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EFFECTIVE BEHAVIOR OF MULTIPLE LINEAR SYSTEMS *

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Dedicated to Prof. Yum-Tong Siu on his 60th birthday

1. Introduction. It is a fundamental problem in algebraic geometry to understand the behavior of a multiple linear system $|nD|$ on a projective complex manifold X for large n. For example, the well-known Riemann-Roch problem is to compute the function

$$n \longmapsto h^0(\mathcal{O}_X(nD)) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(nD)).$$

In the introduction to his collected works [33], Zariski cited the Riemann-Roch problem as one of the four "difficult unsolved questions concerning projective varieties (even algebraic surfaces)". The other natural problems about $|nD|$ are to find the fixed part and base points (see [32]), the very ampleness, the properties of the associated rational map and its image variety, the finite generation of the ring of sections, $\cdots$.

For a genus g curve X, Riemann-Roch theorem gives good and *effective* solutions to these problems.

- Assume that $\deg D > 0$. If $n \geq \frac{2g-1}{\deg D}$, then $h^1(\mathcal{O}_X(nD)) = 0$. So, in general,

$$h^0(nD) = \begin{cases} n \deg D + 1 - g, & \text{if } n \deg D > 2g - 2, \\ \text{a periodic function of } n, & \text{if } \deg D = 0, \\ 0, & \text{if } \deg D < 0. \end{cases}$$

- If $n \geq \frac{2g}{\deg D}$, then $|nD|$ is base point free.
- If $n \geq \frac{2g+1}{\deg D}$, then $|nD|$ is very ample.

When X is a surface, the Riemann-Roch problem is also equivalent to the computation of $h^1(\mathcal{O}_X(nD))$. This problem was studied first by the Italian geometers in the 19th century. Castelnuovo [9] proved that if $|D|$ is a base point free linear system of dimension ≥ 2, then there is a constant s such that

$$h^0(\mathcal{O}_X(nD)) = \chi(\mathcal{O}_X(nD)) + s$$

for n sufficiently large, i.e., $h^1(\mathcal{O}_X(nD))$ is a constant.

In [32], Zariski established the fundamental theory on the behavior of an arbitrary multiple linear system $|nD|$ on an algebraic surface (see the next section for the details.) By using *Zariski decomposition*, he showed that one only needs to know the behavior of $|nA + T|$, where A is a nef divisor and T is a fixed divisor (see Theorem 2.3). Zariski proved the boundedness of $h^1(\mathcal{O}_X(nA + T))$, the fixed part B_n and the

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isolated base points of $|nA+T|$ when n is sufficiently large. An important conjecture on the periodicity was proved later by Cutkosky and Srinivas [10] in 1993. However, all of these results are ineffective on n.

In the language of Beltrametti and Sommese [4], these problems are about the k-very ampleness.

Definition 1.1. (1) Let k be a nonnegative integer. A divisor D (or the linear system $|D|$) on X is called k-very ample if any $k+1$ points (not necessarily distinct) give $k+1$ independent conditions on $|D|$. Precisely, for any zero dimensional subscheme $\Delta \subset X$ with $\deg \Delta := h^0(\mathcal{O}_\Delta) \leq k+1$,

$$h^0(I_\Delta(D)) = h^0(\mathcal{O}_X(D)) - \deg \Delta,$$

where I_Δ is the ideal sheaf of Δ.

(2) D is called (-1)-very ample if $H^1(\mathcal{O}_X(D)) = 0$.

The Riemann-Roch problem is about (-1)-very ampleness; "0-very ample" is equivalent to "base point free"; "1-very ample" is just "very ample".

If X is a curve of genus g, $k \geq -1$ and D is a divisor satisfying

$$\deg D \geq k + 2g,$$

then D is k-very ample. In particular, if $\deg D > 0$, then $|nD|$ is k-very ample provided

$$n \geq \frac{k+2g}{\deg D}.$$

In recent years, the effective version of some important theorems attracted much attention. For example, Fujita's conjecture and the effective Matsusaka's big theorem (see, for example, [1, 26, 27, 28, 12, 13, 15, 17, 19] ...). They are about the 0- and 1-very ampleness of the adjoint linear system $|nH + K_X|$ and $|nH|$ for an ample divisor H, here K_X is the canonical divisor of X. I would like to mention the latest bounds of Angehrn-Siu [1] and Siu [28] for a d-dimensional complex manifold: $|K_X + nH|$ is base point free if

$$n \geq \frac{1}{2}(d^2 + d + 2).$$

(The power 2 in the bound is improved to $\frac{4}{3}$ by Siu's student Heier [18].) $|nH|$ is very ample if

$$n \geq \frac{\left(2^{3d-1}5d\right)^{4^{d-1}} \left(3(3d-2)^d H^d + K_X H^{d-1}\right)^{4^{d-1}3d}}{\left(6(3d-2)^d - 2d - 2\right)^{4^{d-1}d-\frac{2}{3}} \left(H^d\right)^{4^{d-1}3(d-1)}}.$$

If X is a surface, then there are also nice solutions: $|nH + K_X|$ is $(n-3)$-very ample (Reider [23], Beltrametti and Sommese [4]). $|nH|$ is very ample when

$$n > \frac{1}{2}\left[\frac{(H(K_X+4H)+1)^2}{H^2} + 3\right], \tag{1.1}$$

(Fernandez del Busto [16]). This bound is improved by Beltrametti and Sommese [5]

$$n > \frac{1}{2}\left(\frac{(H(K_X+2H)+1)^2}{H^2} + 7\right). \tag{1.2}$$

But the optimal effective Matsusaka's theorem for a surface is still open (Ein [14], Open Problem 4).

It is also of great interest to find the effective behavior of a multiple linear system $|nD|$. The purpose of this note is to give effective version of some well-known theorems on multiple linear systems due to Zariski [32], Castelnuovo [9], Artin [2, 3], Benveniste [6], Cutkosky and Srinivas [10, 11]. We also try to find the effective behavior of the rational map defined by $|nD|$.

For two divisors A and T, let

$$\begin{aligned}\mathfrak{M}(A,T) &= \frac{((K_X-T)A+2)^2}{4A^2} - \frac{(K_X-T)^2}{4},\\ \mathfrak{m}(A,T) &= \min\{\, n\in\mathbb{Z} \mid n > \mathfrak{M}(A,T)\}.\end{aligned}$$

Now we state our main result.

Theorem 1.2. *Let A be a nef and big divisor on an algebraic surface X, let T be any fixed divisor, and let k be a nonnegative integer. Assume that either $n > k+\mathfrak{M}(A,T)$, or $n \geq \mathfrak{M}(A,T)$ when $k=0$ and $T\sim K_X+\lambda A$ for some $\lambda\in\mathbb{Q}$. Suppose $|nA+T|$ is not $(k-1)$-very ample, i.e., there is a zero dimensional subscheme Δ on X with minimal degree $\deg\Delta \leq k$ such that it does not give independent conditions on $|nA+T|$. Then there is an effective divisor $D\neq 0$ containing Δ such that*

$$\begin{cases} TD - D^2 - K_X D \leq k, \\ DA = 0. \end{cases} \tag{1.3}$$

The effective version of the theorems in [32, 2, 3, 6, 9, 10, 11] are obtained by direct applications of this theorem for various T. For example, if $A=H$ is ample, then (1.3) has no solution $D\neq 0$. Thus we get an effective version of Serre's theorem.

Corollary 1.3. *If H is ample and $n > k+\mathfrak{M}(H,T)$ for some integer $k\geq 0$, then $|nH+T|$ is $(k-1)$-very ample. Equivalently, $|nH+T|$ is $(n-\mathfrak{m}-1)$-very ample when $n\geq\mathfrak{m}$.*

The bound in this corollary is optimal in many cases. If $T=K_X$, then

$$\mathfrak{M}(A,K_X) = \frac{1}{A^2}.$$

So $\mathfrak{m}(A,K_X)=2$ (or 1 if $A^2>1$). Thus the corollary implies also that $|nH+K_X|$ is $(n-3)$-very ample (or $(n-2)$-very ample if $H^2>1$).

If $T=0$, then

$$\mathfrak{M}(A,0) = \frac{(K_XA+2)^2}{4A^2} - \frac{K_X^2}{4},$$

and this corollary for $k=2$ is an effective version of Matsusaka Big Theorem. Our bound is better than (1.1) and (1.2). We will present an example to show that this bound is the best possible.

In general, we set

$$\tau(A,T) = \begin{cases} \min\limits_{D}\{\, TD - K_XD - D^2 \,\}, & \text{if } A \text{ is not ample,}\\ +\infty, & \text{if } A \text{ is ample,}\end{cases}$$

where D runs over all effective divisors $D \neq 0$ such that $DA = 0$. τ is well defined (Lemma 4.2). Then we have

Corollary 1.4. *Assume that* $\tau = \tau(A, T) \geq 1$ *and* $n \geq \mathfrak{m} = \mathfrak{m}(A, T)$. *Then* $|nA + T|$ *is*

$$\min\{\, \tau - 2,\ n - \mathfrak{m} - 1\,\}$$

very ample.

Some well-known conditions on linear systems are those satisfying $\tau \geq 1$ (see § 2). For example, $\tau(A, K_X) = \min\{\, -D^2\,\} \geq 1$ (Fujita's condition). $\tau(A, 0) = 2$ if and only if $p_a(D) \leq 0$ for any D (Artin's condition). Laufer-Ramanujan's condition is that $TD \geq K_X D$ for any D (reduced and irreducible), which implies also that $\tau \geq 1$.

As a consequence, the behavior of $|nA|$ is controlled by the curves C_i orthogonal to A, namely $AC_i = 0$. If Artin's condition is satisfied, then the behavior of $|nA|$ is quite similar to that of the canonical multiple linear system $|nK_X|$ of a minimal surface of general type.

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2. Zariski's results and generalizations. In this section, we recall Zariski's fundamental results and their generalizations. In our language, these results are essentially about (-1)- and 0-very ampleness.

Let X be a smooth projective complex surface, K_X be its canonical divisor and D be any divisor on X.

Definition 2.1. D is called *nef* (numerically effective) if for any curve C on X, $DC \geq 0$. D is called *big* if $D^2 > 0$. D is called *pseudo-effective* if for any ample divisor H, $DH \geq 0$.

Theorem 2.2. (Zariski decomposition [32]). *Let D be a pseudo-effective divisor on X. There exist uniquely $\mathbb{Q}$-divisors A and F on X, such that $D = A+F$ satisfying the following conditions:*

1. *$F = 0$ or the intersection matrix of the irreducible components of F is negative definite;*
2. *A is nef and F is effective;*
3. *each irreducible component C of F satisfies $AC = 0$.*

The decomposition is called *Zariski decomposition*. The following basic theorem has been used to reduce the general case $|nD|$ to the case $|nA + T|$, where T is any fixed divisor on X.

Theorem 2.3. (Zariski [32]). *As in Theorem 2.2, $D = A + F$ is the Zariski decomposition.*

1. *$\kappa(D) = 2$ if and only if $A^2 > 0$.*
2. *If D is effective, then for all $n \geq 0$,*

$$h^0(\mathcal{O}_X(nD)) = h^0(\mathcal{O}_X([nA]))$$

3. *If sA is an integral divisor, and $n = as + b$ with $0 \leq b < s$, then*

$$h^0(\mathcal{O}_X(nD)) = h^0(\mathcal{O}_X(asA + bD)).$$

(4) *As in* (3), *if* $\kappa(D) \geq 0$, *then*

$$h^1(\mathcal{O}_X(nD)) = h^1(\mathcal{O}_X(asA + bD)) - \frac{F^2}{2}n^2 + \frac{FK_X}{2}n - \left(b\frac{FK_X}{2} - b^2\frac{F^2}{2}\right).$$

THEOREM 2.4. (Zariski [32]). *As in Theorem 2.3, assume that T is any divisor on X.*

(1) $h^1(\mathcal{O}_X(nA+T))$ *is bounded.*

(2) *Let D be effective and let B_n be the fixed part of $|nD|$. Then*

$$B_n = \widetilde{B}_n + nF,$$

where $\widetilde{B}_n$ is a bounded (*rational*) *effective divisor.*

(3) *If $|D|$ has no fixed part and $n \gg 0$, then $|nD|$ has no base point and $h^1(\mathcal{O}_X(nA+T))$ is a constant.*

Shafarevich [25] gave a new proof of the base point freeness in (3).

THEOREM 2.5. (Zariski [32], Cutkosky-Srinivas [10, 11]). $h^1(\mathcal{O}_X(nA+T))$ *and* $\widetilde{B}_n$ *are periodic when* $n \gg 0$.

This theorem has been proved by Zariski [32] for the case $A^2 = 0$ and by Cutkosky and Srinivas [10, 11] for the case $A^2 > 0$.

For a fixed D, we let

$$R_m = R_m[D] = H^0(X, \mathcal{O}_X(mD)), \quad R[D] = \oplus_{m=0}^{\infty} R_m[D].$$

$R[D]$ is a graded ring.

Zariski gave in [32] a criterion for $R[D]$ to be finitely generated.

THEOREM 2.6. (Zariski [32]) $R[D]$ *is finitely generated if and only if* $\kappa(D) \leq 1$, *or* $\kappa(D) = 2$ *and some multiple* $|h(D-F)|$ *has no fixed part.*

DEFINITION 2.7. (1) A curve $C = C_1 + \cdots + C_r$ on an algebraic surface X is called *negative definite* if the intersection matrix (C_iC_j) of C is negative definite.

(2) A curve C is called *rational* if for any effective divisor $D = n_1C_1 + \cdots + n_rC_r \neq 0$, we have $p_a(D) \leq 0$.

(3) If $A^2 > 0$, then the maximal reduced divisor $C = C_1 + \cdots + C_r$ with $CA = 0$ is called the *exceptional curve* of A. We denote it by $E(A) = C$.

THEOREM 2.8. (Artin [2, 3]) *Let $C = C_1 + \cdots + C_r$ be a negative definite connected curve on an algebraic surface X.*

(1) *There is a unique effective divisor $Z = n_1C_1 + \cdots + n_rC_r$ such that $ZC_i \leq 0$ and Z is minimal.* (*Z is called the* **fundamental cycle** *of C*). *In fact, $Z \geq C$.*

(2) $p_a(Z) \geq 0$, *and* $p_a(Z) = 0$ *if and only if C is rational.*

THEOREM 2.9. (Artin's projective contraction theorem [2, 3]) *A negative definite curve $C = C_1 + \cdots + C_r$ on a projective surface X is rational if and only if C can be contracted to rational singular points on a projective surface Y.* (*The singular points on Y are called rational if* $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$).

Benveniste [6] generalized a result of Zariski ([32], Theorem 6.1).

THEOREM 2.10. (Benveniste [6]) *Suppose $C = C_1 + \cdots + C_r$ is a connected component of $E(A)$ for some nef and big divisor A. If C is rational and $n \gg 0$, then C can not be a fixed component of $|nA|$.*

3. Some technique results. Reider's method is usually used to study the adjoint linear system $|K_X+L|$ for a nef and big divisor L. In our case, L is not necessarily nef. Because there is no reference of this method for the general case, we shall present in this section the generalization of Reider's method so that Bogomolov's inequality can be used in the general case.

We use the notion "k points" for any zero-dimensional subscheme of length k, not requiring the points to be distinct.

Given a subscheme $Z' \subset Z$, the "complement" Z'' of Z' in Z is the canonical closed subscheme $Z'' \subset Z$ with an ideal sheaf $\mathcal{I}_{Z''} = [\mathcal{I}_Z : \mathcal{I}_{Z'}]$, i.e., for any open set $U \subset X$,

$$\mathcal{I}_{Z''}(U) := \{g \in \mathcal{O}_X(U) \mid g\mathcal{I}_{Z'}(U) \subset \mathcal{I}_Z(U)\}.$$

We call Z'' the *residual subscheme* of Z' in Z and denote it by

$$Z'' = Z - Z'.$$

Assume that Z is a local complete intersection, and Z'' is the residual of $Z' \subset Z$ in Z. Then Z' is the residual of Z'' in Z. Furthermore, we have

$$\deg Z' + \deg Z'' = \deg Z.$$

Note that in the surface case, the 4 equivalent conditions in the following theorem imply that Δ is a local complete intersection.

THEOREM 3.1. *Let Δ be a zero-dimensional subscheme of X (including empty set) and let L be a divisor on X. Then the following conditions are equivalent.*

(1) *There is a rank two vector bundle E with a non zero global section δ satisfying*

$$Z(\delta) = \Delta, \quad \det E = L. \tag{A}$$

(2) *There are 3 curves F_1, F_2 and F_3 such that F_1 and F_2 have no common components, and*

$$\begin{cases} \Delta = F_1 \cap F_2 - F_1 \cap F_2 \cap F_3, \\ L \equiv F_1 + F_2 - F_3. \end{cases} \tag{B}$$

(3) *There exists a rank two vector bundle $\mathcal{E}$ with a global section s such that $\dim Z(s) = 0$ and*

$$\begin{cases} \Delta = Z(s) - Z(s) \cap F, \\ L \equiv \det \mathcal{E} - F. \end{cases} \tag{C}$$

(4) *Either $\Delta = \emptyset$ or there is an element η in $H^1(\mathcal{I}_\Delta(K_X + L))^\vee$ such that for any subscheme (including empty set) $\Delta' \subsetneq \Delta$, η is not in the image of the following natural inclusion map:*

$$H^1(\mathcal{I}_{\Delta'}(K_X + L))^\vee \hookrightarrow H^1(\mathcal{I}_\Delta(K_X + L))^\vee.$$

Equivalently,

$$\bigcup_{\Delta' \subsetneq \Delta} H^1(\mathcal{I}_{\Delta'}(K_X + L))^\vee \subsetneq H^1(\mathcal{I}_\Delta(K_X + L))^\vee. \tag{D}$$

(See [31] for the details of the proof).

REMARK 3.2. In the above correspondence, if $\Delta = \emptyset$, then the following trivial cases correspond to each other:

(1) $E = \mathcal{O}_X \oplus \mathcal{O}_X(L)$;
(2) $f = af_1 + bf_2$ for some sections a and b of line bundles;
(3) $f \in \operatorname{im}(s)$ is in the image of s (we do not go to the details of this condition);
(4) $\eta = 0$.

We would like to mention the implication from (2) to (1) which will be used in the proof of Lemma 4.9. Denote by f_i the global section of $\mathcal{O}_X(F_i)$ defining F_i. Let $\mathcal{F}$ be the syzygy sheaf of (f_1, f_2, f_3),

$$0 \longrightarrow \mathcal{F} \longrightarrow \oplus_{i=1}^3 \mathcal{O}_X(-F_i) \xrightarrow{f} \mathcal{O}_X, \tag{3.1}$$

where f is defined by $f(x, y, z) = f_1 x + f_2 y + f_3 z$, and let $E = \mathcal{F}(F_1 + F_2)$. One can prove that $\det E = F_1 + F_2 - F_3 = L$ and E has a global section δ such that

$$Z(\delta) = F_1 \cap F_2 - F_1 \cap F_2 \cap F_3 = \Delta.$$

DEFINITION 3.3. We say that Δ satisfies Cayley-Bacharach property with respect to $|K_X + L|$ if for any F in $|K_X + L|$ and for any subscheme $\Delta' \subset \Delta$ with $\deg \Delta' = \deg \Delta - 1$, F contains Δ' implies that F contains Δ. Equivalently, for any such Δ',

$$\dim H^1(\mathcal{I}_{\Delta'}(K_X + L))^\vee < \dim H^1(\mathcal{I}_\Delta(K_X + L))^\vee. \tag{D'}$$

(D) implies (D').

LEMMA 3.4. *If Δ is reduced or* $\deg \Delta \leq 2$, *then (D') is equivalent to (D).*

Proof. In the two cases, Δ admits at most a finite number of subschemes Δ' with $\deg \Delta' = \deg \Delta - 1$, so (D') implies (D). Indeed, if Δ is reduced, the finiteness is obvious. If Δ is a non-reduced zero-dimensional scheme of degree 2, and if p is a point on Δ, then it is easy to prove that $I_\Delta = (x, y^2)$, where x and y are some local coordinates of X near $p = (0, 0)$. So Δ contains only one subscheme p of degree 1. ∎

COROLLARY 3.5. *Let L be a fixed divisor on X and $k \geq 1$ a fixed integer. Then the following conditions are equivalent.*

(1) *(A) (equivalently (B) or (C)) has no solution for any $\Delta \neq \emptyset$ with* $\deg \Delta \leq k$.
(2) *For any zero dimensional subscheme $\Delta \neq \emptyset$ of degree $\leq k$,*

$$H^1(\mathcal{O}_X(K_X + L))^\vee = H^1(\mathcal{I}_\Delta(K_X + L))^\vee.$$

(3) *Any zero dimensional subscheme $\Delta \neq \emptyset$ of degree $\leq k$ gives* $\deg \Delta$ *independent conditions on $|K_X + L|$. Namely, $|K_X + L|$ is $(k-1)$-very ample.*

The author and E. Viehweg ([29, 31, 30]) prove that the Cayley-Bacharach theorem for an n-dimensional projective manifold is equivalent to the k-very ampleness of $|K_X + L|$.

THEOREM 3.6. (Bogomolov [7]) *Let E be a rank two vector bundle on an algebraic surface X. If $c_1(E)^2 > 4c_2(E)$, then there is an invertible subsheaf $\mathcal{O}_X(M) \subset E$ such that*

(1) $(2M - c_1(E))H > 0$ *for any ample divisor* H*;*
(2) $(2M - c_1(E))^2 \geq c_1^2(E) - 4c_2(E)$*;*
(3) *for any nef divisor* A,

$$MA \geq \frac{1}{2}c_1(E)A + \frac{1}{2}\sqrt{A^2}\sqrt{c_1^2(E) - 4c_2(E)}\ .$$

(3) follows from (2) and Hodge index theorem.

LEMMA 3.7. *Let* E *be a rank two vector bundle on* X*, and let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be two different maximal invertible subsheaves of* E*. Then there exists an effective divisor* D *on* X *such that*

$$c_1(E) - c_1(\mathcal{M}_1) - c_1(\mathcal{M}_2) \equiv D.$$

Furthermore, $D = 0$ *if and only if* $E = \mathcal{M}_1 \oplus \mathcal{M}_2$.

See [29] for the proof of this lemma.

THEOREM 3.8. *Let* L *be a divisor on* X *such that* $L^2 > 0$ *and* $LH \geq 0$ *for some ample divisor* H*. Assume that* Δ *is empty or a zero dimensional subscheme of* X *satisfying one of the equivalent conditions of Theorem 3.1. If* $L^2 > 4\deg\Delta$*, then either*

(1) $\eta = 0$ (*and so* $\Delta = \emptyset$*, this corresponds to the trivial cases, see Remark 3.2*)*; or*
(2) $\eta \neq 0$*, and there exists an effective divisor* $D \neq 0$ *passing through* Δ *such that for any nef and big divisor* A,

$$DL - \deg\Delta \leq D^2 < \frac{\ell}{2}DA \leq \frac{\ell}{4}\left(AL - \sqrt{A^2}\sqrt{L^2 - 4\deg\Delta}\right),$$

where $\ell = AL/A^2 > 0$.

Proof. From the assumption and Hodge index theorem, we see that $LH > 0$. By Riemann-Roch theorem, we can prove easily that for a sufficiently large n, $h^0(nL) > 0$. Hence for any nef and big divisor A, $LA \geq 0$. Since $L^2 > 0$, $LA > 0$.

We assume that the equivalent condition (1) of Theorem 3.1 is true. Namely there is a rank two vector bundle E with a non zero global section δ such that $Z(\delta) = \Delta$, and $\det E = L$. Thus E admits a maximal invertible subsheaf $\mathcal{M}_1 \cong \mathcal{O}_X$,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{I}_\Delta(L) \longrightarrow 0.$$

From the assumption, $c_1^2(E) = L^2 > 4\deg\Delta = 4c_2(E)$. Hence E is not semistable. By Theorem 3.6, E admits a new maximal invertible sheaf $\mathcal{M}_2 = \mathcal{O}(M)$ satisfying the three inequalities in Theorem 3.6. From Theorem 3.6 (1) and $LH > 0$ for any ample divisor, we see that $M \not\equiv 0$, so $\mathcal{M}_1 \neq \mathcal{M}_2$. Now by Lemma 3.7, there exists an effective divisor $D \equiv L - M$ passing through Δ. Substitute $M = L - D$ into the second and third inequality of Theorem 3.6, we get

$$DL - \deg\Delta \leq D^2,$$
$$DA \leq \frac{1}{2}\left(LA - \sqrt{A^2}\sqrt{L^2 - 4\deg\Delta}\right).$$

By Lemma 3.7, if $\eta \neq 0$, then $D \neq 0$. In fact, we only need to prove that $D^2 < \frac{\ell}{2}DA$. If $DA = 0$, then $D^2 < 0 = \frac{\ell}{2}DA$ by Hodge index theorem. If $DA > 0$, also by Hodge index theorem and (2),

$$D^2A^2 \leq (DA)^2 < DA \cdot LA/2,$$

so $D^2 < \ell DA/2$. □

Corollary 3.9. (Beltrametti and Sommese [4]) *As in Theorem 3.8, if L is nef and big and $\Delta \neq \emptyset$, then we have*

$$DL - \deg \Delta \leq D^2 < \frac{1}{2}DL < \deg \Delta,$$

Lemma 3.10. *Let $C = C_1 + \cdots + C_r$ be a negative definite curve on X.*

(1) *The classes of the C_i are independent in $NS(X) \otimes \mathbb{Q}$.*

(2) *Let $\mathcal{E} = x_1C_1 + \cdots + x_rC_r$. If $\mathcal{E}C_i \leq 0$ (resp. < 0) for any i, then all x_i are nonnegative (resp. positive). (Hence $\mathcal{E}$ is an effective divisor).*

(3) *If $A^2 > 0$, then the number of curves C satisfying $AC = 0$ is finite. Hence $E(A)$ is well-defined.*

(4) *There is a nef and big divisor A such that $E(A) = C$.*

Proof. (1) The proof is well-known.

(2) Write $\mathcal{E}$ in the form $\mathcal{E} = A - B$, where A and B are effective divisors, without common components. We have $\mathcal{E}B \leq 0$ by assumption, hence $AB - B^2 \leq 0$. Since $AB \geq 0$ and $B^2 \leq 0$, it follows that $B^2 = 0$, and hence $B = 0$ since the subspace generated by $C_1, \cdots, C_r$ is negative definite.

(3) By Hodge index theorem, these curves span a negative subspace of $NS(X) \otimes \mathbb{Q}$. Thus the number is less or equal to the dimension of $NS(X) \otimes \mathbb{Q}$.

(4) Let H be a very ample divisor on X. Then we can find integers $x_1, \cdots, x_r$ such that

$$|\det(C_iC_j)| \cdot HC_k + (x_1C_1 + \cdots + x_rC_r)C_k = 0, \quad \text{for } k = 1, \cdots, r.$$

By (2), x_i are positive. Let $A = |\det(C_iC_j)|H + x_1C_1 + \cdots + x_rC_r$. Then A is nef and big and $E(A) = C$. □

4. Effective bounds. In this section, we fix a nef and big divisor A and an arbitrary divisor T. Denote by $C_1, \cdots, C_r$ the exceptional curves of A.

Theorem 4.1. *Let A be a nef and big divisor, let T be any fixed divisor, and let k be a nonnegative integer. Assume that either $n > k + \mathfrak{M}(A,T)$, or $n \geq \mathfrak{M}(A,T)$ when $k = 0$ and $T \sim K_X + \lambda A$ for some $\lambda \in \mathbb{Q}$. Suppose $|nA + T|$ is not $(k-1)$-very ample, i.e., there is a zero dimensional subscheme Δ on X with minimal degree $\deg \Delta \leq k$ such that it does not give independent conditions on $|nA + T|$. Then there is an effective divisor $D \neq 0$ containing Δ such that*

$$\begin{cases} D^2 + K_XD + k \geq TD, \\ DA = 0. \end{cases} \tag{4.1}$$

Proof. Let $L = nA + T - K_X$, i.e., $|nA + T| = |K_X + L|$. We claim that if $n > k + \mathfrak{M}(A,T)$, or $k = 0$, $n = \mathfrak{M}(A,T)$ and $T \sim K_X + \lambda A$, then

$$L^2 > 4k, \quad 0 \leq \frac{1}{2}\left(AL - \sqrt{A^2}\sqrt{L^2 - 4k}\right) < 1. \tag{4.2}$$

Indeed,

$$L^2 - 4k = A^2n^2 + 2A(T - K_X)n + (T - K_X)^2 - 4k,$$

the bigger root of the above quadratic polynomial of n is

$$n_k = \left(-A(T - K_X) + \sqrt{(A(T - K_X))^2 - (T - K_X)^2A^2 + 4kA^2}\right)/A^2.$$

Let $h := (A(T - K_X))^2 - A^2(T - K_X)^2$. By Hodge index theorem, $h \geq 0$, with equality if and only if there is a rational number λ such that $T \sim K_X + \lambda A$. Since

$$k + \mathfrak{M}(A, T) - n_k = \frac{1}{A^2} \cdot \left(1 - \frac{1}{2}\sqrt{h + 4kA^2}\right)^2 \geq 0,$$

we have $L^2 > 4k$ when $n > k + \mathfrak{M}(A, T) \geq n_k$.

Let

$$f(x) = x^2 - AL \cdot x + \frac{h}{4} + k \cdot A^2.$$

The smaller root of $f(x)$ is

$$x_1 = \frac{1}{2}\left(AL - \sqrt{A^2}\sqrt{L^2 - 4k}\right).$$

On the other hand,

$$f(0) \geq 0, \quad f(1) = A^2 \cdot (k + \mathfrak{M}(A, T) - n) < 0 \;\text{ when } n > k + \mathfrak{M}(A, T),$$

so $0 \leq x_1 < 1$ when $n > k + \mathfrak{M}(A, T)$.

Note that if $k = 0$, $h = 0$ and $n = \mathfrak{M}(A, T)$, then $\mathfrak{M}(A, T) > n_0$ and $f(0) = f(1) = 0$. So $x_1 = 0$ and (4.2) is also true. (In fact, this is the unique case where $x_1 < 1$ and $f(1) \geq 0$.) This proves the claim.

If $|nA+T|$ is not $(k-1)$-very ample, then there exists a minimal zero dimensional subscheme Δ (may be empty) with $\deg \Delta \leq k$ satisfying the conditions of Theorem 3.1 corresponding to the non-trivial cases (see Remark 3.2). Apply Theorem 3.8 (2) to Δ and $L = nA+T-K_X$, we get an effective divisor $D \neq 0$ containing Δ such that

$$\begin{gathered} DL - \deg \Delta \leq D^2, \\ 0 \leq DA \leq \frac{1}{2}\left(AL - \sqrt{A^2}\sqrt{L^2 - 4\deg \Delta}\right) \leq x_1 < 1. \end{gathered}$$

It implies $DA = 0$ and $D^2 + K_XD + k \geq TD$. $\square$

In general, if $x > 0$ and $f(x) < 0$, i.e.,

$$n > n_k + \frac{1}{A^2}\left(x + \frac{f(0)}{x} - 2\sqrt{f(0)}\right) = -\frac{A(T - K_X)}{A^2} + \frac{x}{A^2} + \frac{f(0)}{xA^2},$$

then $x_1 < x$, so $DA < x$.

The following are natural conditions on A or $E(A)$ such that (4.1) has no nonzero solutions D:

(A) A is ample (see Corollary 1.3). (Matsusaka's condition, $\tau = +\infty$).

(B) $TC_i \geq K_X C_i + k$, for any $i = 1, \cdots, r$. (Laufer-Ramanujam's condition, $\tau \geq k+1$).

(C) $E(A)$ is a rational curve. (Artin's condition, $\tau(A, 0) = 2$).

LEMMA 4.2. *For any integer $k \geq 0$, there are at most a finite number of effective divisors D satisfying* (4.1).

Proof. Let $C_1, \cdots, C_r$ be all of the curves satisfying $AC_i = 0$. We have known that these curves span a negative definite subspace W of $NS(X) \otimes \mathbb{Q}$. It is easy to see that the first inequality in (4.1) gives a bounded domain in W. Thus if $D = \sum_{i=1}^{r} n_i C_i$ satisfies the conditions in the lemma, then $(n_1, \cdots, n_r)$ must be in a bounded domain of $\mathbb{Q}^n$. This implies the lemma since n_i are integers. □

COROLLARY 4.3. *Assume that A is nef and big, and $k \geq 0$.*

(1) *If Laufer-Ramanujam condition is true for k and $n > k + \mathfrak{M}(A, T)$, then $|nA + T|$ is $(k-1)$-very ample.*

(2) *If $E(A)$ is a rational curve, then $h^1(nA) = 0$ for $n > \mathfrak{M}(A, 0)$, and $|nA|$ is base point free for $n > 1 + \mathfrak{M}(A, 0)$.*

Proof. This follows from Corollary 1.4. □

In general T may not satisfy Laufer-Ramanujam condition. We will modify it such that Laufer-Ramanujam condition is true.

Let $\sigma_i = \max\{KC_i - TC_i + k,\ 0\}$ for $i = 1, \cdots, r$. Because (C_iC_j) is a negative definite matrix, we can find an integral divisor

$$\mathcal{E}_k = \mathcal{E}_k(A, T, C_1 + \cdots + C_r) = x_1 C_1 + \cdots + x_r C_r$$

such that $\mathcal{E}_k C_i = -|\det(C_iC_j)| \cdot \sigma_i$. By Lemma 3.10, $\mathcal{E}_k$ is effective. Let $T' = T - \mathcal{E}_k$. Then Laufer-Ramanujam condition for k is true for T', i.e.,

$$(T - \mathcal{E}_k)C_i \geq K_X C_i + k, \quad i = 1, \cdots, r. \tag{4.3}$$

By the previous corollary, when $n > k + \mathfrak{M}(A, T - \mathcal{E}_k)$, $|nA + T - \mathcal{E}_k|$ is $(k-1)$-very ample.

THEOREM 4.4. *Assume that A is nef and big. Let T be any divisor.*

(1) *If $n > \mathfrak{M}(A, T - \mathcal{E}_0)$, then*

$$h^1(nA + T) = h^1(\mathcal{O}_{\mathcal{E}_0}(nA + T)).$$

Because $\mathcal{O}_{\mathcal{E}_0}(A)$ is a numerically trivial bundle, by ([10], *Theorem 8*), *$h^1(\mathcal{O}_{\mathcal{E}_0}(nA + T))$ is a periodic function of n.*

(2) *If $n > 1 + \mathfrak{M}(A, T - \mathcal{E}_1)$, then the fixed part B_n of $|nA + T|$ is bounded by $\mathcal{E}_1$, and B_n is a periodic divisor of n by* [11].

(3) *Let C' be a connected component of $E(A) = C' + C''$, and let $\mathcal{E}_1 = \mathcal{E}_1' + \mathcal{E}_1''$ be the corresponding decomposition* (*here $T = 0$*). *If C' is rational and $n > 1 + \mathfrak{M}(A, -\mathcal{E}_1'')$, then C' can not be the fixed part of $|nA|$. Namely, the fixed part of $|nA|$ is contained in $\mathcal{E}_1''$.* (*See Theorem 2.10 or* [6]).

Proof. (1) Since $n > \mathfrak{M}(A, T - \mathcal{E}_0)$, by the above corollary, $|nA + T - \mathcal{E}_0|$ is (-1)-very ample, i.e., $H^1(nA + T - \mathcal{E}_0) = 0$. In fact, if $n > \mathfrak{M}(A, T - \mathcal{E}_0)$, then $(K_X - nA - T + \mathcal{E}_0)A < 0$, so

$$H^2(nA + T - \mathcal{E}_0) \cong H^0(K_X - nA - T + \mathcal{E}_0)^\vee = 0.$$

From the long exact sequence of the following

$$0 \longrightarrow \mathcal{O}(nA+T-\mathcal{E}_0) \longrightarrow \mathcal{O}(nA+T) \longrightarrow \mathcal{O}_{\mathcal{E}_0}(nA+T) \longrightarrow 0,$$

we can see that

$$h^1(nA+T) = h^1(\mathcal{O}_{\mathcal{E}_0}(nA+T)).$$

Note that $\mathcal{E}_0$ is supported on these C_i with $C_iA = 0$. It has been proved in [10] that $h^1(\mathcal{O}_{\mathcal{E}_0}(nA+T))$ is a periodic function of n. This completes the proof.

(2) We take $k = 1$. If $n > 1 + \mathfrak{M}(A, T - \mathcal{E}_1)$, then $|nA + T - \mathcal{E}_1|$ is 0-very ample, so it has no fixed component. Thus the fixed part B_n of $|nA + T|$ is contained in $\mathcal{E}_1$, i.e., $\mathcal{E}_1 - B_n$ is effective. This completes the proof.

(3) If $\mathcal{E}_1' = 0$ or $\mathcal{E}_1'' = 0$, then (3) follows from (2) and the previous corollary (2). Otherwise, we claim that $|nA - \mathcal{E}_1''|$ is 0-very ample. Thus the fixed part of $|nA|$ is contained in $\mathcal{E}_1''$.

Indeed, with the notations of Theorem 4.1, if we write $D = D' + D''$ and let $T = 0$, then from

$$(-\mathcal{E}_1)C_j \geq K_X C_j + 1, \quad \text{for all } j$$

and $D' \cap D'' = \emptyset$, we get

$$\begin{cases} \mathcal{E}_1'' D'' + K_X D'' + 1 \leq 0, & \text{if } D'' \neq 0, \\ D'^2 + K_X D' = 2p_a(D'') - 2 \leq -2, & \text{if } D' \neq 0, \\ D''^2 \leq -1, & \text{if } D'' \neq 0. \end{cases}$$

Since $D = D' + D'' \neq 0$,

$$D^2 + DK_X + 1 - (-\mathcal{E}_1'')D = (D'^2 + D'K_X) + D''^2 + (D''K_X + 1 + \mathcal{E}_1''D'') < 0.$$

Now the claim is a consequence of Theorem 4.1. □

THEOREM 4.5. *Assume that $A^2 > 0$ and $|A|$ has no fixed part. Let T be any divisor.*

(1) *If $n > 1 + \mathfrak{M}(A, 0)$, then $|nA|$ has no base points.* (*Zariski* [32].)

(2) *If $n > 1 + \mathfrak{M}(A, 0)$ and $n > 1 + \mathfrak{M}(A, T - \mathcal{E}_1)$, then $h^1(nA+T) = h^1(\mathcal{O}_{\mathcal{E}_1}(T))$ is a constant, and $h^2(nA + T) = 0$. So*

$$h^0(nA+T) = \chi(nA+T) + s,$$

where $s = h^1(\mathcal{O}_{\mathcal{E}_1}(T))$ is a constant. (*Castelnuovo* [9]).

(3) *The fixed part B_n of $|nA + T|$ is a fixed divisor for $n \gg 0$.*

Proof. (1) Because $|A|$ has no fixed part, A is nef. If p is a base point of $|nA|$, then there is a curve D passing through p such that $DA = 0$. Because $|nA|$ has also no fixed part, this means that we can find a curve in $|nA|$ disjoint with D because $DA = 0$. This is impossible since p should be their common point.

(2) In this case, we can find a divisor D in $|nA|$ such that D is disjoint with the exceptional curve $E(A)$ of A. Thus $\mathcal{O}_{\mathcal{E}_1}(D) = \mathcal{O}_{\mathcal{E}_1}$. As in the proof of Theorem 4.4,

$$h^1(nA+T) = h^1(\mathcal{O}_{\mathcal{E}_1}(nA+T)) = h^1(\mathcal{O}_{\mathcal{E}_1}(D) \otimes \mathcal{O}_{\mathcal{E}_1}(T)) = h^1(\mathcal{O}_{\mathcal{E}_1}(T)).$$

Note that $n > 1+\mathfrak{M}(A, T-\mathcal{E}_1)$ implies $n > -A(T-\mathcal{E}_1-K_X)/A^2 = -A(T-K_X)/A^2$, so $A(K_X - nA - T) < 0$. Thus $h^2(nA+T) = h^0(K_X - nA - T) = 0$.

(3) Let M_n be the moving part of $|nA+T|$. Because $|A|$ has no fixed part, $|M_n+A|$ has also no fixed part. Since $(n+1)A + T = M_n + A + B_n$, we have $B_{n+1} \le B_n$. It implies that when $n \gg 0$, B_n is a fixed divisor. □

Zariski's criterion (Theorem 2.6) for the finite generation of $R[D]$ gives a criterion for projective contractability of a negative definite curve (see also [24]).

Corollary 4.6. (Criterion for Projective Contractability) *Let $C = C_1+\cdots+C_r$ be a negative definite curve. Then C can be contracted to (normal singular) points on a projective surface if and only if there is a nef and big divisor A such that $C = E(A)$ and $|nA|$ has no fixed part for some n.*

Corollary 4.7. (Artin [2]) *A negative definite and rational curve C on an algebraic surface can be contracted projectively.*

Proof. By Lemma 3.10, there is a nef and big divisor A such that $E(A) = C$. By Corollary 4.3, $|nA|$ is base point free for large n. Thus C can be contracted projectively. □

Theorem 4.8. *Let A be a nef and big divisor with exceptional curve $E(A) = C_1 + \cdots + C_r$, and let $T = 0$. Denote by $E_1, \cdots, E_s$ the connected components of $E(A)$. Let $\mathcal{E}_{0,i} = \mathcal{E}_0(A, 0, E_i)$, let Z_i be the fundamental cycle of E_i, and let*

$$\widetilde{\mathcal{E}}_{0,i} = \begin{cases} \mathcal{E}_{0,i}, & \text{if } \mathcal{E}_{0,i} \neq 0, \\ Z_i, & \text{if } \mathcal{E}_{0,i} = 0. \end{cases} \qquad \widetilde{\mathcal{E}}_0 = \sum_{i=1}^{s} \widetilde{\mathcal{E}}_{0,i}.$$

Assume that either $|A|$ has no base point, or $E(A)$ is rational. If

$$n > 2 + \mathfrak{M}(A, 0),$$

then

(1) *Φ_{nA} is a birational morphism onto a projective surface $\Sigma_n = \Phi_{nA}(X)$.*
(2) *On $X \setminus \cup_{i=1}^r C_i$, Φ_{nA} is an isomorphism.*
(3) *Φ_{nA} contracts the curves $E(A)$ to some (singular) points of Σ_n.*
(4) *Furthermore, if $E(A)$ is rational, then Φ_{nA} has connected fibers. In general, if $n > \mathfrak{M}(A, -\widetilde{\mathcal{E}}_0)$, then Φ_{nA} has also connected fibers.*

Proof. In fact, we only need to prove (4) that Φ_{nA} has connected fibers. The locus over which $|nA|$ is not very ample is contained in $E(A)$, so we only need to prove that $\Phi_{nA}(E_i) \neq \Phi_{nA}(E_j)$ when $i \neq j$.

By construction, $\mathcal{E}_{0,i} = \mathcal{E}_0(A, 0, E_i) = 0$ if and only if E_i consists of (-1)- and (-2)-curves. By definition, $\widetilde{\mathcal{E}}_{0,i} \neq 0$, and

$$-\widetilde{\mathcal{E}}_0 C_j \ge K_X C_j, \quad \text{for } j = 1, \cdots, r.$$

So $|nA - \widetilde{\mathcal{E}}_0|$ satisfies Laufer-Ramanujian condition for $k = 0$. Hence it is (-1)-very ample, i.e., $H^1(nA - \widetilde{\mathcal{E}}_0) = 0$.

Now we consider the exact sequence

$$0 \to \mathcal{O}(nA - \widetilde{\mathcal{E}}_0) \to \mathcal{O}(nA) \to \mathcal{O}_{\widetilde{\mathcal{E}}_0}(nA) \to 0. \tag{4.4}$$

Since $|nA|$ has no base point and $A\cdot E(A)=0$, the generic divisor $B\in|nA|$ does not contain any C_i, and hence disjoint with $E(A)$. We obtain

$$\mathcal{O}_{\widetilde{\mathcal{E}}_0}(nA)=\mathcal{O}_{\widetilde{\mathcal{E}}_0}(B)=\mathcal{O}_{\widetilde{\mathcal{E}}_0}=\bigoplus_{i=1}^{s}\mathcal{O}_{\widetilde{\mathcal{E}}_{0,i}}.$$

The long exact sequence of (4.4) gives us a surjective map

$$H^0(nA)\to\bigoplus_{i=1}^{s}H^0(\mathcal{O}_{\widetilde{\mathcal{E}}_{0,i}}),$$

so for each $i\neq j$, there is a section s in $H^0(nA)$ such that $s(\widetilde{\mathcal{E}}_{0,i})\neq 0$, $s(\widetilde{\mathcal{E}}_{0,j})=0$. Hence $|nA|$ separates E_i and E_j, which means that $\Phi_{nA}(E_i)\neq\Phi_{nA}(E_j)$.

If $E(A)$ is rational, then one can prove similarly that $H^1(nA-Z_i-Z_j)=0$ and $\Phi_{nA}(E_i)\neq\Phi_{nA}(E_j)$ provided

$$n>\mathfrak{M}(A,-Z_i-Z_j)=2+\mathfrak{M}(A,0)-\frac{m_i+m_j}{4}\quad\text{for any}\quad i\neq j,\tag{4.5}$$

where $m_i=-Z_i^2$ is the multiplicity of the normal rational singular point with exceptional curve E_i. On the other hand, condition (4.5) follows from our assumption $n>2+\mathfrak{M}(A,0)$. □

Let A be a divisor on X, and let

$$R_m:=H^0(X,\mathcal{O}_X(mA)),\quad R[A]:=\bigoplus_{m=0}^{\infty}R_m.$$

$R[A]$ admits naturally a graded ring structure. The generation of this ring was studied by Zariski [32] for any divisor A, and by Mumford [22], Kodaira [20, 21] and Bombieri [8] for the canonical divisor of surfaces of general type.

Lemma 4.9. *Assume that $A^2>0$, ℓ and p are two positive integers such that $|\ell A|$ has no base point, and $H^1(mA)=V$ is fixed for any $m\geq p$. Let $k=\ell^2A^2$ and let*

$$\mathfrak{N}(A,\ell,p):=\begin{cases}\max\left\{2\ell+p-1,\ 3\ell+\dfrac{K_XA}{A^2}\right\}, & \text{if}\ \ V=0,\\ \max\left\{2\ell+p-1,\ k+\mathfrak{M}(A,0)+\ell\right\}, & \text{if}\ \ V\neq 0,\end{cases}$$

If $m>\mathfrak{N}(A,\ell,p)$, then we have

$$R_m=R_\ell R_{m-\ell}.\tag{4.6}$$

Proof. Note first that A is nef and big. Let $E(A)=C_1+\cdots+C_r$. We choose three generic curves F_1, F_2 and F_3 in $|\ell A|$ such that they have no common zero point and do not contain any C_i. Then we see that $E(A)$ is disjoint with all F_i. We denote by f_i the global section defining F_i. Let $\mathcal{F}$ be the syzygy sheaf of (f_1,f_2,f_3) (see (3.1)),

$$0\longrightarrow\mathcal{F}\longrightarrow\oplus_{i=1}^{3}\mathcal{O}_X(-F_i)\longrightarrow\mathcal{O}_X\longrightarrow 0,\tag{4.7}$$

and let $E = \mathcal{F}(F_1 + F_2)$. We have known that $\det E = F_1 + F_2 - F_3 = \ell A$ and E has a global section δ such that $Z(\delta) = F_1 \cap F_2$,

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{I}_{F_1 \cap F_2}(\ell A) \longrightarrow 0. \tag{4.8}$$

From the conditions, we claim that

a) $H^1((m-\ell)A) = H^1((m-2\ell)A) = V$ if $m > 2\ell + p - 1$;

b) $H^2((m-3\ell)A) = H^0(K_X - (m-3\ell)A) = 0$ if $m > 3\ell + \dfrac{K_X A}{A^2}$;

c) $H^1(\mathcal{I}_{F_1\cap F_2}((m-\ell)A)) = V$ if $V = 0$ or $m > \ell + k + \mathfrak{M}(A, 0)$.

Indeed, we only need to prove c). From the long exact sequence of

$$0 \longrightarrow \mathcal{O}_X((m-3\ell)A) \longrightarrow \mathcal{O}_X((m-2\ell)A)^{\oplus 2} \overset{(f_1,f_2)}{\longrightarrow} \mathcal{I}_{F_1\cap F_2}(m-\ell) \longrightarrow 0,$$

we obtain

$$V^{\oplus 2} = H^1((m-2\ell)A)^{\oplus 2} \longrightarrow H^1(\mathcal{I}_{F_1\cap F_2}((m-\ell)A)) \longrightarrow H^2((m-3\ell)A) = 0.$$

Thus if $V = 0$, then $H^1(\mathcal{I}_{F_1\cap F_2}((m-\ell)A) = 0$. Now we consider the case $V \neq 0$. Suppose

$$V = H^1(\mathcal{O}_X((m-\ell)A)) \subsetneq H^1(\mathcal{I}_{F_1\cap F_2}((m-\ell)A)).$$

Then $F_1 \cap F_2$ violates the $(k-1)$-very ampleness of $(m-\ell)A$ ($k = \deg(F_1 \cap F_2)$), so there exists a minimal non empty subscheme $\Delta \subset F_1 \cap F_2$ violating the $(k-1)$-very ampleness. Because $m - \ell > k + \mathfrak{M}(A, 0)$, by Theorem 4.1 we have a curve $D \neq 0$ passing through Δ such that (4.1) is true, hence the support of Δ is contained in both F_1 and $E(A)$, which contradicts the choice of F_1. This completes the proof of c).

Similarly, consider the long exact sequence of (4.8)$\otimes\mathcal{O}((m-2\ell)A)$, we get

$$V = H^1((m-2\ell)A) \to H^1(E((m-2\ell)A)) \to H^1(\mathcal{I}_{F_1\cap F_2}((m-\ell)A)) = V.$$

So

$$h^1(\mathcal{F}(mA)) = h^1(E((m-2\ell)A)) \leq 2\dim V. \tag{4.9}$$

The long exact sequence of (4.7)$\otimes\mathcal{O}_X(mA)$ gives us the following

$$\begin{aligned} &\oplus_{i=1}^3 H^0((m-\ell)A) \overset{(f_1,f_2,f_3)}{\longrightarrow} H^0(mA) \longrightarrow H^1(\mathcal{F}(m)) \to \\ \overset{\alpha}{\to}\ &\oplus_{i=1}^3 H^1((m-\ell)A) \longrightarrow H^1(mA) \longrightarrow H^2(\mathcal{F}(m)). \end{aligned}$$

We can see by (4.9) that α is injective, equivalently (f_1, f_2, f_3) is surjective, namely

$$R_m = f_1 R_{m-\ell} + f_2 R_{m-\ell} + f_3 R_{m-\ell}.$$

This completes the proof. ◻

The following theorem is about the projective normality of $\Phi_{mA}(X)$ (see [8], Theorem 3A).

Theorem 4.10. *Let A be a nef divisor with $A^2 > 0$ such that either $|A|$ has no fixed part or $E(A)$ is rational. Let*

$$\ell = 1 + \mathfrak{m}(A, 0),$$

$$p = \begin{cases} \mathfrak{m}(A, 0), & \text{if } E(A) \text{ is rational,} \\ 1 + \max\{\mathfrak{m}(A, 0), \mathfrak{m}(A, -\mathcal{E}_1)\}, & \text{if } |A| \text{ has no fixed part.} \end{cases}$$

Assume that

$$2m > \begin{cases} \max\left\{ 2\ell+p+1,\ 3\ell+3+\dfrac{K_XA}{A^2} \right\}, & \text{if } E(A) \text{ is rational,} \\ \max\{ 2\ell+p+1,\ k+\mathfrak{M}(A,0)+\ell+1 \}, & \text{if } |A| \text{ has no fixed part.} \end{cases}$$

Then for any $n \geq 1$,

$$R_{nm} = R_m^n.$$

Proof. Under the conditions, $|\ell A|$ and $|(\ell+1)A|$ have no base point, and $h^1(mA) = V$ is fixed when $m \geq p$. By the previous lemma, if $m > \mathfrak{N}(A, \ell+1, p)$, then we have

$$R_m = R_\ell R_{m-\ell}, \quad R_m = R_{\ell+1} R_{m-\ell-1}.$$

Now we claim that for any $n \geq 1$,

$$R_{nm} = R_m^n.$$

Indeed, we can write $m = s\ell + t(\ell+1)$ for some non negative s and t, (e.g., $s = \left[\frac{m}{\ell}\right](\ell+1) - m$ and $t = m - \left[\frac{m}{\ell}\right]\ell$). We can assume that $n \geq 2$. From the assumption $nm \geq 2m > \mathfrak{N}(A, \ell+1, p)$, so the previous lemma gives us that

$$R_{nm} = R_{(n-1)m+s\ell+t(\ell+1)} = R_{(n-1)m} R_\ell^s R_{\ell+1}^t = R_{(n-1)m} R_m.$$

By induction on n, we have $R_{nm} = R_m^n$ for any $n \geq 1$. □

Example 4.11. Let $\pi : X \to \mathbb{P}^2$ be a double cover ramified over a smooth curve B of degree $2d$, and let $H = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ be the pullback divisor of a line. Then nH is very ample if and only if $n \geq d$. (Note that if $n < d$, then $|nH| = \pi^*|\mathcal{O}_{\mathbb{P}^2}(n)|$, which implies that the map defined by $|nH|$ factorizes through the double cover π, so nH is not very ample). On the other hand, $K_X \equiv (d-3)H$ and $H^2 = 2$. Thus $\mathfrak{M}(H,0) = d - \frac{5}{2}$, hence $\mathfrak{m}(H,0) = d-2$. In particular, $n > 2 + \mathfrak{M}(H,0)$ iff $n \geq d$. Therefore, our bound in Corollary 1.3 can not be improved for X and H.

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