in \Omega : a(x) < 1/2\}$, $B = \{x \in \Omega : a(x) > 1/2\}$. 

...
Theorem 1.1. Let $u_\epsilon$ be a minimizer solution to $(P_\epsilon)$. Then, as $\epsilon \to 0$,

$$
u_\epsilon \to \begin{cases} 1, & \text{uniformly on any compact subset of } A, \\ 0, & \text{uniformly on any compact subset of } B. \end{cases}$$

Theorem 1.1 implies that, if $\partial A \cap \partial B \neq \emptyset$, then any minimizer solution of $(P_\epsilon)$ undergoes a sharp transition near $\partial A \cap \partial B$.

The next theorem tells us that $(P_\epsilon)$ has solutions which have no transition layers near some designated components of the set $\{x \in \Omega : a(x) = 1/2\}$.

Theorem 1.2. Let $\Omega_1$ and $\Omega_2$ be two open sets so that $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_i \subset \subset \Omega$, $i = 1, 2$ $a(x) < 1/2$ if $x \in \partial \Omega_i$ and $a(x) > 1/2$ if $x \in \partial \Omega_2$. Here $\Omega_1$ or $\Omega_2$ can be empty. Then $(P_\epsilon)$ has a solution $v_\epsilon$ satisfying, as $\epsilon \to 0$,

$$
v_\epsilon \to \begin{cases} 1, & \text{uniformly on any compact subset of } (A \setminus \overline{\Omega_2}) \cup \overline{\Omega_1}, \\ 0, & \text{uniformly on any compact subset of } (B \setminus \overline{\Omega_1}) \cup \overline{\Omega_2}. \end{cases}$$

Note that if $\Omega_1 = \Omega_2 = \emptyset$, Theorem 1.2 becomes Theorem 1.1. From the proof of Theorem 1.2 in section 2, we will see that $v_\epsilon$ is a “local” minimizer solution of $(P_\epsilon)$.

To describe our results on solutions with spikes, we need some preparations. Assume that $b \in (0, 1/2)$ is a constant. It follows from [NTW] and [PS] that the following problem has a unique solution $U_b$:

$$
\begin{cases} -\Delta u = f_b(u) := u(u - b)(1 - u) & \text{in } \mathbb{R}^N, \\ u > 0, u(0) = \max_{x \in \mathbb{R}^N} u(x). \end{cases}
$$

Moreover, $U_b$ is non-degenerate in the sense that the kernel of the operator $-\Delta \varphi - f_b(U_b)\varphi$, $\varphi \in H^1(\mathbb{R}^N)$, is spanned by $\{\frac{\partial U_b}{\partial x_i} : i = 1, 2, \ldots, N\}$. Furthermore, since we have removed the translation invariance by requiring the solution to have its maximum at the origin, $U_b$ is radially symmetric about the origin 0 and for $m_b := |f_b'(0)|^{1/2} > 0$ we have

$$|DU_b(x)|, \quad U_b(x) \leq Ce^{-m_b|x|} \quad \text{in } \mathbb{R}^N.$$ 

In the following, we denote $U_{b, \epsilon, x}(y) = U_b(\frac{y-x}{\epsilon})$.

Suppose that $a(x) > 1/2$ for $x \in \partial \Omega$. Then it follows from Theorem 1.2 that $(P_\epsilon)$ has a “local” minimizer solution $u_\epsilon$ such that $u_\epsilon \to 0$ uniformly on any compact subset of $\Omega$. If $u_\epsilon$ is not the minimal positive solution of $(P_\epsilon)$ in the sense that any positive solution of $(P_\epsilon)$ satisfies $u \geq u_\epsilon$, then we can obtain a unique minimal positive solution $w_\epsilon^*$ of $(P_\epsilon)$ by a standard limiting procedure: $w_\epsilon^* = \lim_{n \to \infty} w_\epsilon^n$, where $w_\epsilon^n$ is the minimal solution of $(P_\epsilon)$ with boundary condition replaced by $w|_{\partial \Omega} = n$. Thus $u_\epsilon \geq w_\epsilon^*$ and hence $w_\epsilon^* \to 0$ as $\epsilon \to 0$. 

Let $\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_N$ be a partition of $\mathbb{R}^N$ such that

$$\begin{align*}
\Omega_1 &= \{x \in \mathbb{R}^N : a(x) < 0\}, \\
\Omega_2 &= \{x \in \mathbb{R}^N : 0 < a(x) < 1/2\}, \\
\Omega_3 &= \{x \in \mathbb{R}^N : 1/2 < a(x) < 1\}, \\
\Omega_4 &= \{x \in \mathbb{R}^N : a(x) = 1\}, \\
\Omega_5 &= \{x \in \mathbb{R}^N : a(x) > 1\}.
\end{align*}$$

Then it follows from Theorem 1.2 that $v_\epsilon$ is a “local” minimizer solution of $(P_\epsilon)$.
If \( a(x) < 1/2 \) for \( x \in \partial \Omega \), then by Theorem 1.2, \((P_\epsilon)\) has a solution \( w_{\epsilon}^{**} \) such that \( w_{\epsilon}^{**} \to 1 \) uniformly on any compact subset of \( \Omega \). Note that both \( w_\epsilon^* \) and \( w_\epsilon^{**} \) have boundary layers but not interior layers.

Our next results show that there are new solutions of \((P_\epsilon)\) with interior peaks superimposed on \( w_\epsilon^* \) or \( w_\epsilon^{**} \).

**Theorem 1.3.** Suppose that \( a(x) > 1/2 \) for \( x \in \partial \Omega \) and \( A = \{x \in \Omega : a(x) < 1/2\} \neq \emptyset \). Let \( x_1, \ldots, x_k \in A \) be any sequence of strict local maximum points of \( a(x) \), or a sequence of strict local minimum points of \( a(x) \), and denote \( a_i = a(x_i) \). Then there is an \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \), \((P_\epsilon)\) has a solution of the form

\[
u_\epsilon^* = w_\epsilon^* + \sum_{i=1}^k U_{\epsilon,x_i,a_i} + \omega_\epsilon,
\]

where as \( \epsilon \to 0 \),

\[
\begin{align*}
x_i^\epsilon & \to x_i, \quad i = 1, \ldots, k, \\
\int_\Omega (\epsilon^2 |D\omega_\epsilon|^2 + |\omega_\epsilon|^2) &= o(\epsilon^N).
\end{align*}
\]

**Theorem 1.4.** Suppose that \( a(x) < 1/2 \) for \( x \in \partial \Omega \) and \( B = \{x \in \Omega : a(x) > 1/2\} \neq \emptyset \). Let \( x_1, \ldots, x_k \in B \) be any sequence of strict local maximum points of \( a(x) \), or a sequence of strict local minimum points of \( a(x) \), and denote \( a_i = a(x_i) \). Then there is an \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \), \((P_\epsilon)\) has a solution of the form

\[
u_\epsilon^{**} = w_\epsilon^{**} - \sum_{i=1}^k U_{\epsilon,x_i,a_i} + \omega_\epsilon,
\]

where as \( \epsilon \to 0 \),

\[
\begin{align*}
x_i^\epsilon & \to x_i, \quad i = 1, \ldots, k, \\
\int_\Omega (\epsilon^2 |D\omega_\epsilon|^2 + |\omega_\epsilon|^2) &= o(\epsilon^N).
\end{align*}
\]

Theorems 1.3 and 1.4 tell us that we can construct a new solution for \((P_\epsilon)\) by putting a peak upward near a strict extremum point of \( a(x) \) in \( A \) to the solution \( w_\epsilon^* \), or by putting a peak downward near a strict extremum point of \( a(x) \) in \( B \) to \( w_\epsilon^{**} \).

**Remark 1.5.** In Theorems 1.3 and 1.4, the requirement that \( x_i, i = 1, \ldots, k, \) are strict local minimum points or maximum points of \( a(x) \) can be relaxed, as in [DaY1]. For example, we can replace a strict local maximum point \( x_i \in A \) by a set \( A_i \subset A \) such that \( \max_{\partial A_i} a(x) < a_i := \sup_{A_i} a(x) \); then in the conclusion, we have, subject to a subsequence, \( x_i^\epsilon \to x_i \in M_i := \{x \in A_i : a(x) = a_i\} \).

**Remark 1.6.** In Theorem 1.3, we can replace the minimal solution \( w_\epsilon^* \) by any “local” minimizer solution \( v_\epsilon \) of \((P_\epsilon)\) obtained by Theorem 1.2 that satisfies \( v_\epsilon \to 0 \) locally uniformly in \( \Omega \). See explanations at the end of section 3.2.
In recent years, a variety of boundary blow-up problems has attracted considerable studies; we refer the interested reader to [APL, AR, BM1-3, CR1-2, DD, DL, Do, Du1-2, DH, DG1-4, Gu, GW, GLL, LM1-2, Ma, MV1-2] and the references therein for some of these problems.

The rest of this paper is organized as follows. In Section 2, we analyze the profiles of minimizer solutions of \((P_\epsilon)\) and its suitable variations, and prove Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorems 1.3 and 1.4.

2. Solutions with interior layers

We prove Theorem 1.2 in this section. As was pointed out in the introduction, Theorem 1.1 is a special case of Theorem 1.2. We will adapt the main ideas in section 2 of [DaY1], and make use of the minimizer solutions defined in the introduction above.

We start with some preparations. Let \(\Omega_1\) and \(\Omega_2\) be two open sets (possibly empty) satisfying \(\Omega_1 \cap \Omega_2 = \emptyset\), \(\Omega_i \subset \subset \Omega\), \(i = 1, 2\), \(a(x) < 1/2\) if \(x \in \partial \Omega_1\), \(a(x) > 1/2\) if \(x \in \partial \Omega_2\). For fixed \(x \in \Omega\), let \(M(x) \in (0, 1)\) and \(m(x) \in (0, 1)\) be determined by

\[
f(x, M(x)) = \max_{t \in [0, 1]} f(x, t) \quad \text{and} \quad f(x, m(x)) = \min_{t \in [0, 1]} f(x, t).
\]

It is easily seen that \(M(x)\) and \(m(x)\) are continuous.

Let \(f_1(x, t) \geq f(x, t)\) be a continuous function defined on \(\overline{\Omega_1} \times \mathbb{R}^1\) such that \(f_1(x, t) = f(x, t)\) for \(t \geq M(x)\), \(f_1(x, t)\) is locally Lipschitz continuous in \(t\) and \(f_1(x, t) > 0\) for \(t \in (-\infty, M(x))\).

Similarly, let \(f_2(x, t) \leq f(x, t)\) be a continuous function defined on \(\overline{\Omega_2} \times \mathbb{R}^1\) such that \(f_2(x, t) = f(x, t)\) for \(t \leq m(x)\), \(f_2(x, t)\) is locally Lipschitz continuous in \(t\) and \(f_2(x, t) < 0\) for \(t \in (m(x), \infty)\).

Define

\[
g(x, t) = \chi_{\Omega_1 \cup \Omega_2}(x) f(x, t) + \chi_{\Omega_1}(x) f_1(x, t) + \chi_{\Omega_2}(x) f_2(x, t),
\]

where \(\chi_E(x) = 1\) for \(x \in E\) and \(\chi_E(x) = 0\) for \(x \notin E\) for any set \(E \subset \mathbb{R}^N\). We see that \(g(x, t)\) is measurable in \(x\) for fixed \(t\), and is locally Lipschitz continuous in \(t\) for fixed \(x\). Moreover, by taking special care with our definitions of \(f_1\) and \(f_2\), we can make sure that \(g(x, t)/t\) is bounded for \(x \in \Omega\) and \(t\) in bounded intervals of \(\mathbb{R}^1\).

We will need two useful results, which are easy modifications of Lemmas 2.2 and 2.3 in [DaY1]. Let \(D\) be a bounded domain in \(\mathbb{R}^N\), and \(h(x, t)\) a function defined on \(\overline{D} \times \mathbb{R}^1\), continuous in \(t\) for each fixed \(x\), measurable in \(D\) for each fixed \(t\), and with the properties that \(h(x, t)/t\) is bounded for \(x \in D\) and \(t\) in bounded intervals of \(\mathbb{R}^1\), and

\[
h(x, t) > 0 \quad \text{for} \quad t < 0; \quad h(x, t) < 0 \quad \text{for} \quad t > 1.
\]
Consider
\[
\inf \left\{ \tilde{J}_e(u, D) = \frac{\epsilon^2}{2} \int_D |Du|^2 - \int_D H(x, u) : u - \eta \in H^1_0(D) \right\},
\]
where \(\eta \in H^1(D)\) satisfies \(0 \leq \eta \leq \beta, \beta \geq 1\) is a constant, and \(H(x, t) = \int_0^t h(x, s)ds\).

**Lemma 2.1.** Suppose that \(h(x, t)\) is as given above and \(u_\epsilon\) is a minimizer of \(\tilde{J}_e(u, D)\). Then \(0 \leq u_\epsilon \leq \beta\) in \(D\).

The proof of Lemma 2.1 is the same as that for Lemma 2.2 in [DaY1].

**Lemma 2.2.** Suppose that \(h_1(x, t)\) and \(h_2(x, t)\) both have the properties of \(h(x, t)\) as given above. Assume that \(\eta_i \in H^1(D), 0 \leq \eta_i \leq \beta\) for \(i = 1, 2, \beta \geq 1\) is a constant, and
\[
h_1(x, t) \geq h_2(x, t) \forall (x, t) \in D \times [0, \beta]; \eta_1(x) \geq \eta_2(x) \forall x \in D.
\]
Let \(u_{\epsilon i}\) be a minimizer of \(\tilde{J}_{\epsilon i}(u, D)\), which denotes \(\tilde{J}_e(u, D)\) with \(h\) replaced by \(h_i\). Then \(u_{\epsilon 1} \geq u_{\epsilon 2}\).

The proof of Lemma 2.2 is the same as that for Lemma 2.3 in [DaY1].

We now establish the existence of a minimizer solution to the problem
\[
-\epsilon^2 \Delta w = g(x, w) \text{ in } \Omega, \quad w|_{\partial \Omega} = \infty. \tag{2.1}
\]

**Lemma 2.3.** For every \(\epsilon > 0\), problem (2.1) has at least one minimizer solution \(w_\epsilon\).

**Proof.** Let \(a^* = \max_{\Omega} a(x)\) and \(a_* = \min_{\Omega} a(x)\). Clearly \(0 < a_* \leq a^* < 1\). Defining
\[
g_{a^*}(x, t) = \begin{cases}
g(x, t), & \text{for } (x, t) \in \overline{\Omega} \times (-\infty, 1], \\
t(t-a^*)(1-t), & \text{for } (x, t) \in \overline{\Omega} \times (1, \infty),
\end{cases}
\]
we see from the construction of \(g(x, t)\) that \(g(x, t) \leq g_{a^*}(x, t)\), and both \(g(x, t)\) and \(g_{a^*}(x, t)\) have the properties of \(h(x, t)\) used in Lemma 2.1.

Consider the problem
\[
-\epsilon^2 \Delta v = g_{a^*}(x, v) \text{ in } \Omega, \quad v|_{\partial \Omega} = \infty. \tag{2.2}
\]
It follows from well-known results (see, e.g., [APL, DG2]) that for \(\epsilon > 0\), (2.2) has a unique positive solution \(\overline{v}_\epsilon\) in the order interval \((1, \infty)\) of \(C^0(\Omega)\). It is clear that \(\overline{v}_\epsilon\) is a supersolution to (2.1). Now for any \(\beta > 1\), if \(w_\epsilon^\beta\) is a minimizer of
\[
\inf \left\{ \frac{\epsilon^2}{2} \int_\Omega |Dw(x)|^2 dx - \int_\Omega G(x, w) dx, \quad \beta - w \in H^1_0(\Omega) \right\},
\]
where \(G(x, t) = \int_0^t g(x, s)ds\) is bounded from above for \((x, t) \in \Omega \times \mathbb{R}\) (which guarantees the existence of a minimizer), we claim that
\[
w_\epsilon^\beta \leq \overline{v}_\epsilon \text{ in } \Omega. \tag{2.3}
\]
Indeed, let $D = \{x \in \Omega : w^\epsilon_\delta(x) > \tau_\epsilon(x)\}$, and define $\phi_\epsilon = (w^\epsilon_\delta - \tau_\epsilon)^+$. We see that $D \subset \subset \Omega$ and $\phi_\epsilon \in H^1_0(D)$. Moreover, since $g_\alpha(x,t)$ is decreasing in $t$ for $t > 1$,

$$-\epsilon^2 \Delta (w^\epsilon_\delta - \tau_\epsilon) \leq g_\alpha(x,w^\epsilon_\delta) - g_\alpha(x,\tau_\epsilon) < 0 \text{ in } D.$$ 

Multiplying $\phi_\epsilon$ to the above inequality and integrating it over $D$, we easily derive a contradiction if $D \neq \emptyset$. This proves our claim (2.3).

Since $w^\epsilon_\delta$ satisfies

$$-\epsilon^2 \Delta w = g(x,w) \text{ in } \Omega, \quad w|_{\partial \Omega} = \beta,$$

the regularity of the operator $-\Delta$ implies that $w^\epsilon_\delta \in C^1(\Omega)$, and due to Lemma 2.1 and Harnack’s inequality we find that $w^\epsilon_\delta > 0$ in $\Omega$. Moreover, by Lemma 2.2 we find that $w^\epsilon_\delta$ is nondecreasing in $\beta$. Now from (2.3) we find, by a standard regularity consideration, that for any increasing sequence $\{\beta_n\}$ with $\beta_n \to \infty$ as $n \to \infty$, $w^\epsilon_{\beta_n} \to w_\epsilon$ in $C^1_{loc}(\Omega)$ as $n \to \infty$. Therefore, $w_\epsilon$ is a minimizer solution to (2.1). This completes the proof of Lemma 2.3. \qed

We next analyze the profile of a minimizer solution $w_\epsilon$ as given in Lemma 2.3, and prove that for $\epsilon$ small enough, $w_\epsilon$ is a solution of $(P_\epsilon)$ possessing the properties described in Theorem 1.2.

**Theorem 2.4.** Let $w_\epsilon$ be a minimizer solution to (2.1). Then, as $\epsilon \to 0$,

$$w_\epsilon \to \begin{cases} 1, & \text{uniformly on any compact subset of } (A \setminus \overline{\Omega}_2) \cup \overline{\Omega}_1, \\ 0, & \text{uniformly on any compact subset of } (B \setminus \overline{\Omega}_1) \cup \overline{\Omega}_2. \end{cases}$$

Clearly, Theorem 1.1 follows directly from Theorem 2.4 if we take $\Omega_1 = \Omega_2 = \emptyset$. On the other hand, since $w_\epsilon \to 1$ uniformly on $\overline{\Omega}_1$ as $\epsilon \to 0$, we see that for $\epsilon > 0$ small, $w_\epsilon(x) > M(x)$ if $x \in \Omega_1$. Thus, $f_1(x,w_\epsilon(x)) = f(x,w_\epsilon(x))$ if $x \in \Omega_1$. Since $w_\epsilon \to 0$ uniformly in $\overline{\Omega}_2$ as $\epsilon \to 0$, we see that for $\epsilon > 0$ small, $w_\epsilon(x) < m(x)$ if $x \in \Omega_2$. Thus, $f_2(x,w_\epsilon(x)) = f(x,w_\epsilon(x))$ if $x \in \Omega_2$. So we see that if $\epsilon > 0$ is small enough, $w_\epsilon$ is also a solution of $(P_\epsilon)$ and Theorem 1.2 follows.

**Proof of Theorem 2.4.** For the sake of clarity, we divide the proof into several steps.

**Step 1.** $w_\epsilon(x) \to 1$ as $\epsilon \to 0$ uniformly on any compact subset of $A \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$.

Let $x_0 \in A \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Choose $\delta > 0$ small such that $B_\delta(x_0) \subset \subset A \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Take $b^* \in (\max_{z \in B_\delta(x_0)} a(z), 1/2)$ and $b_\epsilon \in (0, \min_{z \in B_\delta(x_0)} a(z))$. Define $f_{b^*}(t) = t(t - b^*)(1 - t)$. Then $g(x,t) = f(x,t) \leq f_{b^*}(t)$ for $(x,t) \in B_\delta(x_0) \times (1, \infty)$. By the same consideration used for (2.2), we see that the problem

$$-\epsilon^2 \Delta u = f_{b^*}(u) \text{ in } B_\delta(x_0), \quad u = 0 \text{ on } \partial B_\delta(x_0)$$

(2.5)
has a unique solution \( u_{\epsilon, b^*} \) satisfying \( u_{\epsilon, b^*} > 1 \), and \( u_{\epsilon, b^*} \to 1 \) in \( C^0_{\text{loc}}(B_\delta(x_0)) \) as \( \epsilon \to 0 \). An argument similar to that leading to (2.3) shows that

\[
w_\epsilon \leq u_{\epsilon, b^*} \quad \text{in} \quad B_\delta(x_0). \tag{2.6}
\]

On the other hand, we have \( f(x, t) \geq f_{b^*, b^*}(t) \) for \( (x, t) \in \overline{B_\delta(x_0)} \times \mathbb{R}^1 \), where

\[
f_{b^*, b^*}(t) = \begin{cases} t(t - b^*)(1 - t), & \text{if } t \in [0, 1], \\
(t(t - b^*)(1 - t), & \text{if } t \in (-\infty, 0) \cup (1, \infty). \end{cases}
\]

Let \( \overline{u}_{\epsilon, b^*, b^*} \) be a minimizer of

\[
\inf \left\{ \frac{\epsilon^2}{2} \int_{B_\delta(x_0)} |Du|^2 - \int_{B_\delta(x_0)} F_{b^*, b^*}(u) : u \in H^1_0(B_\delta(x_0)) \right\}, \tag{2.8}
\]

where \( F_{b^*, b^*}(t) = \int_0^t f_{b^*, b^*}(s)ds \). Since \( \int_0^1 f_{b^*, b^*}(t)dt > 0 \), as observed in [DaY1], it follows from Theorem 2' and the proof of Lemma 2.1 in [CS], and our Lemma 2.1 above, that

\[
\overline{u}_{\epsilon, b^*, b^*}(x) \to 1 \text{ as } \epsilon \to 0 \text{ uniformly for } x \in B_{\delta/2}(x_0). \tag{2.9}
\]

Since \( w_\epsilon \) is a minimizer solution to (2.1) and \( f(x, t) = g(x, t) \) for \( x \in B_\delta(x_0) \), there exist sequences \( \{\beta_n\} \) and \( \{w_{\epsilon, n}\} \) with \( \beta_n \to \infty \) and \( w_{\epsilon, n} \to w_\epsilon \) in \( C^i_{\text{loc}}(\Omega) \) as \( n \to \infty \), where \( w_{\epsilon, n} \) is a minimizer of

\[
\inf \left\{ \frac{\epsilon^2}{2} \int_\Omega |Du|^2 - \int_\Omega G(x, u) : \beta_n - u \in H^1_0(\Omega) \right\},
\]

and hence its restriction to \( B_\delta(x_0) \) is a minimizer to

\[
\inf \left\{ \frac{\epsilon^2}{2} \int_{B_\delta(x_0)} |Du|^2 - \int_{B_\delta(x_0)} G(x, u) : w_{\epsilon, n} - u \in H^1_0(B_\delta(x_0)) \right\}. \tag{2.10}
\]

By Lemma 2.2 above, since \( w_{\epsilon, n}^\beta_n(x) > 0 \) for \( x \in \partial B_\delta(x_0) \), we conclude that

\[
w_{\epsilon, n}^\beta_n(x) \geq \overline{u}_{\epsilon, b^*, b^*}(x), \quad \forall x \in B_\delta(x_0). \tag{2.11}
\]

Clearly (2.11) implies

\[
w_{\epsilon}(x) \geq \overline{u}_{\epsilon, b^*, b^*}(x), \quad \forall x \in B_\delta(x_0). \tag{2.12}
\]

Moreover, it follows from (2.6), (2.9) and (2.12) that \( w_{\epsilon}(x) \to 1 \) as \( \epsilon \to 0 \) uniformly for \( x \in B_{\delta/2}(x_0) \). This implies the conclusion of Step 1.

**Step 2.** \( w_{\epsilon}(x) \to 1 \) as \( \epsilon \to 0 \) uniformly on any compact subset of \( \Omega \).

For any \( x_0 \in \Omega \) and \( B_\delta(x_0) \subset \Omega \), we consider the problem

\[
-\epsilon^2 \Delta v = g_{a^*}(x, v) \quad \text{in} \quad B_\delta(x_0), \quad v|_{\partial B_\delta(x_0)} = \infty, \tag{2.13}
\]
where \( g_{\bullet}(x, v) \) is as in (2.2). Applying the same reasoning used for (2.2) we see that (2.13) has a unique solution \( v_{\epsilon, a^*} \) satisfying \( \max_{B_\delta(x_0)} v_{\epsilon, a^*} > 1 \) and \( v_{\epsilon, a^*} \to 1 \) in \( C^0_{\text{loc}}(B_\delta(x_0)) \) as \( \epsilon \to 0 \). Moreover, similar to (2.3), we have

\[
w_\epsilon \leq v_{\epsilon, a^*} \quad \text{in} \quad B_\delta(x_0).
\]

On the other hand, from our choice of \( f_1 \), we can find some constant \( \kappa > 0 \) small such that,

\[
g(x, t) \geq g_{\kappa, a^*}(t) := \begin{cases} \kappa(1 - t), & \forall (x, t) \in B_\delta(x_0) \times (-\infty, 1), \\ t(t - a^*)(1 - t), & \forall (x, t) \in B_\delta(x_0) \times (1, \infty). \end{cases}
\]

Replacing \( f_{b, b^*} \) by \( g_{\kappa, a^*}(t) \) in the proof of (2.12), we obtain

\[
w_\epsilon \geq \tilde{u}_{\epsilon, \kappa, a^*} \quad \text{in} \quad B_\delta(x_0),
\]

where \( 0 < \tilde{u}_{\epsilon, \kappa, a^*} < 1 \) is a minimizer of

\[
\inf \left\{ \frac{\epsilon^2}{2} \int_{B_\delta(x_0)} |Du|^2 - \int_{B_\delta(x_0)} G_{\kappa, a^*}(u) : u \in H^1_0(B_\delta(x_0)) \right\},
\]

where \( G_{\kappa, a^*}(t) = \int_0^t g_{\kappa, a^*}(s) ds \). It is well known (see, e.g., [Da]) that \( \tilde{u}_{\epsilon, \kappa, a^*} \to 1 \) in \( C^0_{\text{loc}}(B_\delta(x_0)) \) as \( \epsilon \to 0 \). Thus, it follows from (2.14) and (2.15) that \( w_\epsilon(x) \to 1 \) as \( \epsilon \to 0 \) uniformly for \( x \in B_{\delta/2}(x_0) \). This implies the conclusion in Step 2.

**Step 3.** \( w_\epsilon(x) \to 0 \) as \( \epsilon \to 0 \) uniformly on any compact subset of \( B \setminus (\overline{B_1} \cup \overline{B_2}) \).

For \( x_0 \in B \setminus (\overline{B_1} \cup \overline{B_2}) \), we can choose \( \delta > 0 \) small so that \( B_\delta(x_0) \subset B \setminus (\overline{B_1} \cup \overline{B_2}) \). Let \( b_1 = (1/2, \min_{x \in B_\delta(x_0)} a(z)) \) and \( b_2 = (\max_{x \in B_\delta(x_0)} a(z), 1) \). We see easily that

\[
g(x, t) = f(x, t) \leq f_{b_1, b_2}(t) := \begin{cases} t(t - b_1)(1 - t), & \forall (x, t) \in B_\delta(x_0) \times [0, 1], \\ t(t - b_2)(1 - t), & \forall (x, t) \in B_\delta(x_0) \times (\mathbb{R}^1 \setminus [0, 1]). \end{cases}
\]

Moreover, by (2.3), we easily see that for any \( \epsilon > 0 \) small and all \( n \), there exists \( C = C(\epsilon) > 1 \) independent of \( n \) such that

\[
w_\epsilon^{\beta_n}(x) < C \quad \forall x \in \partial B_\delta(x_0).
\]

Thus, Lemma 2.2 implies that

\[
w_\epsilon^{\beta_n} \leq y_{\epsilon, C} \quad \forall x \in B_\delta(x_0),
\]

where \( y_{\epsilon, C} \) is a minimizer of

\[
\inf \left\{ \frac{\epsilon^2}{2} \int_{B_\delta(x_0)} |Du|^2 - \int_{B_\delta(x_0)} F_{b_1, b_2}(u) : C - u \in H^1_0(B_\delta(x_0)) \right\}
\]

with \( F_{b_1, b_2}(t) = \int_0^t f_{b_1, b_2}(s) ds \). This implies

\[
w_\epsilon \leq y_{\epsilon, C} \quad \forall x \in B_\delta(x_0).
\]
We now claim
\[ y_{\epsilon,C} \to 0 \text{ in } C^0_{\text{loc}}(B_{\delta}(x_0)) \text{ as } \epsilon \to 0. \] (2.20)

Since \( y_{\epsilon,C} \) satisfies
\[ -\epsilon^2 \Delta y = f_{b_1,b_2}(y) \text{ in } B_{\delta}(x_0), \quad y|_{\partial B_{\delta}(x_0)} = C, \] (2.21)
we easily see from the maximum principle that \( y_{\epsilon,C} \leq C \text{ in } B_{\delta}(x_0) \). Setting \( z_{\epsilon} = C - y_{\epsilon,C} \), we see that \( z_{\epsilon} \) satisfies the Dirichlet problem
\[ -\epsilon^2 \Delta z = \tilde{f}_{b_1,b_2}(z) \text{ in } B_{\delta}(x_0), \quad z|_{\partial B_{\delta}(x_0)} = 0, \] (2.22)
where \( \tilde{f}_{b_1,b_2}(s) = -f_{b_1,b_2}(C - s) \), and it is a minimizer of
\[ \inf \left\{ \frac{\epsilon^2}{2} \int_{B_{\delta}(x_0)} |Du|^2 - \int_{B_{\delta}(x_0)} \tilde{F}_{b_1,b_2}(u) : u \in H^1_0(B_{\delta}(x_0)) \right\} \]
with \( \tilde{F}_{b_1,b_2}(t) = \int_0^t \tilde{f}_{b_1,b_2}(s)ds \).

Since \( \tilde{f}_{b_1,b_2}(s) > 0 \) for \( s \in [0,C-1) \cup (C-b_1,C) \) and \( \tilde{f}_{b_1,b_2}(s) < 0 \) for \( s \in (C-1,C-b_1) \) with \( \tilde{f}_{b_1,b_2}(C-1) = \tilde{f}_{b_1,b_2}(C-b_1) = \tilde{f}_{b_1,b_2}(C) = 0 \) and \( \int_{C-1}^{C-1} \tilde{f}_{b_1,b_2}(s)ds > 0 \), we see from the arguments leading to (2.9) that
\[ z_{\epsilon} \to C \text{ in } C^0_{\text{loc}}(B_{\delta}(x_0)) \text{ as } \epsilon \to 0, \]
and (2.20) follows. From (2.19) and (2.20) we find \( w_{\epsilon}(x) \to 0 \) as \( \epsilon \to 0 \) uniformly for \( x \in B_{\delta/2}(x_0) \). The conclusion in Step 3 thus follows.

**Step 4.** \( w_{\epsilon}(x) \to 0 \) as \( \epsilon \to 0 \) uniformly on any compact subset of \( \Omega_2 \).

For \( x_0 \in \Omega_2 \), we have \( g(x,t) = f_2(x,t) < 0 \) for \( t \in (0,\infty) \), and \( f_2(x,t) = f(x,t) \) for \( t \in (-\infty,m(x)] \). Thus, we can choose \( \bar{g}(t) \) satisfying \( \bar{g}(0) = 0, \bar{g}'(0) < 0, t\bar{g}(t) < 0 \) for all \( t \neq 0, \bar{g} \) is \( C^1 \), \( \lim_{t \to \infty} \bar{g}(t)/t^2 = -\infty \), and
\[ g(x,t) \leq \bar{g}(t) \forall (x,t) \in \partial \Omega_2 \times (-\infty, \infty). \]
For any \( \epsilon > 0 \) small and all \( n \), since \( w_{\epsilon}^{\beta_n} \leq C \) for \( x \in \partial B_{\delta}(x_0) \), where \( C = C(\epsilon) \) is independent of \( n \), it follows from Lemma 2.2 that for all \( n \),
\[ w_{\epsilon}^{\beta_n}(x) \leq \bar{y}_{\epsilon,C}(x) \forall x \in B_{\delta}(x_0), \] (2.23)
where \( \bar{y}_{\epsilon,C} \) is a minimizer of
\[ \inf \left\{ \frac{\epsilon^2}{2} \int_{B_{\delta}(x_0)} |Du|^2 - \int_{B_{\delta}(x_0)} \bar{G}(u) : C - u \in H^1_0(B_{\delta}(x_0)) \right\} \] (2.24)
with \( \bar{G}(t) = \int_0^t \bar{g}(s)ds \). This implies
\[ w_{\epsilon}(x) \leq \bar{y}_{\epsilon,C}(x) \forall x \in B_{\delta}(x_0). \] (2.25)
Clearly $\tilde{y}_{\epsilon,C}$ is a solution to 

$$-\epsilon^2 \Delta y = \bar{g}(y), \quad y|_{\partial B_{\delta}(x_0)} = C.$$ 

By [DG1], we have $\tilde{y}_{\epsilon,C} \to 0$ in $C^{0}_{\text{loc}}(B_{\delta}(x_0))$ as $\epsilon \to 0$ and hence $u_{\epsilon}(x) \to 0$ as $\epsilon \to 0$ uniformly for $x \in B_{\delta/2}(x_0)$. This implies the conclusion in Step 4.

**Step 5. Behavior of $w_{\epsilon}$ near $\partial \Omega_1$ and $\partial \Omega_2$.**

To complete the proof of the theorem, it remains to prove that for $x_0 \in \partial \Omega_1$ and $x_0 \in \partial \Omega_2$, $w_{\epsilon}(x) \to 1$ and $w_{\epsilon}(x) \to 0$ respectively, as $\epsilon \to 0$ uniformly for $x \in B_{\delta/2}(x_0)$.

For $x_0 \in \partial \Omega_1$, we have $a(x_0) < 1/2$ and there is a small $\delta > 0$ such that $a(x) < 1/2$ for $x \in B_{\delta}(x_0)$. We also have $g(x,t) \geq f(x,t) \geq f_{b,b^*}(t)$. By minor modifications of the arguments in Step 1, we obtain that $w_{\epsilon}(x) \to 1$ as $\epsilon \to 0$ uniformly for $x \in B_{\delta/2}(x_0)$. In a similar fashion, the proof of the case that $x_0 \in \partial \Omega_2$ can be done by modifying arguments in Step 3. \[\square\]

3. Solutions with interior spikes and boundary layers

In this section, we will use a reduction method to prove Theorems 1.3 and 1.4. This method has been widely used in many singularly perturbed elliptic problems with homogeneous Dirichlet or Neumann boundary conditions. In a recent paper [DuY], it was also used to construct peak solutions to boundary blow-up problems. We will mainly follow the strategy of [DuY], but there are extra difficulties here. These difficulties arise in the proof of Theorem 1.3 since $f_t(x,w^*_\epsilon(x))$ changes sign for $x \in \Omega$ when $\epsilon > 0$ is small. Such difficulties do not arise in [DuY], nor do they arise in the proof of Theorem 1.4 here, because when $w^*_\epsilon > 1$, $f_t(x,w^*_\epsilon(x))$ is negative and bounded away from 0. Since the proofs of Theorems 1.3 and 1.4 are similar except these extra difficulties in the proof of Theorem 1.3, we will omit the proof of Theorem 1.4.

For the proof of Theorem 1.3, our strategy is to find a solution of $(P_\epsilon)$ of the form $w^*_\epsilon + w$ with $w \in H^1_0(\Omega)$, where $w^*_\epsilon$ is the minimal solution of $(P_\epsilon)$ as described in the introduction. This amounts to solving the problem

$$-\epsilon^2 \Delta w = h(x,w) \quad \text{in } \Omega, \quad w|_{\partial \Omega} = 0, \quad (3.1)$$

where

$$h(x,t) = h_\epsilon(x,t) = f(x,w^*_\epsilon(x) + t) - f(x,w^*_\epsilon(x)).$$

As the rest of this section is rather long, we break it into three subsections.

3.1. Preparations. In this section, we set up a variational framework for (3.1). Since we will only prove Theorem 1.3, we always assume that $a(x) > 1/2$ on $\partial \Omega$.

The following Lemma 3.1 is useful here and it also plays an important role in the reduction method to be used in the proof of Theorem 1.3.
Lemma 3.1. There exist $\epsilon_0 > 0$ and $\kappa^* > 0$ such that for all $\epsilon \in (0, \epsilon_0)$,
\[
\int_\Omega \left( \epsilon^2 |Dv|^2 - f_\epsilon(x, w_\epsilon^*)v^2 \right) dx \geq \kappa^* \int_\Omega v^2 dx, \quad \forall v \in C_0^\infty(\Omega).
\]

The proof of Lemma 3.1 is rather involved and long; we postpone it to the next subsection, which is devoted entirely to this proof.

For $\ell \in (\max_{x \in T} a(x), 1)$, we define
\[
f_\ell(s) := s(1 - s).
\]

Lemma 3.2. Fix $\ell \in (\max_{x \in T} a(x), 1)$ and let $t^* > 1$ satisfy
\[
f_\ell(t^*) \leq \min_{[0, 1]} f_\ell(t), \quad -(t^*)^2 + (M_2/2)t^* + M_1 < 0,
\]
where
\[
M_1 = \max_{x \in T, t \in [0, 1]} |f_\ell(x, t)|, \quad M_2 = \max_{x \in T, t \in [0, 1]} |f_\ell(x, t)|.
\]

Then $h(x, t) < 0$ for all $t \geq t^*$ and every $x \in \Omega$.

Proof. If $w_\epsilon^*(x) \geq 1$, we see that $f(w_\epsilon^*(x) + t) < f(w_\epsilon^*(x))$ for any $t > 0$. Therefore, $h(x, t) < 0$ for $t \geq t^*$.

Suppose now $0 < w_\epsilon^*(x) < 1$. If $t \geq t^*$, we have $w_\epsilon^*(x) + t > t \geq t^*$. Therefore
\[
f(x, w_\epsilon^* + t) < f(x, t^*) \leq f_\ell(t^*) \leq \min_{t \in [0, 1]} f_\ell(t) \leq f(x, w_\epsilon^*(x)).
\]
Thus, we always have $h(x, t) < 0$ for $t \geq t^*$. This completes the proof. \( \square \)

In order to use a variational approach, we need to modify $h(x, t)$ a little. Write
\[
h(x, t) = f_\ell(x, w_\epsilon^*)t - t^2[t - (1/2)f_\ell(x, w_\epsilon^*)].
\]

We choose a $C^2$ function $g^*(t)$ such that
(i) $0 \leq g^*(t) \leq t^2$,
(ii) $g^*(t) = 0$ for $t \leq 0$, $g^*(t) = t^2$ for $t \in [0, t^*]$, $g^*(t)$ is nondecreasing,
(iii) $|\frac{d}{dt} g^*(t)| \leq Ct^{\alpha-i}$ for $t \geq t^*$ and $i = 0, 1, 2$,

where $\alpha \in (0, 1)$ is chosen so that
\[
2 + 2\alpha < 2^*, \quad 2^* = \infty \text{ if } N = 2; \quad 2^* = 2N/(N - 2) \text{ if } N > 2.
\]

We now see that
\[
g(x, t) = g_\epsilon(x, t) := -g^*(t)[t - (1/2)f_\ell(x, w_\epsilon^*(x))]
\]
is a $C^2$-function.

Consider the problem
\[
-\epsilon^2 \Delta v - f_\ell(x, w_\epsilon^*(x))v = g(x, v) \quad \text{in } \Omega, \quad v|_{\partial \Omega} = 0.
\quad (3.2)
\]
We have the following result.

**Lemma 3.3.** Let $v \in H^1_0(\Omega)$ be a solution to (3.2). Then for all small $\epsilon > 0$, $0 \leq v \leq t^*$ in $\Omega$ and hence $v$ is a solution to (3.1).

**Proof.** We first show that if $\epsilon > 0$ is small and $v$ is a nontrivial solution to (3.2), then $v \geq 0$. Multiplying (3.2) by $\tilde{v}(x) := (-v)^+ \in H^1_0(\Omega)$ and integrating it over $\Omega$, we obtain

$$\int_\Omega [\epsilon^2 |D\tilde{v}|^2 - f_t(x, w^*_\epsilon)\tilde{v}^2] \, dx = 0. \quad (3.3)$$

It follows from Lemma 3.1 that $\tilde{v}$ is a solution to (3.1).

Suppose for contradiction that (3.2) has a solution $v$ satisfying $v(x) > t^*$ for some $x \in \Omega$. Note that the interior regularity implies that $v \in C^1(\Omega) \cap W^{2,p}_{loc}(\Omega)$ for all $p > 1$. Let $x_0 \in \Omega$ be such that $v(x_0) = \max_{\Omega} v > t^*$. Then by Bony’s maximum principle, there exists a sequence $x_n \to x_0$ such that

$$g(x_0, v(x_0)) + f_t(x_0, w^*_\epsilon(x_0))v(x_0) = \lim_{n \to \infty} \left(-\epsilon^2 \Delta v(x_n)\right) \geq 0. \quad (3.4)$$

For the case that $\mu := v(x_0) - (1/2)f_t(x_0, w^*_\epsilon(x_0)) > 0$, we see that if $f_t(x_0, w^*_\epsilon(x_0)) < 0$, then

$$g(x_0, v(x_0)) + f_t(x_0, w^*_\epsilon(x_0))v(x_0) < g(x_0, v(x_0)) - g^*(v(x_0))\mu \leq 0.$$

This contradicts (3.4). If $f_t(x_0, w^*_\epsilon(x_0)) \geq 0$, then necessarily $w^*_\epsilon(x_0) \in (0, 1)$, and hence

$$g(x_0, v(x_0)) + f_t(x_0, w^*_\epsilon(x_0))v(x_0)
= -g^*(v(x_0))\left[v(x_0) - (1/2)f_t(x_0, w^*_\epsilon(x_0))\right] + f_t(x_0, w^*_\epsilon(x_0))v(x_0)
\leq -g^*(t^*)\left[v(x_0) - (1/2)M_2\right] + M_1 v(x_0)
= [M_1 - (t^*)^2]v(x_0) + (1/2)M_2(t^*)^2
\leq -(t^*)^3 + (1/2)M_2(t^*)^2 + M_1 t^* < 0.$$

This contradicts (3.4) too.

For the case that $\mu < 0$, we have, by Lemma 3.2,

$$g(x_0, v(x_0)) + f_t(x_0, w^*_\epsilon(x_0))v(x_0)
\leq -v(x_0)^2\mu + f_t(x_0, w^*_\epsilon(x_0))v(x_0)
= h(x_0, v(x_0)) < 0.$$

Hence we again have a contradiction to (3.4). This completes the proof. \qed

Now we define, for each $\epsilon \in (0, \epsilon_0)$, the space $H = H_\epsilon$ as the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_\epsilon := \left(\int_\Omega (\epsilon^2 |Du|^2 - f_t(x, w^*_\epsilon)u^2) \, dx\right)^{1/2}.$$
We easily see that $H \subset H^{1}_{0}(\Omega)$. Indeed, for $u \in H$, we have
\[
\int_{\Omega} \epsilon^{2} |Du|^{2} dx = \|u\|^{2}_{\epsilon} + \int_{\Omega} f_{t}(x, w_{\epsilon}^{*})u^{2} dx \leq \|u\|^{2}_{\epsilon} + \int_{\{w_{\epsilon}^{*}(x) \leq 1\}} f_{t}(x, w_{\epsilon}^{*})u^{2} dx
\]
\[
\leq \|u\|^{2}_{\epsilon} + M_{1} \int_{\{w_{\epsilon}^{*}(x) \leq 1\}} u^{2} dx \leq \|u\|^{2}_{\epsilon} + M_{1} \int_{\Omega} u^{2} dx
\]
\[
\leq \left(1 + \frac{M_{1}}{\kappa^{*}}\right)\|u\|^{2}_{\epsilon}.
\]

Let
\[
I(u) = (1/2) \int_{\Omega} (\epsilon^{2}|Du|^{2} - f_{t}(x, w_{\epsilon}^{*})u^{2}) dx - \int_{\Omega} G(x, u) dx,
\]
where $G(x, t) = \int_{0}^{t} g(x, s) ds$.

By slight modifications of the arguments in the proof of Lemma 2.4 in [DuY], we obtain the following result.

**Lemma 3.4.** $I(u)$ is well-defined for $u \in H$.

Clearly, any critical point $w$ of $I$ in $H$ satisfies
\[
\int_{\Omega} \left(\epsilon^{2}Dw \cdot D\phi - f_{t}(x, w_{\epsilon}^{*})w\phi - g(x, w)\phi\right) dx = 0, \quad \forall \phi \in C_{0}^{\infty}(\Omega).
\]

Therefore, by standard regularity consideration, $w$ is $C^{1}$ in the interior of $\Omega$ and satisfies (3.2) in $\Omega$. Since $H \subset H^{1}_{0}(\Omega)$, the boundary condition is satisfied in the weak sense.

### 3.2. Proof of Lemma 3.1.

This subsection is devoted to the proof of Lemma 3.1, namely, there exists $\kappa^{*} > 0$ independent of $\epsilon$ such that for all $\epsilon > 0$ small
\[
\int_{\Omega} [\epsilon^{2}|Dv|^{2} - f_{t}(x, w_{\epsilon}^{*})v^{2}] dx \geq \kappa^{*} \int_{\Omega} v^{2} dx, \quad \forall v \in C_{0}^{\infty}(\Omega).
\]
At the end of this subsection, we will give some detailed explanations about Remark 1.6.

Since 0 is a subsolution and any positive integer $n$ is a supersolution to the problem
\[
-\epsilon^{2}\Delta w = f(x, w) \quad \text{in} \quad \Omega, \quad w|_{\partial\Omega} = n, \quad (3.6)
\]
we see from a standard sub- and supersolution argument as in [DG1] that this problem possesses a minimal positive solution $w_{\epsilon}^{n} \in C^{1}(\overline{\Omega})$ for each $n \geq 1$ and $\epsilon > 0$. It is easily seen (as in [DG1]) that
\[
w_{\epsilon}^{n} \leq w_{\epsilon}^{*} \text{ and } w_{\epsilon}^{n} \to w_{\epsilon}^{*} \text{ in } C^{1}_{loc}(\Omega) \text{ as } n \to \infty.
\]
For the eigenvalue problem
\[
-\epsilon^{2}\Delta k - f_{t}(x, w_{\epsilon}^{n})k = \lambda k \quad \text{in} \quad \Omega, \quad k|_{\partial\Omega} = 0,
\]
it is well known that there is a first eigenvalue $\lambda_{\epsilon}^{n}$ with corresponding eigenfunction $k_{\epsilon}^{n} > 0$ in $\Omega$, and
\[ \lambda_n^{\epsilon} = \inf_{v \in H_0^1(\Omega) \setminus \{0\}} \left\{ \int_\Omega [\epsilon^2|Dv|^2 - f_t(x, w_n^\epsilon)v^2]dx / \int_\Omega v^2dx \right\}. \] 

(3.7)

We may assume that \( \|k_n^\epsilon\|_\infty = 1 \).

From the properties of \( f_t(x, t) \), we can find a constant \( C_0 > 0 \) independent of \( n \) and \( \epsilon \) such that \( C_0 - f_t(x, w_n^\epsilon) \geq 0 \) in \( \overline{\Omega} \). It follows that \( \lambda_n^{\epsilon} + C_0 > 0 \). This implies that \( \lambda_n^{\epsilon} \) has a lower bound independent of \( \epsilon \) and \( n \).

We want to show that there exists some \( \kappa^* > 0 \) such that

\[ \lambda_n^{\epsilon} \geq \kappa^* \]  
for all \( n \geq 1 \) and all small \( \epsilon > 0 \). \hfill (3.8)

If (3.8) can be proved, then from (3.7) we deduce

\[ \int_\Omega [\epsilon^2|Dv|^2 - f_t(x, w_n^\epsilon)v^2]dx \geq \kappa^* \int_\Omega v^2dx, \forall v \in C_0^\infty(\Omega). \]

Letting \( n \to \infty \), we obtain

\[ \int_\Omega [\epsilon^2|Dv|^2 - f_t(x, w_n^\epsilon)v^2]dx \geq \kappa^* \int_\Omega v^2dx, \forall v \in C_0^\infty(\Omega), \]

as required.

We use a contradiction argument to show (3.8). Suppose that there exists a sequence \( \{\epsilon_n\} \) with \( \epsilon_n \to 0 \) as \( n \to \infty \) such that \( \lambda_n^{\epsilon_n} < 1/n \). For convenience, we use the notations \( \{\lambda_n\} := \{\lambda_n^{\epsilon_n}\} \), \( \{w_n\} := \{w_n^{\epsilon_n}\} \) and \( \{k_n\} := \{k_n^{\epsilon_n}\} \). We have

\[ -\epsilon_n^2 \Delta k_n = f_t(x, w_n)k_n + \lambda_nk_n \text{ in } \Omega, \quad k_n|_{\partial \Omega} = 0. \]  

(3.9)

Let \( x_n \in \Omega \) satisfy \( k_n(x_n) = 1 \). We first observe that

\[ \lim_{n \to \infty} d(x_n, \partial \Omega) = 0. \]  

(3.10)

Otherwise, we can find a constant \( c > 0 \) and a subsequence (still denoted by \( \{x_n\} \)) such that \( d(x_n, \partial \Omega) \geq c \) for all \( n \) large. Since \( w_n \to 0 \) in \( C^0_{\text{loc}}(\Omega) \) as \( n \to \infty \), we see from the properties of \( f \) that for \( n \) large,

\[ f_t(x_n, w_n(x_n)) + \lambda_n < 0. \]  

(3.11)

Standard elliptic regularity (see [GT]) implies that \( k_n \in C^1(\overline{\Omega}) \cap W^{2,p}(\Omega) \) for all \( p > 1 \). Hence by Bony’s maximum principle, for each \( n \), there exists a sequence \( x_n^m \to x_n \) such that

\[ f_t(x_n, w_n(x_n)) + \lambda_n = \lim_{m \to \infty} [-\epsilon_n^2 \Delta k_n(x_n^m)] \geq 0, \]

a contradiction to (3.11).

By passing to a subsequence if necessary, we have two cases to consider:

(i) \( \epsilon_n^{-1}d(x_n, \partial \Omega) = \infty \) as \( n \to \infty \),
Thus min \(\xi_n \in \partial \Omega\) be such that \(d(x_n, \xi_n) = d(x_n, \partial \Omega)\). By passing to a subsequence, we may assume that \(\lim_{n \to \infty} \xi_n = \xi \in \partial \Omega\). Since \(a(x) > 1/2\) for \(x \in \partial \Omega\), the continuity of \(a(x)\) implies that there exists a small \(\tau_0 > 0\) such that \(a(x) > 1/2\) for \(x \in B_{\varepsilon\tau_0}(\xi) \cap \Omega\).

For any \(\tau \in (0, \tau_0)\), we denote \(a_1^\tau = \inf_{B_{\varepsilon\tau}\cap \Omega} a(x)\) and \(a_2^\tau = \sup_{B_{\varepsilon\tau}\cap \Omega} a(x)\). Then \(a_2^\tau \geq a_1^\tau > 1/2\). Now we consider the problem

\[-\varepsilon_n^2 \Delta W = f_{a_1^\tau, a_2^\tau}(W) \text{ in } B_{\tau}, \quad W|_{\partial B_{\tau}} = \infty,\]  

where \(B_{\tau} = \{x \in \mathbb{R}^N : |x| < \tau\}\),

\[f_{a_1^\tau, a_2^\tau}(t) = \begin{cases} \frac{t(t-a_1^\tau)(1-t)}{2}, & \text{if } t \in [0, 1], \\ \frac{(t-a_2^\tau)(1-t)}{2}, & \text{if } t \in \mathbb{R}^1 \setminus [0, 1]. \end{cases}\]

We see from [APL, DG3] that (3.12) has a minimal (hence radial) positive solution \(W_{\tau,n}(x) = W_{\tau,n}(r)\) (where \(r = |x|\)) with \(W_{\tau,n} \to 0\) in \(C^0_{\text{loc}}(B_{\tau})\) as \(\varepsilon_n \to 0\). Moreover, using a sweeping out argument as in [GW2, Lemma 3.3], we find that for any \(\gamma \in (0, 1/2)\) there exists \(\Gamma = \Gamma(\gamma) > 0\) independent of \(n\) such that for \(n\) sufficiently large,

\[W_{\tau,n}(x) \leq \gamma \quad \forall x \in B_{\tau-\varepsilon_n\Gamma}.\]  

**Claim 1:** For all large \(n\), \(W_{\tau,n}(0) = \min_{B_{\tau}} W_{\tau,n}\) and \(W_{\tau,n}'(r) \geq 0\) for \(r \in (0, \tau)\).

Suppose that \(n\) is large so that (3.13) holds, and let \(r_1 \in (0, \tau)\) be the first point bigger than 0 such that \(W_{\tau,n}(r_1) = 1\). Then

\[\frac{1}{2} |W_{\tau,n}'(r_1)|^2 + \int_0^{r_1} \frac{N-1}{s} |W_{\tau,n}'(s)|^2 ds + \varepsilon_n^{-2} \int_{W_{\tau,n}(0)}^1 f_{a_1^\tau, a_2^\tau}(s) ds = 0.\]

This implies that \(W_{\tau,n}(0) \leq \mu\), where \(\mu \in (0, a_1^\tau)\) is the unique number such that \(\int_\mu^1 f_{a_1^\tau, a_2^\tau}(s) ds = 0\). Suppose there exists \(r_1 \in (0, \tau)\) satisfying \(W_{\tau,n}(r_1) = \min_{B_{\tau}} W_{\tau,n} < W_{\tau,n}(0)\). Then a similar argument yields

\[\int_{W_{\tau,n}(0)}^{\min_{B_{\tau}} W_{\tau,n}} f_{a_1^\tau, a_2^\tau}(s) ds \leq 0.\]

This is clearly impossible since \(\min_{B_{\tau}} W_{\tau,n} < W_{\tau,n}(0) \leq \mu\) and \(f_{a_1^\tau, a_2^\tau}(s) < 0\) for \(s \in (0, a_1^\tau)\).

Thus \(\min_{B_{\tau}} W_{\tau,n} = W_{\tau,n}(0)\).

Suppose that there exists \(r_2 \in (0, \tau)\) where \(W_{\tau,n}\) attains a local maximum. Then \(W_{\tau,n}'(r_2) = 0\) and \(W_{\tau,n}''(r_2) < 0\). (Note that \(W_{\tau,n}''(r_2) = 0\) is impossible for otherwise the uniqueness theorem for ODEs implies \(W_{\tau,n}(r) \equiv 0\).) Hence \(f_{a_1^\tau, a_2^\tau}(W_{\tau,n}(r_2)) > 0\), which implies \(W_{\tau,n}(r_2) \in (a_1^\tau, 1)\). Since \(W_{\tau,n}(\tau) = \infty\), we see that there is \(r_3 \in (r_2, \tau)\) where \(W_{\tau,n}\) attains its minimum over \((r_2, \tau)\). Hence \(W_{\tau,n}(r_3) < W_{\tau,n}(r_2) < 1\). We can further
deduce $W_{\tau,n}(r^3_n) \leq \mu$ by arguments similar to those leading to $W_{\tau,n}(0) \leq \mu$ above. Thus, we derive
\[
\int_{W_{\tau,n}(r^3_n)} f_{a_1^\ast, a_2^\ast}(s) ds \leq \int_{\mu} f_{a_1^\ast, a_2^\ast}(s) ds < \int_{\mu} f_{a_1^\ast, a_2^\ast}(s) ds = 0.
\]
On the other hand,
\[
\int_{W_{\tau,n}(r^3_n)} f_{a_1^\ast, a_2^\ast}(s) ds = -\epsilon_n^2 \int_{r^3_n} \frac{N - 1}{s} |W_{\tau,n}'(s)|^2 ds \leq 0.
\]
This contradiction proves Claim 1.

Introducing
\[
X = \epsilon_n^{-1}(\tau - r), \quad \tilde{W}_{\tau,n}(X) = W_{\tau,n}(r),
\]
we easily see from the regularity of $-\Delta$ that
\[
\tilde{W}_{\tau,n} \to z_\tau \text{ in } C^1_{loc}(0, \infty) \text{ as } n \to \infty,
\]
where $(z_\tau)'(t) \leq 0$ and $z_\tau$ is a solution of the problem
\[
-z'' = f_{a_1^\ast, a_2^\ast}(z) \text{ in } (0, \infty), \quad z(0) = \infty, \quad z(\infty) = z^0_\tau.
\]
From (3.13) and $(z_\tau)'(t) \leq 0$ we find that $0 \leq z^0_\tau \leq \gamma$. Thus $(z_\tau)''(t) \to -f_{a_1^\ast, a_2^\ast}(z^0_\tau)$ as $t \to \infty$, and hence $f(z^0_\tau) = 0$. This implies that $z^0_\tau = 0$.

Now for each $n$, we let $B_n \subset \Omega$ be the ball which is tangent to $\partial \Omega$ at $\xi_n$ with center $\zeta_n$ and radius $\tau$. Since $\partial \Omega$ is $C^2$, this is possible for small $\tau$. Due to $\xi_n \to \xi$ as $n \to \infty$, we see that for $n$ sufficiently large, $B_n \subset \subset B_{8\tau}(\xi) \cap \Omega$. Since $w_n$ is a minimal positive solution to (3.6), we claim that the restriction of $w_n$ to $B_n$ is a minimal positive solution to the problem
\[
-\epsilon_n^2 \Delta w = f(x, w) \text{ in } B_n, \quad w = w_n \text{ on } \partial B_n.
\]
Indeed, if the minimal positive solution $\tilde{w}_n$ of (3.16) (whose existence follows from a standard sub- and supersolution argument) is not $w_n$, then we easily see that $\tilde{w}_n$ extended by $w_n$ in $\Omega \setminus B_n$ is a supersolution to (3.6). Since 0 is a subsolution to (3.6), we see that it has a minimal positive solution in the order interval $[0, w_n]$ and it is different from $w_n$. This contradicts the minimality of $w_n$. Thus,
\[
w_n(x) \leq \overline{W}_{\tau,n}(x) \quad \forall x \in B_n,
\]
where $\overline{W}_{\tau,n}$ is the minimal (hence radial) positive solution to the problem
\[
-\epsilon_n^2 \Delta W = f_{a_1^\ast, a_2^\ast}(W) \text{ in } B_n, \quad W|_{\partial B_n} = \infty.
\]
Define $\tilde{W}_{\tau,n}(r) = \overline{W}_{\tau,n}(x)$, where $r = |x - \zeta_n|$; we see that $\tilde{W}_{\tau,n}$ is a minimal positive solution to (3.12). Thus, $\tilde{W}_{\tau,n} = W_{\tau,n}$. 

Since $d(x_n, \partial \Omega) \to 0$ as $n \to \infty$, by our choice of $\xi_n$ and $B_n$, we see that $x_n \in B_n$ for $n$ large. If $\epsilon_n^{-1}d(x_n, \partial \Omega) = \epsilon_n^{-1}|x_n - \xi_n| = X_n \to \infty$, then we can use (3.14), $z'(t) \leq 0$ and $z_0' = 0$ to deduce

$$w_n(x_n) \leq \tilde{W}_{\tau,n}(x_n) = \tilde{W}_{\tau,n}(X_n) \to 0.$$ Hence $w_n(x_n) \to 0$ as $n \to \infty$. This implies that, if Case (i) occurs, we can derive a contradiction by arguments similar to those used to rule out the case (3.10). Thus Case (i) leads to a contradiction.

We now consider Case (ii). Making the transformations $X^n = \epsilon_n^{-1}(x - \xi_n)$ and $\tilde{w}_n^*(X^n) = w_n(x)$, we can show by a standard compactness argument (such as in [DG2, GW2]) that, subject to a subsequence and a suitable translation and rotation of the coordinate axes,

$$\tilde{w}_n^* \to z_\infty \text{ in } C^{1\text{loc}}(T_1) \text{ as } n \to \infty,$$

where $z_\infty$ is a positive solution to the problem

$$-\Delta z = z(z - a_0)(1 - z) \text{ in } T_1, \quad z|_{\partial T_1} = \infty \quad (3.20)$$

with $a_0 = a(\xi)$ and $T_1 = \{x = (x_1, \ldots, x_n) : x_1 > 0\}$. Here to see that $z|_{\partial T_1} = \infty$ we can use a comparison argument as follows: For each large $n$ we choose an annulus $A_n \supset \Omega$ whose small sphere touches $\partial \Omega$ at $\xi_n$ and we require all the $A_n$ to be of the same size. Let $f(t) = \min_{x \in \Omega} f(x, t)$ and let $v_n$ be the minimal solution of the problem

$$-\epsilon_n^2 \Delta v = f(v) \text{ in } A_n, \quad v|_{\partial A_n} = n.$$ Then it is easy to show that $w_n \geq v_n$ in $\Omega$. The minimality of $v_n$ implies that it is radially symmetric and hence it is easy to see that if we blow-up $v_n$ by the same change of variables as we did for $w_n$, then $v_n$ has a limit $v_\infty$ that satisfies

$$-\Delta v_\infty = f(v_\infty) \text{ in } T_1.$$ Since $v_n$ is radially symmetric, it is not difficult to see that $v_\infty|_{\partial T_1} = \infty$. On the other hand, it follows from $w_n \geq v_n$ that $z_\infty \geq v_\infty$ and hence we must have $z_\infty|_{\partial T_1} = \infty$.

We want to show that $z_\infty(x_1, \ldots, x_N) = z_\infty(x_1)$ in $T_1$ and $z_\infty(t)$ is a positive solution of the problem

$$-z'' = f_{a_0}(z) \text{ in } (0, \infty), \quad z(0) = \infty, \quad z(\infty) = 0. \quad (3.21)$$

To this end, we need some preparations.

**Claim 2:** Any positive solution $z(t)$ of

$$-z'' = f_{a_0}(z) \text{ in } (0, \infty), \quad z(0) = \infty \quad (3.22)$$ satisfies $z'(t) < 0$ for $t > 0$.

Since $z(0) = \infty$, we can find a sequence $t_n \to 0$ such that $1 < z(t_n) \to \infty$ and $z'(t_n) < 0$. If we can show that $z'(t) < 0$ for $t > t_n$ for every $n$, then clearly $z'(t) < 0$ for all $t > 0$. 


The first integral gives, for any $t_2 > t_1 > 0$,

$$z'(t_2)^2/2 - z'(t_1)^2/2 = \int_{z(t_2)}^{z(t_1)} f_{a_0}(s)ds. \quad (3.23)$$

Let $t_0$ stand for any fixed $t_n$ and define $T:= \sup\{\beta > t_0 : z'(t) < 0 \ \forall t \in [t_0, \beta]\}$. It suffices to show that $T = \infty$. Otherwise, $t_0 < T < \infty$ and $z'(T) = 0$, $z''(T) > 0$. (Note that $z''(T) = 0$ is impossible for otherwise the uniqueness theorem for ODEs implies $z \equiv 0$.) We can then define $T^*: = \sup\{\beta > T : z'(t) > 0 \ \forall t \in [T, \beta]\}$. We divide our discussion below into two cases:

(a) $T^* = \infty$, (b) $T < T^* < \infty$.

In case (a), denote $\alpha_\infty = \lim_{t \to \infty} z(t)$. If $\alpha_\infty = \infty$, then in (3.23) taking $t_1 = T$ and $t_2 = t > T$, we obtain

$$z'(t)^2/2 = \int_{z(t)}^{z(T)} f_{a_0}(s)ds = H(z(t)),$$

where $H(u) = \int_u^{z(T)} f_{a_0}(s)ds$. Since $z''(T) > 0$, we have $f_{a_0}(z(T)) < 0$ and hence $z(T) < a_0$ or $z(T) > 1$. If $z(T) < a_0$, we choose $t_\ast \in (t_0, T)$ such that $z(t_\ast) = 1$ and take $t_2 = T$, $t_1 = t_\ast$ in (3.23) to obtain

$$\int_{z(T)}^{1} f_{a_0}(s)ds = -z'(t_\ast)^2/2 < 0.$$

This implies that $z(T) < \mu$, where $\mu \in (0, a_0)$ is uniquely determined by $\int_{\mu}^{1} f_{a_0}(s)ds = 0$. It is now clear that whether $z(T) < \mu$ or $z(T) > 1$, $H(u)$ satisfies $H(z(T)) = 0$, $H'(z(T)) \neq 0$ and $H(u) > 0$ for $u > z(T)$. Thus,

$$z'(t) = [2H(z(t))]^{1/2}, \ \forall t > T,$$

and

$$\int_{z(T)}^{z(t)} [2H(z)]^{-1/2}dz = t - T.$$

Letting $t \to \infty$ we obtain

$$\int_{z(T)}^{\infty} [2H(z)]^{-1/2}dz = \infty.$$

On the other hand, by the properties of $f_{a_0}(t)$ and $H'(z(T)) \neq 0$, it is easily seen that $\int_{z(T)}^{\infty} [2H(z)]^{-1/2}dz < \infty$. This contradiction shows that $\alpha_\infty = \infty$ cannot happen.

If $\alpha_\infty < \infty$, we have $\lim_{t \to \infty} z''(t) = -f_{a_0}(\alpha_\infty)$ and hence we necessarily have $f_{a_0}(\alpha_\infty) = 0$. Since $\alpha_\infty > z(T)$, we must have $\alpha_\infty = a_0$ or $\alpha_\infty = 1$.

If $\alpha_\infty = a_0$, then in (3.23) we take $t_1 = T$ and let $t_2 \to \infty$ to obtain

$$0 = \int_{z(T)}^{a_0} f_{a_0}(s)ds.$$

But this is impossible since $f_{a_0}(s) < 0$ in $(z(T), a_0)$. 

If $\alpha_\infty = 1$, we similarly obtain

$$0 = \int_{z(T)}^{1} f_{a_0}(s) ds.$$  

This is also impossible since $z(T) < 1$ implies $z(T) < \mu$ as observed before. Thus case (a) always leads to a contradiction.

Now let us consider case (b). Since $z''(T^*) < 0$ and $z''(T) > 0$, we have $f_{a_0}(z(T^*)) > 0$ and $f_{a_0}(z(T)) < 0$. It follows that $z(T^*) \in (a_0, 1)$ and $z(T) \in (0, a_0)$, and as observed before this implies $z(T) < \mu$. Taking $t_1 = T$ and $t_2 = T^*$ in (3.23), we obtain

$$\int_{z(T^*)}^{z(T)} f_{a_0}(s) ds = 0.$$  

But on the other hand we have

$$\int_{z(T)}^{z(T^*)} f_{a_0}(s) ds < \int_{z(T)}^{1} f_{a_0}(s) ds < 0.$$  

Thus case (b) also leads to a contradiction. This proves Claim 2.

**Claim 3:** Problem (3.22) has exactly two positive solutions $z_0$ and $z_1$, and as $t \to \infty$, $z_0$ decreases to 0, $z_1$ decreases to 1.

Let $z(t)$ be a positive solution of (3.22). By Claim 2, we know that $\alpha_\infty := \lim_{t \to \infty} z(t)$ always exists. Clearly $\alpha_\infty \in [0, \infty)$ and $\lim_{t \to \infty} z''(t) = -f_{a_0}(\alpha_\infty)$. It follows that $f_{a_0}(\alpha_\infty) = 0$ and hence $\alpha_\infty \in \{0, a_0, a\}$.

If $\alpha_\infty = a_0$, in (3.23) we take $t_1 = t_*$, where $t_* > 0$ is such that $z(t_*) = 1$, and let $t_2 \to \infty$; it results

$$\int_{a_0}^{1} f_{a_0}(s) ds = -z'(t_*)^2/2 < 0,$$

which is impossible since $f_{a_0}(s) > 0$ in $(a_0, 1)$.

If $\alpha_\infty = 0$, we take $t_1 = t$ and let $t_2 \to \infty$ in (3.23) to obtain

$$-z'(t)^2/2 = \int_{z(t)}^{z(t)} f_{a_0}(s) ds =: F(z(t)).$$

Clearly $F(0) = 0$, $F'(0) = 0$, and since $a_0 > 1/2$, $F(z) < 0$ for $z > 0$. Thus

$$z'(t) = -\left[-2F(z(t))\right]^{1/2},$$

and

$$\int_{z(t)}^{\infty} \left[-2F(z)\right]^{-1/2} dz = t. \quad (3.24)$$

Conversely, one easily checks that (3.24) determines a unique $z(t)$, which satisfies (3.22) and is decreasing to 0 as $t \to \infty$. Thus the case $\alpha_\infty = 0$ yields a unique solution $z_0$ as described in Claim 3.
If \( \alpha_\infty = 1 \), we take \( t_1 = t \) and let \( t_2 \to \infty \) in (3.23) and obtain

\[
-z'(t)^2/2 = \int_1^{z(t)} f_{a_0}(s)ds =: G(z(t)).
\]

Clearly \( G(1) = 0, G'(1) = 0 \) and \( G(z) < 0 \) for \( z > 1 \). Therefore

\[
z'(t) = -\left[ -2G(z(t)) \right]^{1/2},
\]

and

\[
\int_1^{t} \left[ -2G(z) \right]^{-1/2}dz = t.
\]

Conversely, one easily checks that (3.25) determines a unique \( z(t) \), which satisfies (3.22) and is decreasing to 1 as \( t \to \infty \). Thus the case \( \alpha_\infty = 1 \) yields a unique solution \( z_1 \) as described in Claim 3. This proves Claim 3.

**Claim 4:** \( z_\infty(x) = z_0(x_1) \), where \( z_0 \) is as given in Claim 3.

We firstly show that \( z_\infty(x) \geq z_0(x_1) \) in \( T_1 \). As in [Do], it is easy to prove that (3.20) has a minimal positive solution. Indeed, denote \( T^n = T_1 \cap B_n(0) \) and let \( z_n \) be the minimal positive solution of the problem

\[
-\Delta z = z(z - a_0)(1 - z) \quad \text{in} \quad T^n, \quad z|_{\partial T^n} = \phi_n,
\]

where \( \phi_n \) is smooth and nonnegative in \( \mathbb{R}^N \), \( \phi_n = n \) in \( B_{n-1}(0) \), \( \phi_n = 0 \) in \( \mathbb{R}^N \setminus B_n(0) \), and \( \phi_n \geq \phi_{n-1} \) in \( \mathbb{R}^N \). Then it is easily seen that \( z_* := \lim z_n \) is a minimal positive solution of (3.20). The minimality of \( z_* \) forces it to be as symmetric as (3.20) allows, and hence \( z_* = z_n(x_1) \). Therefore \( z_* \) solves (3.22). By Claim 3 we have \( z_* = z_1 \) or \( z_0 \). Since \( F(z) < G(z) \) for \( z > 1 \), from (3.24) and (3.25) we see that \( z_1 > z_0 \). Therefore \( z_* \geq z_0 \) and \( z_\infty \geq z_* \geq z_0 \).

Next we show that \( z_\infty \leq z_0 \). From (3.17) we obtain

\[
\tilde{w}_n^*(X^n) \leq \tilde{W}_{\gamma_n}(X_n) \quad \forall x \in \tilde{B}_n,
\]

where

\[
\tilde{B}_n = \{ X^n : \epsilon_n X^n + \xi_n \in B_n \}, \quad X_n = \epsilon_n^{-1}(\tau - |x - \xi_n|).
\]

By (3.14) and (3.19), we deduce that

\[
z_\infty(x) \leq z_\tau(x_1) \quad \text{in} \quad T_1.
\]

Since \( z_0^0 = 0 \) and the function \( f_{a_1^\tau, a_2^\tau} \) behaves similarly to \( f_{a_0} \), as in Claim 3 we see that \( z_\tau \) is the unique solution satisfying (3.15) and it is uniquely determined by (3.24) with \( F(z) \) replaced by \( F_\tau(z) := \int_0^1 f_{a_1^\tau, a_2^\tau}(s)ds \). Since \( f_{a_1^\tau, a_2^\tau}(z) \to f_{a_0}(z) \) as \( \tau \to 0 \) uniformly on compact sets of \( [0, \infty) \), and the convergence of the integral \( \int_0^\infty \left[ -2F(z) \right]^{-1/2}dz \) at \( \infty \) is uniform in \( \tau \) for all small \( \tau \), we see from the correspondent of (3.24) for \( z_* \) that

\[
z_\tau \to z_0 \quad \text{locally uniformly in} \quad (0, \infty) \quad \text{as} \quad \tau \to 0.
\]
Therefore it follows from (3.26) that \( z_\infty \leq z_0 \). This completes the proof of Claim 4.

Denote \( \tilde{k}_n(X^n) = k_n(x) \). We see from (3.9) that \( \tilde{k}_n \) satisfies
\[
-\Delta \tilde{k}_n - f_t(\epsilon_n X^n + \xi_n, \tilde{w}_n^\star) \tilde{k}_n = \lambda_n \tilde{k}_n \quad \text{in } \tilde{\Omega}_n, \quad \tilde{k}_n|_{\partial \tilde{\Omega}_n} = 0,
\]
where \( \tilde{\Omega}_n := \{ \epsilon_n^{-1}(x - \xi_n) : x \in \Omega \} \). A standard compactness argument then implies that, subject to a subsequence,
\[
\tilde{k}_n \to \tilde{k} \quad \text{in } C^1_{loc}(T_1) \text{ as } n \to \infty,
\]
where \( \tilde{k} \geq 0 \) satisfies \( \| \tilde{k} \|_{\infty} \leq 1 \) and
\[
\Delta \tilde{k} + f'_{a_0}(z_\infty(x_1)) \tilde{k} \in [0, C_0] \quad \text{in } T_1.
\]
Moreover, if we denote \( \eta_n = \epsilon_n^{-1}(x_0 - \xi_n) \) and \( \eta = \lim_{n \to \infty} \eta_n \), we see that \( 0 \leq |\eta| \leq C \) and \( \tilde{k}(\eta) = 1 \).

We want to show that such \( \tilde{k} \) cannot exist and hence reach a contradiction. We need another preparation.

**Claim 5:** There exists a solution \( q(t) \) of the equation
\[
-z'' = f'_{a_0}(z_\infty) z \quad \text{in } (0, \infty),
\]
which is positive on \((0, \infty)\) and \( \lim_{t \to 0} q(t) = \infty, \lim_{t \to \infty} q(t) = \infty. \)

We easily see that \(-z_\infty'(t)\) is a positive solution of (3.29) in \((0, \infty)\) and
\[
\lim_{t \to 0} [-z_\infty'(t)] = \infty, \quad \lim_{t \to \infty} [-z_\infty'(t)] = 0.
\]
Choose \( \sigma^* > 0 \) small such that \( z_\infty(t) > 1 \) for all \( t \in (0, \sigma^*), \) and let \( z_{\sigma^*} \) denote the solution of (3.29) in \((0, \infty)\) satisfying \( z_{\sigma^*}(\sigma^*) = 1 \) and \( z_{\sigma^*}'(\sigma^*) = 0. \) Since \( z_{\sigma^*}''(\sigma^*) = -f'_{a_0}(z_\infty(\sigma^*))z_{\sigma^*}'(\sigma^*) \) is decreasing in a small interval \((\sigma, \sigma^*) \subset (0, \sigma^*). \) If \((\sigma, \sigma^*) \) is the largest possible such interval, we claim that \( \sigma = 0. \) Otherwise, we must have \( \sigma \in (0, \sigma^*) \) and
\[
z_{\sigma^*}(\sigma) > 0, \quad z_{\sigma^*}'(\sigma) = 0, \quad z_{\sigma^*}''(\sigma) \leq 0.
\]
But this is in contradiction to \( z_{\sigma^*}''(\sigma) = -f'_{a_0}(z_\infty(\sigma))z_{\sigma^*}'(\sigma) > 0. \)

We next show that \( z_{\sigma^*}(t) > 0 \) for \( t > \sigma^*, \) and \( z_{\sigma^*}(t) \to \infty \) as \( t \to \infty. \) Since
\[
\left( (z_\infty)'z_{\sigma^*}' - z_{\sigma^*}(z_\infty)'' \right)' \equiv 0 \quad \text{in } (0, \infty),
\]
we obtain:
\[
(z_\infty)'z_{\sigma^*}' - z_{\sigma^*}(z_\infty)'' \equiv -(z_\infty)''(\sigma^*) = j(z_\infty(\sigma^*)) < 0.
\]
Hence \( (z_{\sigma^*}'(t) > 0 \) and \( z_{\sigma^*}'(t) > z_\infty'(t)/z_{\sigma^*}'(\sigma^*) > 0 \) for \( t > \sigma^* \).

Since \( z_\infty'(t) \to 0 \) and \( z_\infty''(t) \to 0 \) as \( t \to \infty, \) (3.30) infers that \( z_{\sigma^*}(t) \) can not stay bounded and monotone for all large \( t. \) This implies that \( z_{\sigma^*} \to \infty \) as \( t \to \infty \) for otherwise
\( z_{\sigma^*} \) has a sequence of local minimum points \( \{t_n\} \) such that \( t_n \to \infty \) and \( z_{\sigma^*}(t_n) \) stays bounded, and it follows that
\[
z'_{\infty}(t_n)z'_{\sigma^*}(t_n) - z_{\sigma^*}(t_n)z''_{\infty}(t_n) = -z_{\sigma^*}(t_n)z''_{\infty}(t_n) \to 0,
\]
contradicting (3.30). Now clearly,
\[
q(t) := z_{\sigma^*}(t) - z'_{\infty}(t)
\]
has the required properties. This proves Claim 5.

We are now ready to show that \( \tilde{k} \) does not exist. Define
\[
r(x) = \tilde{k}(x)/q(x_1)
\]
then \( r(x) \to 0 \) as \( x_1 \to \infty \) and as \( x_1 \to 0 \), uniformly in \( (x_2, \ldots, x_N) \). Therefore, we can find
\[
x^n = (x^n_1, x^n_2, \ldots, x^n_N)
\]
with \( \{x^n_1\} \) bounded away from 0 and \( \infty \) such that
\[
0 < \sup \{r(x) : 0 < x_1 < \infty\} = \lim_{n \to \infty} r(x^n).
\]
We may assume that \( x^n_1 \to x_1^0 > 0 \).

Let \( \tilde{k}_n(x) = \tilde{k}(x - (0, x^n_2, \ldots, x^n_N)) \). We find that \( \tilde{k}_n \) satisfies (3.28), and by a standard regularity and compactness consideration, by passing to a subsequence, \( \tilde{k}_n \to \tilde{k}^* \) uniformly on compact subsets of \( T_1 \), and \( \tilde{k}^* \) satisfies (3.28) with \( 0 < \tilde{k}^* \leq 1 \) on \( T_1 \), and furthermore,
\[
r^*(x) := \tilde{k}^*(x)/q(x_1)
\]
has the property
\[
0 < \sup \{r^*(x) : 0 < x_1 < \infty\} = r^*(x_1^0, 0, \ldots, 0),
\]
i.e. \( r^*(x) \) achieves its positive maximum on \( T_1 \) at the interior point \( (x_1^0, 0, \ldots, 0) \). This contradicts the maximum principle, since a simple calculation shows that \( r^* \) satisfies
\[
\Delta r + 2q^{-2}\nabla q \cdot \nabla r \geq 0, \quad r|_{\partial T_1} = 0.
\]
This completes the proof of Lemma 3.1.

We now explain Remark 1.6. The proof of Theorem 1.3 in the next subsection only uses the facts that \( w^*_\epsilon \) is a solution to \( (P_\epsilon) \), satisfies (3.5) and converges to 0 locally uniformly in \( \Omega \) as \( \epsilon \to 0 \). Hence Theorem 1.3 is still valid if \( w^*_\epsilon \) is replaced by any “local” minimizer solution \( v_\epsilon \) of \( (P_\epsilon) \) that converges to 0 locally uniformly in \( \Omega \), provided that (3.5) holds for such \( v_\epsilon \).

To show that (3.5) holds if \( w^*_\epsilon \) is replaced by any \( v_\epsilon \) obtained from Theorem 1.2 by taking \( \Omega_1 = \emptyset \) and \( \Omega_2 \) such that \( a(x) > 1/2 \) on \( \overline{\Omega} \setminus \Omega_2 \), we follow the proof of Lemma 3.1 with the following changes:
(i) Replace $w^n$ by $w^\beta_n$, where $w^\beta_n \to v_\epsilon$ as $n \to \infty$ is given in the proof of Theorem 2.4.

(ii) Replace the minimal positive solution $W_{\tau,n}$ of (3.12) by a global minimizer solution $w_{\tau,n}$ of

$$-\epsilon^2 \Delta w = f_{a_{\tau_1}a_{\tau_2}}(w) \text{ in } B_\tau, \quad w|_{\partial B_\tau} = \beta_n.$$ 

Since $g(x,u) \leq f_{a_{\tau_1}a_{\tau_2}}(u)$ for $x \in B_\tau$ and $u \in \mathbb{R}^1$, we can use Lemma 2.2 to conclude

$$w^\beta_n(x) \leq w_{\tau,n}(r) \text{ if } x \in B_n \text{ and } r = |x - \zeta_n|,$$

where $B_n$ is as in (3.16). The rest follows easily.

### 3.3. Construction of solutions with interior spikes

In this subsection, we use the reduction method to prove Theorem 1.3. We will combine the “cut-off” technique used in [DuY] with the reduction arguments in [DaY1]. Many technical details here are the same as those in section 3 of [DG3]. Therefore, we will only give the main steps.

For any small $\delta > 0$, we use $B_\delta(x_i)$ to denote the open ball of radius $\delta$ and center $x_i$, where $x_i, i = 1, \ldots, k$, are the strict local minimum points of $a(x)$ in $A$ as assumed in Theorem 1.3; if $x_i, i = 1, \ldots, k$, are strict local maximum points of $a(x)$ in $A$, the proof is parallel.

Define

$$D_\delta = \{Z = (z^1, z^2, \ldots, z^k) : z^i \in B_\delta(x_i), \ i = 1, 2, \ldots, k\}.$$ 

We fix $\theta > 0$ small enough so that

$$|x_i - x_j| > 8\theta, \ d(x_i, \partial\Omega) > 8\theta \quad \text{for all } 1 \leq i, j \leq k, i \neq j,$$

and

$$a(x) > a(x_i) \text{ for } x \in B_{4\theta}(x_i) \setminus \{x_i\}, i = 1, \ldots, k.$$ 

We then choose $\xi_i = \xi_{i,\theta} \in C^1(\bar{\Omega})$ such that

$$0 \leq \xi_i(x) \leq 1 \text{ in } \Omega, \ \xi_i(x) = 1 \text{ if } |x-x_i| \leq \theta, \ \xi_i(x) = 0 \text{ if } |x-x_i| \geq 2\theta.$$ 

We let

$$\tilde{U}_{a_{i,\epsilon},z^i}(y) := \xi_i(y)U_{a_{i,\epsilon},z^i}(y) = \xi_i(y)U_{a_{i,\epsilon}}(y - z^i_\epsilon).$$

For each $Z \in D_\delta$, define

$$E_{\epsilon,Z} = \{\omega \in H : \langle \omega, \frac{\partial U_{a_{i,\epsilon},z^i}}{\partial z^i_\epsilon} \rangle_\epsilon = 0, \ 1 \leq i \leq k, \ 1 \leq h \leq N\},$$

where $z^i_h$ denotes the $h$-th component of $z^i = (z^i_1, \ldots, z^i_N) \in \mathbb{R}^N$ and

$$\langle u, \omega \rangle_\epsilon = \int_\Omega (\epsilon^2 Du \cdot D\omega + u\omega)dx.$$ 

We observe that $E_{\epsilon,Z}$ is a closed subspace of $H$. 

We will construct a solution for (3.2), which has the form
\[ v = \sum_{i=1}^{k} U_{a_i, z_i} + \omega, \]
where \( \omega \in E_{\epsilon, Z} \) with \( Z = (z_1^\epsilon, \ldots, z_k^\epsilon) \), and
\[ \|\omega\|_\epsilon = \left( \int_{\Omega} \epsilon^2 |D\omega|^2 - f_t(x, w^*_\epsilon)\omega^2 \right)^{1/2} = o(\epsilon^{N/2}). \]

Define \( \tilde{I} = \tilde{I}_\epsilon \) by
\[ \tilde{I}(Z, \omega) = I\left( \sum_{i=1}^{k} \tilde{U}_{a_i, z_i} + \omega, Z = (z_1^\epsilon, \ldots, z_k^\epsilon) \in D_\delta, \omega \in E_{\epsilon, Z} \right). \]

**Proposition 3.5.** There exist \( \epsilon_1 > 0 \) and \( 0 < \delta_0 < \theta \) such that for any \( \epsilon \in (0, \epsilon_1] \) and \( 0 < \delta < \delta_0 \) there is a \( C^1 \)-map \( Z \to \omega(Z) = \omega_\epsilon(Z) \in H \) defined for \( Z \in D_\delta \), such that \( \omega(Z) \in E_{\epsilon, Z} \),
\[ \frac{d}{d\rho} \tilde{I}(Z, \omega(Z) + \rho\psi)|_{\rho=0} = 0, \quad \forall \psi \in E_{\epsilon, Z}. \] (3.31)

Moreover, we have the following estimate:
\[ \|\omega_\epsilon(Z)\|_\epsilon = O\left( \sum_{i=1}^{k} |a(z_i) - a_i| + o_\epsilon(1) \right) \epsilon^{N/2}, \]
where \( o_\epsilon(1) \to 0 \) as \( \epsilon \to 0 \).

Clearly (3.31) is equivalent to that \( \omega(Z) \) is a critical point of \( \tilde{I}(Z, \omega) \) in \( E_{\epsilon, Z} \) for fixed \( Z \in D_\delta \). For convenience of notation, we write
\[ W_{\epsilon, Z} = \sum_{i=1}^{k} \tilde{U}_{a_i, z_i}. \]

As in [DG3], we expand \( \tilde{I}(Z, \omega) \) near \( \omega = 0 \) as follows:
\[ \tilde{I}(Z, \omega) = \tilde{I}(Z, 0) + \langle K_\epsilon(Z), \omega \rangle + \frac{1}{2} \langle Q_\epsilon(Z)\omega, \omega \rangle + R_{\epsilon, Z}(\omega), \]
where
\[ \langle K_\epsilon(Z), \omega \rangle = \int_{\Omega} \left( \epsilon^2 D W_{\epsilon, Z} \cdot D\omega - f_t(y, w^*_\epsilon) W_{\epsilon, Z}\omega - g(y, W_{\epsilon, Z})\omega \right) dy, \] (3.32)
\[ \langle Q_\epsilon(Z)\omega, \psi \rangle = \int_{\Omega} \left( \epsilon^2 D\omega \cdot D\psi - f_t(y, w^*_\epsilon)\omega\psi - g_t(y, W_{\epsilon, Z})\omega\psi \right) dy, \] (3.33)
\[ R_{\epsilon, Z}(\omega) = - \int_{\Omega} \left( G(y, W_{\epsilon, Z} + \omega) - G(y, W_{\epsilon, Z}) - g(y, W_{\epsilon, Z})\omega - (1/2)g_t(y, W_{\epsilon, Z})\omega^2 \right) dy. \] (3.34)

Here \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( H \) induced by its norm, and \( K_\epsilon(Z) \in H \) is uniquely determined by the right hand side of (3.32); similarly \( Q_\epsilon : H \to H \) is uniquely determined by the right hand side of (3.33).
Clearly
\[
\frac{d}{d\rho} \tilde{f}(Z, \omega + \rho \psi)|_{\rho=0} = \langle K(\rho), Q(\rho) \omega + R(\rho, \omega), \psi \rangle, \quad \forall \omega, \psi \in H.
\] (3.35)

In order to prove Proposition 3.5, we need several lemmas.

**Lemma 3.6.** There exist \( \epsilon_2 > 0 \) and \( C > 0 \) such that for \( \epsilon \in (0, \epsilon_2] \) and \( Z \in D_{\theta/2}, \)
\[
\| R_{\epsilon,Z}(\omega) \| \leq C \epsilon^{-N \alpha/2} \| \omega \|^{1+\alpha}, \quad \| R_{\epsilon,Z}(\omega) \| \leq C \epsilon^{-N \alpha/2} \| \omega \|^{\alpha}, \quad \forall \omega \in H,
\]
where \( \alpha \in (0,1) \) is given in the definition of \( g(t) \).

**Proof.** The proof for the above estimates follows exactly the proof of Lemma 2.7 in [DuY]; the only difference is that we should replace \( |f''(u^*)| \leq C[-f'(u^*)]^{1/2} \) by
\[
|f_{it}(x, w^*_t)| \leq C[M_1 + 1 - f_i(x, w^*_t)]^{1/2},
\]
and make the corresponding changes. We need Lemma 3.1.

**Lemma 3.7.** There exist \( \epsilon_3 \in (0, \epsilon_2], \delta_1 \in (0, \theta) \) and \( c_1 > 0 \), such that if \( \epsilon \in (0, \epsilon_3] \), then for any \( Z \in D_{\delta_1}, \)
\[
\| P_{\epsilon,Z}Q(\epsilon)(Z) \omega \| \leq c_1 \| \omega \|, \quad \forall \omega \in E_{\epsilon,Z},
\] (3.36)
where \( P_{\epsilon,Z} \) denotes the orthogonal projection from the Hilbert space \( H \) to its closed subspace \( E_{\epsilon,Z} \).

**Proof.** This is similar to the proof of Lemma 3.7 in [DG3], but we will need Lemma 3.1 here. We first notice that, since \( \tilde{U}_{a_i} = (0, 1) \) for \( i = 1, \ldots, k \), and since \( \xi_i \) and \( \xi_j \) have disjoint supports when \( i \neq j \), we have \( W_{\epsilon,Z}(x) \in (0, 1) \). Thus,
\[
f_i(x, w^*_t) + g_t(x, W_{\epsilon,Z}) = h_t(x, W_{\epsilon,Z}) = f_i(x, w^*_t + W_{\epsilon,Z}),
\]
and
\[
\langle Q(\epsilon)(Z) \omega, \psi \rangle = \int_{\Omega} \left( \epsilon^2 D \omega \cdot D \psi - f_i(y, w^*_t + W_{\epsilon,Z}) \omega \psi \right) dy.
\]

We now use a contradiction argument to prove (3.36). Suppose that there are \( \epsilon_j \to 0, \delta_j \to 0 \) and \( Z_j = (z^{i,j}, \ldots, z^{j,k}) \in D_{\delta_j}, \omega_j \in E_{\epsilon_j,Z_j} \) such that
\[
\| P_{\epsilon_j,Z_j}Q_{\epsilon_j}(Z_j) \omega_j \| = o_j(1) \| \omega \|, \quad \| \omega \| = \epsilon_j^{N/2},
\] (3.37)
where \( o_j(1) \to 0 \) as \( j \to \infty \). We may also assume that \( \| \omega \| = \epsilon_j^{N/2} \).

For each \( i \), define \( \Omega_{i,j} = \{ y : \epsilon_j y + z^{i,j} \in \Omega \}, \) \( \mathring{\omega}_{i,j}(y) := \omega_j(\epsilon_j y + z^{i,j}) \) for \( y \in \Omega_{i,j}, \) and extend \( \mathring{\omega}_{i,j} \) to be zero for \( y \notin \Omega_{i,j} \). Then \( \{ \mathring{\omega}_{i,j} \} \) is bounded in \( H^1(\mathbb{R}^N) \). Thus by passing to a subsequence we may assume that
\[
\mathring{\omega}_{i,j} \to \mathring{\omega}_i \in H^1(\mathbb{R}^N) \text{ weakly as } j \to \infty.
\]
Exactly as in the proof of Lemma 3.7 in [DG3], we can use (3.37) to deduce that \( \tilde{w}_i = 0 \) for \( i = 1, \ldots, k \), that is, \( \tilde{w}_{i,j} \) converges to 0 weakly in \( H^1(\mathbb{R}^N) \) as \( j \to \infty \). By Sobolev imbedding theorems on bounded domains, we deduce from this fact that for any fixed \( R > 0 \),

\[
\int_{B_R(0)} \tilde{w}_{i,j}^2 dy = o_j(1), \quad i = 1, 2, \ldots, k.
\]

This will be used below to derive a contradiction. We start with, due to (3.37),

\[
o_j(\|\omega_j\|^2) = o_j(\|\omega_j\|^2) - \int_{\Omega} f_t(y, w_{\epsilon_j}^*) + W_{\epsilon_j, z_j} \omega_j^2 dy
\]

\[
= \int_{\Omega} \epsilon_j^2 |D\omega_j|^2 dy - \int_{\Omega} f_t(y, w_{\epsilon_j}^*) \omega_j^2 dy
\]

\[
= \int_{\Omega} \epsilon_j^2 |D\omega_j|^2 dy - \int_{\Omega} f_t(y, w_{\epsilon_j}^*) \omega_j^2 dy + \int_{\Omega} [f_t(y, w_{\epsilon_j}^*) - f_t(y, w_{\epsilon_j}^*) + W_{\epsilon_j, z_j}] \omega_j^2 dy
\]

\[
= \|\omega_j\|^2 - \int_{\Omega} f_t(y, \eta_j) W_{\epsilon_j, z_j} \omega_j^2 dy
\]

where \( \eta_j \in (w_{\epsilon_j}^*, w_{\epsilon_j}^* + W_{\epsilon_j, z_j}) \). On \( B_{\epsilon_j R}(z_{i,j}) \), \( w_{\epsilon_j}^* \) and \( W_{\epsilon_j, z_j} \) have \( L^\infty \) bounds independent of \( j \), and hence \( f_t(x, \eta_j) \) has \( L^\infty \) bounds independent of \( j \). Thus,

\[
\left| \int_{\bigcup_{i=1}^k B_{\epsilon_j R}(z_{i,j})} f_{tt}(x, \eta_j) W_{\epsilon_j, z_j} \omega_j^2 dx \right| \leq C\epsilon_j^N \Sigma_{i=1}^k \int_{B_R(0)} (\tilde{w}_{i,j})^2 dy = o(\epsilon_j^N). \tag{3.38}
\]

For \( j \) large,

\[
\Omega = \{\eta_j(x) \geq 1\} \cup \left(\{\eta_j(x) < 1\} \setminus \bigcup_{i=1}^k B_{\epsilon_j R}(z_{i,j})\right) \cup \left(\bigcup_{i=1}^k B_{\epsilon_j R}(z_{i,j})\right).
\]

Therefore,

\[
- \int_{\Omega} f_t(y, \eta_j) W_{\epsilon_j, z_j} \omega_j^2 dy
\]

\[
= - \left[ \int_{\{\eta_j(x) \geq 1\}} + \int_{\{\eta_j(x) < 1\} \setminus \bigcup_{i=1}^k B_{\epsilon_j R}(z_{i,j})} + \int_{\bigcup_{i=1}^k B_{\epsilon_j R}(z_{i,j})} \right] f_t(y, \eta_j) W_{\epsilon_j, z_j} \omega_j^2 dy
\]

\[
=: I_1 + I_2 + I_3.
\]

Since \( f_t(x, t) < 0 \) for \( t \geq 1 \), we have

\[
I_1 \geq 0.
\]

Since \( |f_t(x, t)| \leq M_2 \) for \( t \in [0, 1] \), and for any \( \tilde{\delta} > 0 \), if we choose \( R > 0 \) sufficiently large

\[
|W_{\epsilon_j, z_j}(x)| \leq \tilde{\delta} \text{ if } x \in \{\eta_j(x) < 1\} \setminus \bigcup_{i=1}^k B_{\epsilon_j R}(z_{i,j})
\]

for all large \( j \), we find

\[
|I_2| \leq M_2 \tilde{\delta} \int \omega_j^2 dy \leq \frac{M_2 \tilde{\delta}}{\kappa^*} \|\omega_j\|^2_{\epsilon_j, j},
\]
by making use of Lemma 3.1. Therefore, for all large \( j \), in view of (3.38),

\[
\begin{align*}
o(\epsilon_j^N) &= o(\|\omega_j\|^2_{\epsilon_j}) \\
&= \|\omega_j\|^2_{\epsilon_j} + I_1 + I_2 + I_3 \\
&\geq \left(1 - \frac{M_2\delta}{K^*}\right)\|\omega_j\|^2_{\epsilon_j} + o(\epsilon_j^N) = \left(1 - \frac{M_2\tilde{\delta}}{K^*}\right)\epsilon_j^N + o(\epsilon_j^N).
\end{align*}
\]

This is a contradiction and the lemma is proved.

\[\square\]

**Lemma 3.8.** There exists \( \epsilon_4 \in (0, \epsilon_3] \) such that for \( \epsilon \in (0, \epsilon_4] \),

\[
\|K_\epsilon(Z)\|_\epsilon = O\left(\Sigma_{i=1}^k |a(z^i) - a_i| + o_\epsilon(1)\right)\epsilon^{N/2},
\]

uniformly for \( Z \in D_{\delta/2} \).

The proof of this lemma is parallel to the proof of Lemma 3.8 in [DG3]. So we omit the details.

**Proof of Proposition 3.5:** By (3.35), we need to solve the following equation for \( \omega \in E_{\epsilon,Z} \) for each given \( Z \in D_\delta \):

\[
P_{\epsilon,Z}K_\epsilon(Z) + P_{\epsilon,Z}Q_\epsilon(Z)\omega + P_{\epsilon,Z}R'_{\epsilon,Z}(\omega) = 0.
\]

This problem is equivalent to the following problem on the whole space \( H = H_\epsilon \):

\[
\begin{cases}
P_{\epsilon,Z}K_\epsilon(Z) + P_{\epsilon,Z}Q_\epsilon(Z)\omega + P_{\epsilon,Z}R'_{\epsilon,Z}(\omega) = 0, \\
(I - P_{\epsilon,Z})\omega = 0,
\end{cases} \quad \omega \in H. \tag{3.39}
\]

By Lemma 3.7, we find that for \( \epsilon \in (0, \epsilon_3] \) and \( Z \in D_{\delta_1} \), the linear operator \( Q'(Z) : H \to H \) given by

\[
Q'(Z)\omega = \left(P_{\epsilon,Z}Q_\epsilon(Z)P_{\epsilon,Z}\omega,(I - P_{\epsilon,Z})\omega\right)
\]

satisfies

\[
\|Q'(Z)\omega\|_\epsilon \geq \min\{1,c_1\}\|\omega\|_\epsilon, \quad \forall \omega \in H.
\]

Therefore, the inverse \( \left(Q'(Z)\right)^{-1} \) exists with norm bounded by \( 1/\min\{1,c_1\} \). We can now rewrite (3.39) as

\[
\omega = G_Z'\omega := \left(Q'(Z)\right)^{-1}\left(-P_{\epsilon,Z}K_\epsilon(Z) - P_{\epsilon,Z}R'_{\epsilon,Z}(\omega),0\right).
\]

By Lemmas 3.6 and 3.8, we can find \( \epsilon_1 \leq \epsilon_3 \) such that for each \( Z \in D_{\delta_0} \) with \( \delta_0 := \min\{\delta_1, \theta/2\} \) and each \( \epsilon \in (0, \epsilon_1] \), \( G_Z' \) maps \( B_\epsilon := \{\omega \in H : \|\omega\|_\epsilon \leq C\|K_\epsilon(Z)\|_\epsilon\} \) to itself and is a contraction mapping:

\[
\|G_Z'\omega_1 - G_Z'\omega_2\|_\epsilon \leq c_2\|\omega_1 - \omega_2\|_\epsilon, \quad \forall \omega_1, \omega_2 \in B_\epsilon,
\]
where \( C > 1 \) and \( c_2 < 1 \) do not depend on \( \epsilon \in (0, \epsilon_1] \) and \( Z \in D_{\delta_0} \). It follows that \( \omega = G_Z^T \omega \) has a unique solution \( \omega_\epsilon(Z) \in B_\epsilon \) for any given \( Z \in D_{\delta_0} \). Moreover, the dependence of \( \omega_\epsilon(Z) \) on \( Z \) is as smooth as \( G_Z^T \) on \( Z \); in particular, it is \( C^1 \). As now

\[
\| \omega_\epsilon(Z) \|_\epsilon \leq C \| K_\epsilon(Z) \|_\epsilon, \quad \forall Z \in D_{\delta_0}, \quad \forall \epsilon \in (0, \epsilon_1],
\]

the required estimate for \( \| \omega_\epsilon(Z) \|_\epsilon \) in Proposition 3.5 follows directly from Lemma 3.8. This completes the proof.

We are now ready to prove Theorem 1.3 by a reduction method based on Proposition 3.5.

**Proof of Theorem 1.3:** Suppose that \( \epsilon \in (0, \epsilon_1], \delta \in (0, \delta_0) \) and \( \omega_\epsilon(Z) \) is given by Proposition 3.5. Let

\[
F_\epsilon(Z) = \tilde{I}(Z, \omega_\epsilon(Z)) = I \left( \sum_{i=1}^{k} \tilde{U}_{a_i, \epsilon, z_i^*} + \omega_\epsilon(Z), \quad Z \in D_\delta.
\]

As indicated in [DaY1], by standard argument in the reduction method, it can be shown that if \( Z_\epsilon \in D_\delta \) is a critical point of \( F_\epsilon \), then \( \sum_{i=1}^{k} \tilde{U}_{a_i, \epsilon, z_i^*} + \omega_\epsilon(Z_\epsilon) \) is a critical point of \( I \), and hence a solution to (3.2).

We will show in the following that \( F_\epsilon \) has a critical point \( Z_\epsilon = (z_1^\epsilon, \ldots, z_k^\epsilon) \) satisfying \( Z_\epsilon \to (x_1, \ldots, x_k) \) in \( \mathbb{R}^{kN} \) as \( \epsilon \to 0 \). By the estimate for \( \| \omega_\epsilon(Z_\epsilon) \|_\epsilon \) in Proposition 3.5, we find that, for such \( Z_\epsilon \), \( \| \omega_\epsilon(Z_\epsilon) \|_\epsilon = o(\epsilon^{N/2}) \). Moreover, by the exponential decay property of \( U_{a_i} \), we easily see that

\[
\|(\xi - 1)U_{a_i, \epsilon, z_i^*}\|_\epsilon = O(e^{-m^*\theta/(2\epsilon)}) = o(\epsilon^{N/2}).
\]

Therefore, if we denote

\[
\omega_\epsilon = \omega_\epsilon(Z_\epsilon) + \sum_{i=1}^{k} (\xi - 1)U_{a_i, \epsilon, z_i^*},
\]

then \( u_\epsilon^* = w_\epsilon^* + \sum_{i=1}^{k} U_{a_i, \epsilon, z_i^*} + \omega_\epsilon \) meets all the requirements of Theorem 1.3.

We now set to show the existence of such \( Z_\epsilon \). By the expansion of \( \tilde{I}(Z, \omega) \) and the estimates in Proposition 3.5 and Lemma 3.6, we have

\[
F_\epsilon(Z) = \tilde{I}(Z, \omega_\epsilon(Z))
\]

\[
= \tilde{I}(Z, 0) + \langle K_\epsilon(Z), \omega_\epsilon(Z) \rangle + \frac{1}{2} \langle Q_\epsilon(Z) \omega_\epsilon(Z), \omega_\epsilon(Z) \rangle + R_{\epsilon, Z}(\omega_\epsilon(Z))
\]

\[
= \tilde{I}(Z, 0) + O(\| K_\epsilon(Z) \|_\epsilon \| \omega_\epsilon(Z) \|_\epsilon) + O(\| \omega_\epsilon(Z) \|_\epsilon^2) + O(\| R_{\epsilon, Z}(\omega_\epsilon(Z)) \|_\epsilon \| \omega_\epsilon(Z) \|_\epsilon)
\]

\[
= \tilde{I}(Z, 0) + O\left( \sum_{i=1}^{k} |a(z_i^*) - a_i|^2 \right) \epsilon^N.
\]

To estimate \( \tilde{I}(Z, 0) \), we follow the lines of the proof for Theorem 3.4 in [DG3] and obtain
\begin{align*}
\bar{I}(Z,0) = \sum_{i=1}^{k} Q_i \epsilon^N - \sum_{i=1}^{k} \int_{\mathbb{R}^N} F(y, U_{i,\epsilon}) dy + o(\epsilon^N),
\end{align*}
where

\begin{align*}
U_{i,\epsilon} &= U_{a_i,\epsilon, z^i}, \
Q_i &= (1/2) \int_{\mathbb{R}^N} f_a(U_{a_i}) U_{a_i} dx.
\end{align*}

By (3.17) in [DaY1], we have

\begin{align*}
\int_{\mathbb{R}^N} F(y, U_{i,\epsilon}) dy &= \int_{\mathbb{R}^N} F_a(U_{a_i}) dx + (a_i - a(z^i)) \int_{\mathbb{R}^N} \left( \frac{1}{3} U_{i,\epsilon}^3 - \frac{1}{2} U_{i,\epsilon}^2 \right) dy + o(\epsilon^N) \\
&= \epsilon^N \int_{\mathbb{R}^N} F_a(U_{a_i}) dx + (a_i - a(z^i)) \epsilon^N \int_{\mathbb{R}^N} \left( \frac{1}{3} U_{a_i}^3 - \frac{1}{2} U_{a_i}^2 \right) dx + o(\epsilon^N),
\end{align*}

where \( F_a(t) = \int_0^t f_a(s) ds. \)

So we finally obtain

\begin{align*}
\mathcal{F}_\epsilon(Z) &= \epsilon^N Q + \epsilon^N \sum_{i=1}^{k} \left( |a_i - a(z^i)| \tilde{c}_i + O\left( |a_i - a(z^i)|^2 \right) + o(1) \right),
\end{align*}

where

\begin{align*}
Q &= \sum_{i=1}^{k} \left( Q_i - \int_{\mathbb{R}^N} F_a(U_{a_i}) dx \right) \\
\tilde{c}_i &= \int_{\mathbb{R}^N} \left( \frac{1}{2} U_{a_i}^2 - \frac{1}{3} U_{a_i}^3 \right) dx > 0
\end{align*}

due to \( U_{a_i} \leq 1. \)

Consider now the problem

\begin{equation}
\max \{ \mathcal{F}_\epsilon(Z) : Z \in \overline{D_\delta} \}. \tag{3.40}
\end{equation}

Let \( Z_\epsilon^\delta = (z_1^{\delta, \epsilon}, \ldots, z_k^{\delta, \epsilon}) \in \overline{D_\delta} \) be a maximum point of problem (3.40), and denote \( Z_0 = (x_1, \ldots, x_N) \). Then we have

\begin{align*}
\mathcal{F}_\epsilon(Z_\epsilon^\delta) &\geq \mathcal{F}_\epsilon(Z_0) = \epsilon^N Q + o(\epsilon^N).
\end{align*}

On the other hand, if \( \delta > 0 \) is suitably small, say \( \delta \in (0, \delta^*] \) with \( \delta^* \leq \delta_0 \), then \( |a_i - a(z_i)| \) is small for \( (z_1, \ldots, z_k) \in D_\delta \) and hence

\begin{align*}
\mathcal{F}_\epsilon(Z_\epsilon^\delta) &\leq \epsilon^N Q + \epsilon^N \sum_{i=1}^{k} \left( a_i - a(z_i^{\delta, \epsilon}) \right) \left( \tilde{c}_i / 2 \right) + o(\epsilon^N). \\
\end{align*}

It follows that

\begin{align*}
\sum_{i=1}^{k} \left( a_i - a(z_i^{\delta, \epsilon}) \right) \left( \tilde{c}_i / 2 \right) &\geq o_\epsilon(1).
\end{align*}

This implies that

\begin{align*}
Z_\epsilon^\delta &\to Z_0 \text{ as } \epsilon \to 0. \tag{3.41}
\end{align*}
Therefore, for fixed $\delta \in (0, \delta^*]$, we can find $\epsilon_\delta \in (0, \epsilon_1)$ such that for $\epsilon \in (0, \epsilon_\delta)$, $Z^\delta_\epsilon$ is an interior point of $D_\delta$ and hence a critical point of $F_\epsilon$. Together with (3.41), we have proved what we wanted.

If $x_1, \ldots, x_k$ are strict local maximum points of $a(x)$, we replace (3.40) by the corresponding minimization problem, and the analysis is similar. This completes the proof of Theorem 1.3.

**References**


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Some detailed proofs for the referee

Lemma 3.7. There exist $\epsilon_5 \in (0, \epsilon_4)$, $\delta_1 \in (0, \theta)$ and $c_1 > 0$, such that if $\epsilon \in (0, \epsilon_5)$, then for any $Z \in D_{\delta_1}$

$$\|P_{\epsilon, Z}Q_{\epsilon}(Z)\omega\|_\epsilon \geq c_1\|\omega\|_\epsilon, \ \forall \omega \in E_{\epsilon, Z},$$

(3.36)

where $P_{\epsilon, Z}$ denotes the orthogonal projection from the Hilbert space $H$ to its closed subspace $E_{\epsilon, Z}$.

Proof. We first notice that, since $\tilde{U}_{a_i, \epsilon, z^i}(x) \in (0, 1)$ for $i = 1, \ldots, k$, and since $\xi_i$ and $\xi_j$ have disjoint supports when $i \neq j$, we have $W_{\epsilon, Z}(x) \in (0, 1)$. Thus,

$$f_i(x, w^*_\epsilon) + g_i(x, W_{\epsilon, Z}) = h_i(x, W_{\epsilon, Z}) = f_i(x, w^*_\epsilon + W_{\epsilon, Z})$$

and

$$\langle Q_{\epsilon}(Z)\omega, \psi \rangle = \int_\Omega \left( e^2D\omega \cdot D\psi - f_i(y, w^*_\epsilon + W_{\epsilon, Z})\omega\psi \right) dy.$$

We now use a contradiction argument to prove (3.36). Suppose that there are $\epsilon_j \to 0$, $\delta_j \to 0$ and $Z_j = (z_1^{i,j}, \ldots, z_k^{i,j}) \in D_{\delta_j}$, $\omega_j \in E_{\epsilon_j, Z_j}$ such that

$$\|P_{\epsilon_j, Z_j}Q_{\epsilon_j}(Z_j)\omega_j\|_{\epsilon_j} = o_j(1)\|\omega\|_{\epsilon_j},$$

(3.37)

where $o_j(1) \to 0$ as $j \to \infty$. We may also assume that $\|\omega\|_{\epsilon_j} = \epsilon_j^{N/2}$.

For each $i$, define

$$\Omega_{i,j} = \{y : \epsilon_j y + z^{i,j} \in \Omega\}, \quad \bar{\omega}_{i,j}(y) := \omega_j(\epsilon_j y + z^{i,j}) \text{ for } y \in \Omega_{i,j},$$

and extend $\bar{\omega}_{i,j}$ to be zero for $y \notin \Omega_{i,j}$. Then $\{\bar{\omega}_{i,j}\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus by passing to a subsequence we may assume that

$$\bar{\omega}_{i,j} \rightharpoonup \bar{\omega}_i \in H^1(\mathbb{R}^N) \text{ weakly as } j \to \infty.$$

Let

$$E^*_i = \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left( DU^h_{i,j} \cdot Dv + U^h_{i,j}v \right) = 0, \quad h = 1, \ldots, N \right\},$$

where $U^h_{i,j}(y) = \xi_i(\epsilon_j y + z^{i,j})D_{y_h}U_{a_i}(y)$. Then from $\omega_j \in E_{\epsilon_j, Z_j}$ it is easily checked that $\bar{\omega}_{i,j} \in E^*_i$. Moreover, for any fixed $i$ and $j$, $E^*_i$ is a closed subspace of $H^1(\mathbb{R}^N)$.

Since

$$\xi_i(\epsilon_j y + z^{i,j})(D_{y_h}U_{a_i})(y) \to (D_{y_h}U_{a_i})(y) \text{ as } j \to \infty,$$

we find by using the exponential decay properties of $U_{a_i}(y)$ and $D_{y_h}U_{a_i}(y)$ that

$$\int_{\mathbb{R}^N} \left( D(D_{y_h}U_{a_i}) \cdot D\bar{\omega}_i + (D_{y_h}U_{a_i})\bar{\omega}_i \right) dy = 0, \quad h = 1, \ldots, N.$$
That is to say that \( \tilde{w}_i \in E^*_i \), where

\[
E^*_i := \left\{ \varphi \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left( D(Dy_i U_{a_i}) \cdot D\varphi + (Dy_i U_{a_i}) \varphi \right) = 0, \quad h = 1, \ldots, N \right\}.
\]

For any \( \psi \in E_{\epsilon, \tilde{z}} \), if we write \( \tilde{\psi}_{i,j}(y) = \psi(\epsilon_j y + z^{i,j}) \) and \( \tilde{W}_{i,j}(y) = W_{\epsilon_j z_j}(\epsilon_j y + z^{i,j}) \), then by (3.33) and (3.37), we have

\[
\int_{\Omega_{i,j}} \left( D\tilde{w}_{i,j} \cdot D\tilde{\psi}_{i,j} - f_i(\epsilon_j y + z^{i,j}, w^*_{i,j}(\epsilon_j y + z^{i,j}) + \tilde{W}_{i,j}(y)) \tilde{\psi}_{i,j} \tilde{\psi}_{i,j} \right) dy
\]

\[
= \epsilon_j^{-N} \int_{\Omega} \left( \epsilon_j^2 D\omega_j \cdot D\psi - f_i(y, w^*_j + W_{\epsilon_j z_j}) \omega_j \psi \right) dy
\]

\[
= \epsilon_j^{-N/2} < Q(\epsilon_j(\tilde{z}_j)) \omega_j, \psi > /\| \omega_j \|_{\epsilon_j}
\]

\[
= o(\epsilon_j^{-N/2}) \| \psi \|_{\epsilon_j}.
\]

Now choose an arbitrary \( \psi \in E^*_i \subset H^1(\mathbb{R}^N) \) and then choose constants \( c^j_h \) such that

\[
\psi^* := \psi - \Sigma_{h=1}^N c^j_h U^{h}_{i,j} \in E^*_i.
\]

To see the existence of such \( c^j_h \) and to find out their properties, it is convenient to recall that, if we denote \( \partial_h U_b(y) = D_{y_h} U_b(y) \), then

\[
-\Delta(\partial_h U_b) = (f_b)'(U_b)(\partial_h U_b),
\]

and for \( h \neq l \),

\[
\int_{\mathbb{R}^N} D(\partial_h U_b) \cdot D(\partial_l U_b) dx = \int_{\mathbb{R}^N} (f_b)'(U_b)(\partial_h U_b)(\partial_l U_b) dx
\]

\[
= \int_{\mathbb{R}^N} (f_b)'(U_b(|x|)) [U'_b(|x|)/|x|] x_h x_l dx = 0,
\]

\[
\int_{\mathbb{R}^N} (\partial_h U_b)(\partial_l U_b) dx = \int_{\mathbb{R}^N} [U'_b(|x|)/|x|] x_h x_l dx = 0.
\]

Therefore we always have

\[
\int_{\mathbb{R}^N} \left( (\partial_h U_b) \cdot D(\partial_l U_b) + (\partial_h U_b)(\partial_l U_b) \right) dx = 0. \quad (3.38)
\]

From \( \psi \in E^*_i \), we find, for each \( h = 1, \ldots, N \),

\[
\lim_{j \to \infty} \int_{\Omega_{i,j}} \left( D U^{h}_{i,j} \cdot D\psi + U^{h}_{i,j} \psi \right) dy
\]

\[
= \int_{\mathbb{R}^N} \left( D(\partial_h U_{a_i}) \cdot D\psi + (\partial_h U_{a_i}) \psi \right) dy = 0.
\]

Together with (3.38), this implies that, for all large \( j \), \( c^j_h \) is uniquely determined and \( c^j_h \to 0 \) as \( j \to \infty \).
Let $\psi_{i,j}(y) = \xi_i^0(\epsilon_j y + z^{i,j})\psi^*(y)$, where $\xi_i^0(y)$ is a $C^1$ cut-off function similar to $\xi_i(y)$ except that $\xi_i^0(y) = 0$ when $|y - x_i| > 3\theta$, $\xi_i^0(y) = 1$ when $|y - x_i| < 2\theta$. Clearly $(\psi_{i,j})_{\epsilon_j z^{i,j}}(y) = \xi_i^0(y)\psi^*_{\epsilon_j z^{i,j}}(y)$ is in $H$. Moreover, since the change of $\psi_{i,j}$ from $\psi^*$ happens outside the supporting sets of $U_{i,j}^h$, we find $\psi_{i,j} \in E_{i,j}^*$. Now in the above equalities deduced from (3.37) we substitute $\tilde{\psi}_{i,j}$ by $\psi_{i,j}$ and find that

\[
\int_{\Omega_{i,j}} (D\tilde{\omega}_{i,j} \cdot D\psi_{i,j} - f_\epsilon(\epsilon_j y + z^{i,j}, w_{\epsilon_j}(\epsilon_j y + z^{i,j}) + \tilde{W}_{i,j}(y))\tilde{\omega}_{i,j}\psi_{i,j}) dy = o(\epsilon_j^{-N/2})\xi_i^0(\psi^*_{\epsilon_j z^{i,j}}) dy = o(1).
\]

On the other hand, due to $c_h^{\mathcal{I}} \to 0$, we have $\psi_{i,j} \to \psi$ in $H^1(\mathbb{R}^N)$, and we also easily see that

\[
w_{\epsilon_j}(\epsilon_j y + z^{i,j}) \to 0, \quad \tilde{W}_{i,j}(y) \to U_{a_i}(y), \quad a(\epsilon_j y + z^{i,j}) \to a(x_i) = a_i
\]

uniformly on every compact subset of $\mathbb{R}^N$. It follows that

\[
\lim_{j \to \infty} \int_{\Omega_{i,j}} (D\tilde{\omega}_{i,j} \cdot D\psi_{i,j} - f_\epsilon(\epsilon_j y + z^{i,j}, w_{\epsilon_j}(\epsilon_j y + z^{i,j}) + \tilde{W}_{i,j}(y))\tilde{\omega}_{i,j}\psi_{i,j}) dy = \int_{\mathbb{R}^N} (D\tilde{w}_i \cdot D\psi - (f_{a_i})'(U_{a_i})\tilde{w}_i \psi) dy.
\]

Therefore, $\tilde{\psi}_i$ satisfies

\[
\int_{\mathbb{R}^N} (D\tilde{\psi}_i \cdot D\psi - (f_{a_i})'(U_{a_i})\tilde{\psi}_i \psi) = 0, \quad \forall \psi \in E_{i}^*.
\] (3.39)

Since $\partial_h U_{a_i}$ is a solution of $-\Delta u - (f_{a_i})'(U_{a_i})u = 0$, (3.39) also holds for $\psi = \partial_h U_{a_i}$, $h = 1, \ldots, N$. Thus we have proved that $\tilde{\psi}_i$ is a solution of

\[
-\Delta u - (f_{a_i})'(U_{a_i})u = 0, \quad u \in H^1(\mathbb{R}^N).
\] (3.40)

By [NTW, PS], $U_{a_i}$ is a nondegenerate solution, that is, any solution of (3.40) belongs to $\text{span}\{\partial_1 U_{a_i}, \ldots, \partial_N U_{a_i}\}$. Therefore,

\[
\tilde{\psi}_i \in \text{span}\{\partial_1 U_{a_i}, \ldots, \partial_N U_{a_i}\}.
\]

Since $\tilde{\psi}_i \in E_{i}^*$, we must have $\tilde{\psi}_i = 0$, that is, $\tilde{\psi}_{i,j}$ converges to 0 weakly in $H^1(\mathbb{R}^N)$ as $j \to \infty$. By Sobolev imbedding theorems on bounded domains, we deduce from this fact that for any fixed $R > 0$,

\[
\int_{B_R(0)} \tilde{\omega}_{i,j}^2 dy = o_j(1), \quad i = 1, 2, \ldots, k.
\]
This will be used below to derive a contradiction. We start with, due to \((3.37)\),

\[
o(\epsilon_j^N) = o(\|\omega_j\|_j^2)
\]

\[
= \int_{\Omega} \epsilon_j^2 |D\omega_j|^2 dy - \int_{\Omega} f_t(y, w_{\epsilon_j}^* + W_{\epsilon_j, z_j})\omega_j^2 dy
\]

\[
= \int_{\Omega} \epsilon_j^2 |D\omega_j|^2 dy - \int_{\Omega} f_t(y, w_{\epsilon_j}^*)\omega_j^2 dy + \int_{\Omega} [f_t(y, w_{\epsilon_j}^*) - f_t(y, w_{\epsilon_j}^* + W_{\epsilon_j, z_j})]\omega_j^2 dy
\]

\[
= \|\omega_j\|_{\epsilon_j}^2 - \int_{\Omega} f_{tt}(y, \eta_j)W_{\epsilon_j, z_j}\omega_j^2 dy
\]

where \(\eta_j \in (w_{\epsilon_j}^*, w_{\epsilon_j}^* + W_{\epsilon_j, z_j})\). On \(B_{\epsilon_j R}(z^{i,j})\), \(w_{\epsilon_j}^*\) and \(W_{\epsilon_j, z_j}\) have \(L^\infty\) bounds independent of \(j\), and hence \(f_{tt}(x, \eta_j)\) has \(L^\infty\) bounds independent of \(j\). Thus,

\[
\left| \int_{\bigcup_{i=1}^k B_{\epsilon_j R}(z^{i,j})} f_{tt}(x, \eta_j)W_{\epsilon_j, z_j}\omega_j^2 dx \right| \leq C \epsilon_j^N \sum_{i=1}^k \int_{B(0)} (\hat{\omega}_{i,j})^2 dy = o(\epsilon_j^N). \tag{3.41}
\]

We easily see that for \(j\) large,

\[
\Omega = \{x : \eta_j(x) \geq 1\} \cup \{x : \eta_j(x) < 1\} \cup \bigcup_{i=1}^k B_{\epsilon_j R}(z^{i,j}) \]  

Therefore,

\[
- \int_{\Omega} f_{tt}(y, \eta_j)W_{\epsilon_j, z_j}\omega_j^2 dy
\]

\[
= - \left[ \int_{\eta_j > 1} + \int_{\eta_j < 1} \right] \int_{\bigcup_{i=1}^k B_{\epsilon_j R}(z^{i,j})} f_{tt}(y, \eta_j)W_{\epsilon_j, z_j}\omega_j^2 dy
\]

\[
:= I_1 + I_2 + I_3.
\]

Since \(f_{tt}(x, t) < 0\) for \(t \geq 1\), we have

\[
I_1 \geq 0. \tag{3.42}
\]

Since \(|f_{tt}(x, t)| \leq M_2\) for \(t \in [0, 1]\), and for any \(\delta > 0\), if we choose \(R > 0\) sufficiently large

\[
|W_{\epsilon_j, z_j}(x)| \leq \delta \quad \text{if} \quad x \in \{\eta_j(x) < 1\} \cup \bigcup_{i=1}^k B_{\epsilon_j R}(z^{i,j})
\]

for all large \(j\), we find

\[
|I_2| \leq M_2 \delta \int_{\Omega} \omega_j^2 dy \leq \frac{M_2 \delta}{k^*} \|\omega_j\|_{\epsilon_j}^2, \tag{3.43}
\]

by making use of Lemma 3.1. Thereof, for all large \(j\),

\[
o(\epsilon_j^N) = o(\|\omega_j\|_{\epsilon_j}^2)
\]

\[
= \|\omega_j\|_{\epsilon_j}^2 + I_1 + I_2 + I_3
\]

\[
\geq \left(1 - \frac{M_2 \delta}{k^*}\right)\|\omega_j\|_{\epsilon_j}^2 + o(\epsilon_j^N) = \left(1 - \frac{M_2 \delta}{k^*}\right)\epsilon_j^N + o(\epsilon_j^N).
\]

This is a contradiction and the lemma is proved.
Lemma 3.8. There exists $0 < \epsilon_6 \in (0, \epsilon_5]$ such that for $\epsilon \in (0, \epsilon_6]$, 
\[ \|K_\epsilon(Z)\|_\epsilon = O\left(\sum_{i=1}^k |a(z^i) - a_i| + o_\epsilon(1)\right) \epsilon^{N/2}, \]
uniformly for $Z \in D_{\theta/2}$.

proof. Let us recall 
\[ < K_\epsilon(z), \omega > = \int_\Omega \left( e^2 DW_{\epsilon,Z} \cdot D\omega - h(y, W_{\epsilon,Z}) \omega \right) dy. \]

For $z \in D_{\theta/2}$, $|y - x_i| \geq \theta$ implies $|y - z^i| \geq \theta/2$ and therefore 
\[ U_{\epsilon_i,\epsilon,z^i}(y), |DU_{\epsilon_i,\epsilon,z^i}(y)| \leq Ce^{-m^*\theta/(2\epsilon)} \] 
when $|y - x_i| \geq \theta$, 
where $m^* = \min_{1 \leq i \leq k} m_{\epsilon_i}$. It follows that 
\[ \int_\Omega e^2 DW_{\epsilon,Z} \cdot D\omega dy = \int_\Omega e^2 \sum_{i=1}^k DU_{\epsilon_i,\epsilon,z^i} \cdot D(\xi_i) dy \]
\[ + \sum_{i=1}^k \int_{|y-x_i|\geq \theta} e^2 \left( U_{\epsilon_i,\epsilon,z^i} D\xi_i \cdot D\omega - \omega DU_{\epsilon_i,\epsilon,z^i} \cdot D\xi_i \right) dy \]
\[ = \int_\Omega e^2 \sum_{i=1}^k DU_{\epsilon_i,\epsilon,z^i} \cdot D(\xi_i) + O\left(e^{-m^*\theta/(2\epsilon)}\|\omega\|_\epsilon\right) \]
\[ = \int_\Omega \left( \sum_{i=1}^k f_{\epsilon_i} \left( U_{\epsilon_i,\epsilon,z^i} \right) \xi_i \right) \omega dy + O\left(e^{-m^*\theta/(2\epsilon)}\|\omega\|_\epsilon\right). \]

Therefore, 
\[ < K_\epsilon(Z), \omega > = \int_\Omega \left( e^2 DW_{\epsilon,Z} \cdot D\omega - h(y, W_{\epsilon,Z}) \omega \right) dy \]
\[ = \int_\Omega \left( \sum_{i=1}^k f_{\epsilon_i} \left( U_{\epsilon_i,\epsilon,z^i} \right) \xi_i - h(y, W_{\epsilon,Z}) \right) \omega dy + O\left(e^{-m^*\theta/(2\epsilon)}\right) \]
\[ = \int_\Omega \left( \sum_{i=1}^k f_{\epsilon_i} \left( U_{\epsilon_i,\epsilon,z^i} \right) \xi_i + f(y, w^*_\epsilon) - f(y, w^*_\epsilon + \sum_{i=1}^k \tilde{U}_{\epsilon_i,\epsilon,z^i}) \right) \omega dy \]
\[ + O\left(e^{-m^*\theta/(2\epsilon)}\|\omega\|_\epsilon\right). \]

Since $\xi_i(y) = 0$ when $|y - x_i| \geq 2\theta$, we find that 
\[ \int_\Omega \left( \sum_{i=1}^k f_{\epsilon_i} \left( U_{\epsilon_i,\epsilon,z^i} \right) \xi_i + f(y, w^*_\epsilon) - f(y, w^*_\epsilon + \sum_{i=1}^k \tilde{U}_{\epsilon_i,\epsilon,z^i}) \right) \omega dy \]
\[ = \sum_{i=1}^k \int_{|y-x_i|<2\theta} \left( f_{\epsilon_i} \left( U_{\epsilon_i,\epsilon,z^i} \right) \xi_i + f(y, w^*_\epsilon) - f(y, w^*_\epsilon + \tilde{U}_{\epsilon_i,\epsilon,z^i}) \right) \omega dy. \]

Therefore we can write 
\[ < K_\epsilon(Z), \omega > = \sum_{i=1}^k (J^i_1 + J^i_2) + O\left(e^{-m^*\theta/(2\epsilon)}\|\omega\|_\epsilon\right), \]
where
\[ J_1^i = \int_{|y-x_i|<2\theta} \left( f_{a_i}(U_{a_i,\epsilon,z^i})\xi_i - f_{a_i}(U_{a_i,\epsilon,z^i}) \right) \omega dy \]
\[ = \int_{|y-x_i|<2\theta} \left( f_{a_i}(U_{a_i,\epsilon,z^i})\xi_i - f_{a_i}(U_{a_i,\epsilon,z^i}) \right) \omega dy, \]
\[ J_2^i = \int_{|y-x_i|<2\theta} \left( f_{a_i}(U_{a_i,\epsilon,z^i}) + f(y, w^*_\epsilon) - f(y, w^*_\epsilon + \tilde{U}_{a_i,\epsilon,z^i}) \right) \omega dy. \]

We have
\[ |J_1^i| \leq \int_{|y-x_i|<2\theta} CU_{a_i,\epsilon,z^i}|\xi_i - 1||\omega|dy = O\left(e^{-m}\theta^2(\epsilon)\right)||\omega||_\epsilon. \]

To estimate \( J_2^i \), we notice that for \( y \) satisfying \(|y-x_i|<2\theta\), \( w^*_\epsilon(y) \) is uniformly small in \( \epsilon \) and
\[ \int_{\{y-x_i|<2\theta\} \setminus \{y-z^i|<\epsilon^{1/2}\}} \left| f_{a_i}(U_{a_i,\epsilon,z^i}) + f(y, w^*_\epsilon) - f(y, w^*_\epsilon + \tilde{U}_{a_i,\epsilon,z^i}) \right| |\omega|dy \]
\[ \leq \int_{\{y-x_i|<2\theta\} \setminus \{y-z^i|<\epsilon^{1/2}\}} CU_{a_i,\epsilon,z^i}|\omega|dy = O\left(e^{-m}\epsilon^{1/2}\right)||\omega||_\epsilon. \]

Since \( Z \in D_{\theta/2} \), for all small \( \epsilon \), we have \( \xi_i(y) = 1 \) when \(|y-z^i|<\epsilon^{1/2}\), and hence
\[ J_2^i = \int_{|y-z^i|<\epsilon^{1/2}} \left( f_{a_i}(U_{a_i,\epsilon,z^i}) + f(y, w^*_\epsilon) - f(y, w^*_\epsilon + U_{a_i,\epsilon,z^i}) \right) \omega dy \]
\[ + O\left(e^{-m}\epsilon^{1/2}\right)||\omega||_\epsilon. \]

For \(|y-z^i|<\epsilon^{1/2}\), we have
\[ |f(y, w^*_\epsilon)| = O(w^*_\epsilon(y)), \]
and
\[ |f(y, w^*_\epsilon + U_{a_i,\epsilon,z^i}) - f(y, U_{a_i,\epsilon,z^i})| = O(w^*_\epsilon(y)). \]

We easily see that \( w^*_\epsilon \leq w_{\epsilon,a^*,a^*} \) in \( \Omega \), where \( w_{\epsilon,a^*,a^*} \) is the minimal positive solution to the problem (see [APL, DG4])
\[ -\epsilon^2 \Delta w = f_{a^*,a^*}(w) \text{ in } \Omega, \quad w = \infty \text{ on } \partial \Omega \]
with \( w_{\epsilon,a^*,a^*} \to 0 \) in \( C_0^0(\Omega) \) as \( \epsilon \to 0 \). (See [APL, DG3].) By the proof of Theorem 2.1 in [DW1], \( w_{\epsilon,a^*,a^*} \) satisfies
\[ -\epsilon \ln(w_{\epsilon,a^*,a^*}(x)) \to d(x, \partial \Omega)|f'_{a^*,a^*}(0)|^{1/2} \text{ as } \epsilon \to 0, \]
uniformly on any compact subset of \( \Omega \). Therefore,
\[ |w^*_\epsilon(y)| = o(\epsilon^N) \text{ for } |y-z^i|<\theta. \]
It follows that

\[
|J_2^j| = \left| \int_{|y-z^i|<\epsilon^{1/2}} \left( f_{a_i}(U_{a_i,\epsilon,z^i}) + f(y,U_{a_i,\epsilon,z^i}) \right) \omega dy \right| + o(\epsilon^N) \|\omega\|_\epsilon
\]

\[
\leq \int_{|y-z^i|<\epsilon^{1/2}} U_{a_i,\epsilon,z^i}(1-U_{a_i,\epsilon,z^i})|a_i - a(y)||\omega|dy + o(\epsilon^N) \|\omega\|_\epsilon
\]

\[
\leq \left( |a_i - a(z^i)| + A_i(\epsilon) \right) \int_{|y-z^i|<\epsilon^{1/2}} U_{a_i,\epsilon,z^i}(1-U_{a_i,\epsilon,z^i})|\omega|dy + o(\epsilon^N) \|\omega\|_\epsilon,
\]

where

\[A_i(\epsilon) := \sup\{|a(x) - a(y)| : x, y \in B_\theta(x_i), |y - x| < \epsilon^{1/2}\} \to 0 \text{ as } \epsilon \to 0.\]

Therefore,

\[
|J_2^j| \leq \left( |a_i - a(z^i)| + A_i(\epsilon) \right) \left( \int_{|y-z^i|<\epsilon^{1/2}} U_{a_i,\epsilon,z^i}^2(1-U_{a_i,\epsilon,z^i})^2dy \right)^{1/2} \|\omega\|_\epsilon
\]

\[
+ o(\epsilon^N) \|\omega\|_\epsilon
\]

\[
= \left( |a_i - a(z^i)| + A_i(\epsilon) \right) \epsilon^{N/2} \left( \int_{\mathbb{R}^N} U_{a_i}(1-U_{a_i})dx + a_\epsilon(1) \right)^{1/2} \|\omega\|_\epsilon
\]

\[
+ o(\epsilon^N) \|\omega\|_\epsilon.
\]

We finally obtain

\[
<K_\epsilon(Z),\omega> = \sum_{i=1}^k (J_1^i + J_2^i) + o(\epsilon^N) \|\omega\|_\epsilon
\]

\[
= \left( \sum_{i=1}^k |a_i - a(z^i)| + A(\epsilon) \right) O(\epsilon^{N/2}) \|\omega\|_\epsilon + o(\epsilon^N) \|\omega\|_\epsilon,
\]

where \(A(\epsilon) = \sum_{i=1}^k A_i(\epsilon) = a_\epsilon(1).\) Therefore

\[
\|K_\epsilon(Z)\|_\epsilon = O\left( \sum_{i=1}^k |a(z^i) - a_i| + A(\epsilon) \right) \epsilon^{N/2},
\]

as required.

**Proof of Theorem 1.3:** Suppose that \(\epsilon \in (0,\epsilon_1], \delta \in (0,\delta_0)\) and \(\omega_\epsilon(Z)\) is given by Proposition 3.5. Let

\[\mathcal{F}_\epsilon(Z) = \bar{I}(Z,\omega_\epsilon(Z)) = I\left( \sum_{i=1}^k \bar{U}_{a_i,\epsilon,z^i} + \omega_\epsilon(Z) \right), \quad Z \in D_\delta.
\]

As indicated in [DaY1], by standard argument in the reduction method, it can be shown that if \(Z_\epsilon \in D_\delta\) is a critical point of \(\mathcal{F}_\epsilon\), then \(\sum_{i=1}^k \bar{U}_{a_i,\epsilon,z^i} + \omega_\epsilon(Z_\epsilon)\) is a critical point of \(I\), and hence a solution to (3.2).

We will show in the following that \(\mathcal{F}_\epsilon\) has a critical point \(Z_\epsilon = (z^*_1, \ldots, z^*_k)\) satisfying \(Z_\epsilon \to (x_1, \ldots, x_k)\) in \(\mathbb{R}^k\) as \(\epsilon \to 0.\) By the estimate for \(\|\omega_\epsilon(Z_\epsilon)\|_\epsilon\) in Proposition 3.5, we
find that, for such $Z_\epsilon$, $\|\omega_\epsilon(Z_\epsilon)\|_\epsilon = o(\epsilon^{N/2})$. Moreover, by the exponential decay property of $U_{a_i}$, we easily see that

$$\|(\xi_i - 1)U_{a_i,\epsilon,z_i^*}\|_\epsilon = O\left(e^{-m\epsilon\theta/(2\epsilon)}\right) = o(\epsilon^{N/2}).$$

Therefore, if we denote

$$\omega_\epsilon = \omega_\epsilon(Z_\epsilon) + \sum_{i=1}^{k} (\xi_i - 1)U_{a_i,\epsilon,z_i},$$

then $u^*_\epsilon = w^*_\epsilon + \sum_{i=1}^{k} U_{a_i,\epsilon,z_i^*} + \omega_\epsilon$ meets all the requirements of Theorem 1.3.

We now set to show the existence of such $Z_\epsilon$. By the expansion of $\tilde{I}(Z,\omega_\epsilon)$ and the estimates in Proposition 3.5 and Lemma 3.6, we have

$$\mathcal{F}_i(Z) = \tilde{I}(Z,\omega_\epsilon(Z))$$

$$= \tilde{I}(Z,0) + <K_\epsilon(Z),\omega_\epsilon(Z)> + \frac{1}{2} <Q_\epsilon(Z)\omega_\epsilon(Z),\omega_\epsilon(Z)> + R_\epsilon,Z(\omega_\epsilon(Z))$$

$$= \tilde{I}(Z,0) + O(||K_\epsilon(Z)||_\epsilon||\omega_\epsilon(Z)||_\epsilon) + O(||\omega_\epsilon(Z)||_\epsilon^2) + O(R_\epsilon,Z(\omega_\epsilon(Z))||_\epsilon||\omega_\epsilon(Z)||_\epsilon)$$

$$= \tilde{I}(Z,0) + O\left(\sum_{i=1}^{k}|a(z^i) - a_i|^2 + o_\epsilon(1)\right)\epsilon^N.$$

Since the support of $\xi_i$ is in $B_{2\theta}(x_i)$ and $|x_i - x_j| > 8\theta$ when $i \neq j$, we find

$$\tilde{I}(Z,0) = I(\sum_{i=1}^{k}\tilde{U}_{a_i,\epsilon,z_i})$$

$$= (1/2) \int_{\Omega} \left(\epsilon^2 D(\sum_{i=1}^{k}\xi_i U_{a_i,\epsilon,z_i})^2 - f_\epsilon(y, w^*_\epsilon)|\sum_{i=1}^{k}\xi_i U_{a_i,\epsilon,z_i}|^2\right) dy$$

$$- \int_{\Omega} G(y, \sum_{i=1}^{k}\xi_i U_{a_i,\epsilon,z_i}) dy$$

$$= \sum_{i=1}^{k} \int_{B_{2\theta}(x_i)} (1/2)\epsilon^2 D(\xi_i U_{a_i,\epsilon,z_i})^2 dy$$

$$- \sum_{i=1}^{k} \int_{B_{2\theta}(x_i)} \left(\frac{1}{2}f_\epsilon(y, w^*_\epsilon)|\xi_i U_{a_i,\epsilon,z_i}|^2 + G(y, \xi_i U_{a_i,\epsilon,z_i})\right) dy.$$

For simplicity of notation, we write

$$U_{i,\epsilon} = U_{a_i,\epsilon,z_i}.$$
Using the exponential decay property of $U_{i, \epsilon}$ and $|DU_{i, \epsilon}|$, we obtain

$$
\int_{B_{2\delta}(x_i)} \epsilon^2 |D(\xi U_{i, \epsilon})|^2 dy \\
= \int_{B_{\delta}(x_i)} \epsilon^2 |DU_{i, \epsilon}|^2 dy + O\left(e^{-m^* \theta/(2\epsilon)}\right) \\
= \int_{\mathbb{R}^N} \epsilon^2 |DU_{i, \epsilon}|^2 dy + O\left(e^{-m^* \theta/(2\epsilon)}\right) \\
= \epsilon^N \int_{\mathbb{R}^N} f_{a_i}(U_{a_i})U_{a_i} dx + O\left(e^{-m^* \theta/(2\epsilon)}\right) \\
= 2Q_i \epsilon^N + o(\epsilon^N), \quad \text{where} \quad Q_i = (1/2) \int_{\mathbb{R}^N} f_{a_i}(U_{a_i})U_{a_i} dx.
$$

Using the exponential decay of $U_{i, \epsilon}$ and our estimates for $w_{\epsilon}^*$, we have

$$
\int_{B_{2\delta}(x_i)} \left(\frac{1}{2} f_t(y, w_{\epsilon}^*)\xi_U^2 + G(y, \xi_U)\right) dy \\
= \int_{B_{\delta}(x_i)} \left(\frac{1}{2} f_t(y, w_{\epsilon}^*)U_{i, \epsilon}^2 + G(y, U_{i, \epsilon})\right) dy + O\left(e^{-m^* \theta/(2\epsilon)}\right) \\
= \int_{B_{\delta}(x_i)} \left(F(y, w_{\epsilon}^* + U_{i, \epsilon}) - F(y, w_{\epsilon}^*) + f(y, w_{\epsilon}^*)U_{i, \epsilon}\right) dy + O\left(e^{-m^* \theta/(2\epsilon)}\right) \\
= \int_{B_{\delta}(x_i)} F(y, U_{i, \epsilon}) dy + O\left(\sup_{B_{\delta}(x_i)} |w_{\epsilon}^*(y)|\right) + O\left(e^{-m^* \theta/(2\epsilon)}\right) \\
= \int_{B_{\delta}(x_i)} F(y, U_{i, \epsilon}) dy + o(\epsilon^N) \\
= \int_{\mathbb{R}^N} F(y, U_{i, \epsilon}) dy + o(\epsilon^N).
$$

Therefore,

$$
\mathcal{F}_\epsilon(Z) = \bar{I}(Z, 0) + O\left(\sum_{i=1}^{k} |a(z^i) - a_i|^2 + o(1)\right) \epsilon^N \\
= \sum_{i=1}^{k} Q_i \epsilon^N - \sum_{i=1}^{k} \int_{\mathbb{R}^N} F(y, U_{i, \epsilon}) dy \\
+ O\left(\sum_{i=1}^{k} |a(z^i) - a_i|^2 + o(1)\right) \epsilon^N.
$$

By (3.17) in [DaY1], we have

$$
\int_{\mathbb{R}^N} F(y, U_{i, \epsilon}) dy \\
= \int_{\mathbb{R}^N} f_{a_i}(U_{i, \epsilon}) dy + (a_i - a(z^i)) \int_{\mathbb{R}^N} \left(\frac{1}{3} U_{i, \epsilon}^3 - \frac{1}{2} U_{i, \epsilon}^2\right) dy + o(\epsilon^N) \\
= \epsilon^N \int_{\mathbb{R}^N} f_{a_i}(U_{a_i}) dx + (a_i - a(z^i)) \epsilon^N \int_{\mathbb{R}^N} \left(\frac{1}{3} U_{a_i}^3 - \frac{1}{2} U_{a_i}^2\right) dx + o(\epsilon^N),
$$
where \( F_{a_i}(t) = \int_0^t f_{a_i}(s) \, ds \).

So we finally obtain

\[
F_\epsilon(Z) = \epsilon^N Q + \epsilon^N \sum_{i=1}^k \left( |a_i - a(z_i)| \hat{c}_i + O\left( |a_i - a(z_i)|^2 \right) + o_\epsilon(1) \right),
\]

where

\[
Q = \sum_{i=1}^k \left( Q_i - \int_{\mathbb{R}^N} F_{a_i}(U_{a_i}) \, dx \right)
\]

and

\[
\hat{c}_i = \int_{\mathbb{R}^N} \left( \frac{1}{2} U_{a_i}^2 - \frac{1}{3} U_{a_i}^3 \right) \, dx > 0
\]
due to \( U_{a_i} \leq 1 \).

Consider now the problem

\[
\max \{ F_\epsilon(Z) : Z \in D_\delta \}. \tag{3.44}
\]

Let \( Z_\epsilon^\delta = (z_1^\delta, \ldots, z_k^\delta) \in D_\delta \) be a maximum point of problem (3.44), and denote \( Z_0 = (x_1, \ldots, x_N) \). Then we have

\[
F_\epsilon(Z_\epsilon^\delta) \geq F_\epsilon(Z_0) = \epsilon^N Q + o(\epsilon^N).
\]

On the other hand, if \( \delta > 0 \) is suitably small, say \( \delta \in (0, \delta^*] \) with \( \delta^* \leq \delta_0 \), then \( |a_i - a(z_i)| \) is small for \( (z_1, \ldots, z_k) \in D_\delta \) and hence

\[
F_\epsilon(Z_\epsilon^\delta) \leq \epsilon^N Q + \epsilon^N \sum_{i=1}^k \left( a_i - a(z_i^\delta) \right) \left( \hat{c}_i/2 \right) + o(\epsilon^N).
\]

It follows that

\[
\sum_{i=1}^k \left( a_i - a(z_i^\delta) \right) \left( \hat{c}_i/2 \right) \geq o_\epsilon(1).
\]

This implies that

\[
Z_\epsilon^\delta \rightarrow Z_0 \text{ as } \epsilon \rightarrow 0. \tag{3.45}
\]

Therefore, for fixed \( \delta \in (0, \delta^*] \), we can find \( \epsilon_\delta \in (0, \epsilon_\gamma) \) such that for \( \epsilon \in (0, \epsilon_\delta) \), \( Z_\epsilon^\delta \) is an interior point of \( D_\delta \) and hence a critical point of \( F_\epsilon \). Together with (3.45), we have proved what we wanted.