Invariant criteria for existence of bounded positive solutions

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Abstract

We consider semilinear elliptic equations $\Delta u \pm \rho(x)f(u) = 0$, or more generally $\Delta u + \varphi(x,u) = 0$, posed in $\mathbb{R}^N$ $(N \geq 3)$. We prove that the existence of bounded positive entire solutions is closely related to the existence of bounded solution for $\Delta u + \rho(x) = 0$ in $\mathbb{R}^N$. Many sufficient conditions which are invariant under the isometry group of $\mathbb{R}^N$ are established. Our proofs use the standard barrier method, but our results extend many earlier works in this direction. Our ideas can also be applied for the existence of large solutions and for the system cases.

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1 Introduction

We consider the semilinear elliptic equation $\Delta u \pm \rho(x)f(u) = 0$, or more generally $\Delta u + \varphi(x,u) = 0$ in $\mathbb{R}^N$, we are interested in the sufficient conditions on $\varphi(x,u)$ for the existence of entire bounded positive solutions. Since no bounded entire super-harmonic function exists in dimension 1 or 2, we assume that $N \geq 3$ in all this note.

In the pioneer work [16], Ni considered the equation $\Delta u + K(x)u^{\frac{N+2}{N-2}} = 0$ in $\mathbb{R}^N$ where $N \geq 3$. Using explicit barrier functions, he proved in particular that, if $K(x)$ is a bounded locally Hölder continuous function such that $|K(x)| \leq C/|x|^l$ for some $l > 2$ in $\mathbb{R}^N$, then we have infinitely many bounded positive solutions, with the property that each of these solutions is bounded from below by a positive constant. Ni indicated also that similar results can be proved if there holds

$$K(x) \sim \frac{C}{r^2(\ln r)^2} \quad \text{as } r = |x| \to \infty.$$

Many works have been done after Ni's result to study equations $\Delta u \pm \rho(x)f(u) = 0$, and the following condition has been often used to generalize Ni's condition :

$$\int_0^\infty r\Psi(r)dr < \infty, \quad \text{where } \Psi(r) = \max_{|x|=r} |\rho(x)|. \tag{1}$$

But this kind of condition has a shortage, that is, it is not invariant under the isometry group $\mathbb{G}$ of $\mathbb{R}^N$, in particular under translations. More precisely, for some ρ verifying (1), we can have $x_0 \in \mathbb{R}^N$ such that it is no longer true for $\rho(x + x_0)$. On the other hand, the existence of entire solutions is clearly unchanged under $\mathbb{G}$, that means if the equation is resolvable with $\rho(x)f(u)$, thanks to the invariance under $\mathbb{G}$ of Laplacian operator, it is the same with $(\rho \circ T)f(u)$ for *any* $T \in \mathbb{G}$. Moreover, we can remark that Ni's original condition on $K(x)$ is invariant under $\mathbb{G}$, so it is interesting and natural to search some more general sufficient existence conditions which are invariant with respect to the group of isometries of $\mathbb{R}^N$.

The main proposal of our work is to prove that, in many situations, (1) can be replaced by the following more general condition :

$$-\Delta U = \rho(x) \quad \text{has a bounded solution in } \mathbb{R}^N. \tag{2}$$

Clearly, (2) is invariant under $\mathbb{G}$. We will remark that in $\mathbb{R}^N$ $(N \geq 3)$, (1) implies (2), while the inverse is wrong in general (see section 2). We will use the condition of type (2) to generalize many existence results for the equation $\Delta u + \varphi(x, u) = 0$ where $|\varphi(x, u)| \leq \rho(x)f(u)$. The idea to relate the resolution of the semilinear equation $\Delta u \pm \rho(x)f(u) = 0$ to the linear problem (2) was already used in [2, 4, 9], but always in some specific cases.

We will use the classical barrier method, since our main purpose does not concern the regularity of $\varphi(x, u)$, so we suppose in general that $\varphi(x, u)$ is a Caratheodory function such that Hess's result in [7] works, even we know that using other lower-upper solution approaches, we can weaken sometimes the conditions on $\varphi(x, u)$. The same remark goes also for ρ and f. More precisely, we just mention the following lemma, which can be proved by using Theorem 1 in [7] and the standard diagonal process (cf. e.g. [16], Theorem 2.10).

Lemma 1 *Let $\varphi(x, u)$ be a Caratheodory function in $\mathbb{R}^N \times (0, \infty)$ and locally bounded, if we have u_1 and u_2 two bounded functions in $\mathbb{R}^N$ such that $\Delta u_1 + \varphi(x, u_1) \geq 0$, $\Delta u_2 + \varphi(x, u_2) \leq 0$ and $u_1 \leq u_2$ in $\mathcal{D}'(\mathbb{R}^N)$, then we have a solution u such that $\Delta u + \varphi(x, u) = 0$ and $u_1 \leq u \leq u_2$.*

The paper is organized as follows: we point out the differences between conditions (1) and (2) in the next section, then we give our main results in section 3. Finally, we use our idea to study some other situations in section 4.

2 Comparison between conditions (1) and (2)

Here we give some simple remarks to point out the defect of condition (1). Firstly, we note that (1) is not invariant under translations, which can be shown as follows: let $\rho_0(x) = \rho_0(|x|)$ be a positive, regular and radially symmetric function satisfying (1) and such that $\rho_0(l) = 1$ for any $l \in \mathbb{N}$. So for $x_0 \in \mathbb{R}^N$, $|x_0| \geq 1/2$, if we denote $\rho_1(x) = \rho_0(x + x_0)$, then $\forall\, r \geq |x_0|$,

$$\Psi_1(r) = \max_{|x|=r} |\rho_1(x)| = \max_{[r-|x_0|, r+|x_0|]} \rho_0(s) \geq 1,$$

hence ρ_1 or Ψ_1 does not satisfy no longer (1), when the length of translation is more than 1/2.

Furthermore, the condition (1) implies easily

$$-\Delta U = \rho(x) \quad \text{has a ground state solution in } \mathbb{R}^N, \tag{3}$$

i.e. U is bounded and $\lim_{|x|\to\infty} U(x) = 0$. Because

$$V(x) = \int_{|x|}^{\infty} \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} \Psi_1(\sigma) d\sigma$$

verifies $-\Delta V = \Psi_1(r)$ in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} V(x) = 0$, so V is a upper solution for (3). But the inverse is not true. In fact we can construct a function ρ_2 satisfying (3) such that for *any*

$T \in \mathbb{G}$, the transformed function $\rho_2 \circ T$ does not verify the condition (1). This means that a real difference does exist between (1) and (2), since (3) is obviously stronger than (2).

More precisely, let $\rho_2(x) = \rho_0(x) + \rho_0(x - e_1)$ with the above given function ρ_0 and an arbitrary unit vector e_1. Since $-\Delta U = \rho_0$ has a ground state solution, so (3) is satisfied for ρ_2. But for any $x_0 \in \mathbb{R}^N$, either $|x_0| \geq 1/2$ or $|x_0 - e_1| \geq 1/2$ holds. Taking $y_0 = x_0$ or $y_0 = x_0 - e_1$ with $|y_0| \geq 1/2$, we have always $\rho_2(x + x_0) \geq \rho_0(x + y_0)$, thus the corresponding $\Psi_2(r) = \max_{|x|=r} |\rho_2(x + x_0)|$ will never satisfy the condition (1)!

Moreover, since the bounded solution of $-\Delta U = \rho(x)$ is not unique by adding a constant and $\rho(x)$ in this note is generally positive, we will often consider the unique positive solution of

$$-\Delta U = \rho(x) \quad \text{in } \mathbb{R}^N, \quad \lim_{r\to\infty} \fint_{B_r} U d\sigma = 0, \tag{4}$$

where $\fint_{B_r}$ denotes the average on the ball of center 0 and radius r. It is clear that the condition (4) is invariant under $\mathbb{G}$, i.e. $\forall\, T \in \mathbb{G}$, the unique solution of (4) corresponding to $\rho \circ T$ is just given by $U \circ T$.

3 Main results

We consider semilinear elliptic equations of the following form

$$\Delta u + \varphi(x, u) = 0 \tag{5}$$

in $\mathbb{R}^N$ ($N \geq 3$). We will prove the following results:

Theorem 1 *Let $\varphi(x, u)$ verify $|\varphi(x, u)| \leq \rho(x)f(u)$ in $\mathbb{R}^N \times (0, \infty)$ where $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ such that (2) holds. Suppose moreover that f is continuous in $(0, \infty)$ satisfying*

$$\lim_{u\to 0_+} \frac{f(u)}{u} = 0 \quad or \quad \lim_{u\to\infty} \frac{f(u)}{u} = 0, \tag{6}$$

then equation (5) possesses infinitely many bounded positive solutions in $\mathbb{R}^N$, and each of these solutions is also bounded from below by a positive constant.

Proof. Let U be the bounded solution in $\mathbb{R}^N$ of problem (4), since $\rho(x) \geq 0$, so U is non negative in $\mathbb{R}^N$. Taking any positive constant C such that $V_\pm = \pm U + C$ satisfy $C_1 \geq V_+ \geq V_- \geq C_2 > 0$ for some $C_1, C_2 > 0$. We will take $aV_\pm(x)$ respectively as super-solution and sub-solution for the equation (5) by choosing appropriately a. For this end, if $\lim_{u\to 0_+} f(u)/u = 0$, we choose $a > 0$ and small enough, such that $f(t)/t \leq C_1^{-1}$, for any $t \leq aC_1$. Hence

$$\Delta(aV_+) + \varphi(x, aV_+) \leq -a\rho(x) + \rho(x)f(aV_+) = -a\rho(x)\left[1 - V_+\frac{f(aV_+)}{aV_+}\right] \leq 0. \tag{7}$$

That means aV_+ is a super-solution of equation (5). A similar argument shows that aV_- will be a sub-solution of (5) for a positive and small enough. Thus by Lemma 1, there exists a solution u of (5) satisfying $aV_+ \geq u \geq aV_- \geq aC_2 > 0$. Since we can choose $a > 0$ arbitrarily small, we see that equation (5) possesses infinitely many bounded positive solutions.

If $\lim_{u\to\infty} f(u)/u = 0$ holds, it suffices to choose $a > 0$ large enough, and then the above arguments work. ∎

Remark If the linear problem $-\Delta U = \rho(x)$ has a ground state solution in $\mathbb{R}^N$, i.e. we have (3), then there exists bounded positive entire solutions u satisfying $\lim_{|x|\to\infty} u(x) = C$ for infinitely many $C > 0$, since in our proof, $\lim_{|x|\to\infty} V_+ = \lim_{|x|\to\infty} V_- = C$. In some particular case, we can get a more precise result (see Theorem 6).

The above theorem is an extension of many earlier works, we cannot mention all of them, so among others, see Ni [16], Kusano-Oharu [10], Brezis-Kamin [2], Lair-Shaker [11] and Naito [15]. In particular, we see that our proof is very short and the condition (6) holds for classical superlinear case $f(u) = u^p$, $p > 1$; sublinear case $f(u) = u^p$, $0 < p < 1$ or singular case $f(u) = u^\gamma$, $\gamma < 0$. We emphasize also that we do not impose any conditions on the monotonicity of function f, on the precise asymptotic behavior of $\varphi(x, u)$ when $|x| \to \infty$, or on the sign of $\varphi(x, u)$.

On the other hand, suppose that a bounded positive solution exists for $-\Delta u = \rho(x)f(u)$ with $\rho(x) \geq 0$ and f is positive in $(0, \infty)$. If u is bounded from below by a positive constant, since $f(u) \geq C > 0$, then $C^{-1}u$ will be a super-solution for (4) since $\rho \geq 0$, hence (4) is resolvable, so the condition (2) is even necessary in this special case.

Our result answers also a question arised by Lair and Shaker (Remark 3, [12]) : For the sublinear equation $\Delta u = \rho(x)u^p$ $(0 < p < 1)$ with $\rho \geq 0$, when

$$\int_0^\infty r\Phi(r)dr < \infty, \quad \text{and} \quad \int_0^\infty r\Psi(r)dr = \infty, \tag{8}$$

where $\Phi(r) = \min_{|x|=r} \rho(x)$ and $\Psi(r) = \max_{|x|=r} \rho(x)$, are there bounded positive solutions in $\mathbb{R}^N$? Our analysis shows that the above condition (8) is neither sufficient nor necessary, so it is not directly related to the existence of bounded solutions.

The next result shows that for any controllable nonlinearity $\varphi(x, u)$, we have always bounded solutions when the equation (4) has a small enough solution. More precisely, we have

Theorem 2 *Let $\varphi(x, u)$ satisfy $|\varphi(x, u)| \leq \rho(x)f(u)$ on $\mathbb{R}^N \times (0, \infty)$ where $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ and f is a continuous function defined on $(0, \infty)$. There exists a positive constant ε_0 depending on f, such that if the problem (4) has a bounded solution U satisfying $U < \varepsilon_0$, then equation (5) possesses infinitely many bounded positive solutions in $\mathbb{R}^N$, which are bounded from below by a positive constant.*

Proof. For any $r_0 > 0$, we define $h(r) = \max_{s\in[r_0,r]} f(s)$ in $[r_0, \infty)$. Without loss of generality, we can suppose that h is positive, otherwise we have trivial positive constant solution. Thus h is positive, nondecreasing and satisfies $f \leq h$ on $[r_0, \infty)$. For arbitrary positive number a, we set $\varepsilon_1 = a/h(a + r_0)$ and we claim that if $\|U\|_\infty \leq \varepsilon_1$, then $v_1(x) = r_0 + a - ac^{-1}U(x)$ with $c = \|U\|_\infty$ is a sub-solution of (5). In fact,

$$\Delta v_1 + \varphi(x, v_1) \geq \Delta v_1 - \rho(x)f(v_1) = \rho(x)\left[ac^{-1} - f\left(r_0 + a - ac^{-1}U(x)\right)\right]. \tag{9}$$

Thus if $\|U\|_\infty \leq \varepsilon_1$, we get then

$$ac^{-1} \geq \frac{a}{\varepsilon_1} = h(a + r_0) \geq h\left(r_0 + a - ac^{-1}U(x)\right) \geq f\left(r_0 + a - ac^{-1}U(x)\right)$$

since U is positive and $h \geq f$ in $[r_0, \infty)$. So we have $\Delta v_1 + \varphi(x, v_1) \geq 0$. Similarly, if we take $v_2(x) = r_0 + a + ac^{-1}U(x)$, and if $c = \|U\|_\infty \leq \varepsilon_2 = a/h(2a + r_0)$, we obtain

$$\Delta v_2 = -ac^{-1}\rho(x) \leq -\rho(x)h(2a + r_0) \leq -\rho(x)f\left(r_0 + a + ac^{-1}U(x)\right) \leq -\varphi(x, v_2),$$

which means that v_2 is a super-solution of (5). Thus when $\|U\|_\infty \leq \varepsilon_2 \leq \varepsilon_1$, since $v_2 \geq v_1$, Lemma 1 gives a solution u of (5) verifying $v_1 \leq u \leq v_2$. Moreover, we have

$$\lim_{r\to\infty} \fint_{B_r} u d\sigma = r_0 + a. \tag{10}$$

Finally, by the monotonicity of h, if we fix a and $\varepsilon_0 = a/h(2a + 2r_0)$, the same proof works for any $r' \in [r_0, 2r_0]$, and (10) shows that (5) possesses infinitely many bounded positive solutions if $\|U\|_\infty \leq \varepsilon_0$. ■

In the following, we will give more precise estimates on ε_0 for some particular cases.

Theorem 3 *Let $\varphi(x, u)$ satisfy $0 \leq \varphi(x, u) \leq \rho(x)f(u)$ on $\mathbb{R}^N \times (0, \infty)$ for some positive nondecreasing and continuous function f defined on $(0, \infty)$ and $\rho \in L^\infty_{loc}(\mathbb{R}^N)$. Suppose moreover that the solution of (4) exists and verifies*

$$\|U\|_\infty < \int_0^\infty \frac{dt}{f(t)}. \tag{11}$$

Then the equation

$$\Delta u = \varphi(x, u) \quad in \quad \mathbb{R}^N \tag{12}$$

possesses infinitely many bounded positive solutions in $\mathbb{R}^N$, which are bounded from below by positive constant.

Proof. By hypothesis, there are positive numbers $a < b < \infty$ such that

$$0 \leq U < \int_a^b \frac{dt}{f(t)} < \infty.$$

We define then a function v with values in $[a, b]$ such that

$$U(x) = \int_{v(x)}^b \frac{dt}{f(t)}.$$

Clearly, v is well defined. Then a direct computation shows that

$$-\rho(x) = \Delta U = -\frac{\Delta v}{f(v)} + \frac{f'(v)}{f^2(v)}|\nabla v|^2 \geq -\frac{\Delta v}{f(v)} \tag{13}$$

in $\mathbb{R}^N$, according to the fact that $f' \geq 0$. This means that $\Delta v \geq \rho(x)f(v) \geq \varphi(x, v)$, hence v is a sub-solution of (12). As $w \equiv b$ is obviously a super-solution and $w \geq v$, we obtain a bounded positive solution u of (12) with $w \geq u \geq v \geq a > 0$.

Now, noticing that by definition $U(x) \geq C(b - v(x))$ in $\mathbb{R}^N$, we get

$$\liminf_{r\to\infty} \fint_{B_r} v(x)d\sigma \geq \lim_{r\to\infty} \fint_{B_r} b - C^{-1}U(x)d\sigma = b.$$

Using $v \leq u \leq b$, we have

$$\lim_{r\to\infty} \fint_{B_r} u(x)d\sigma = b.$$

Since b can be chosen as any positive constant large enough, we can claim that there exist infinitely many bounded positive entire solutions, which proves the theorem. ■

An immediate consequence of the above theorem is the following corollary for "non superlinear" case.

Corollary 1 *Let φ, ρ and f be as in Theorem 3. If f satisfies $\limsup_{t\to\infty} f(t)/t < \infty$ and the linear problem (4) has a bounded solution, then the equation (12) has infinitely many bounded positive entire solutions.*

In [9], Kenig and Ni proved that for any nonnegative function $K(x) \in L^q_{loc}(\mathbb{R}^N)$ with $q > N/2$ and $N \geq 3$, the equation $\Delta u = K(x)u$ has always an entire positive solutions.

Theorem 4 *Let $\varphi(x,u)$ satisfy $0 \leq \varphi(x,u) \leq \rho(x)f(u)$ on $\mathbb{R}^N \times (0,\infty)$ for some positive nondecreasing, continuous function $f(u)$ defined on $(0,\infty)$ and $\rho \in L^\infty_{loc}(\mathbb{R}^N)$. Suppose moreover that the solution U of (4) exists and verifies*

$$\|U\|_\infty < \sup_{t>0} \frac{t}{f(t)},$$

then the equation

$$-\Delta u = \varphi(x,u) \quad in\ \mathbb{R}^N \tag{14}$$

possesses infinitely many bounded positive entire solutions, which are bounded from below by positive constant.

Proof. We use the same strategy as in the proof of Theorem 2. Let $\alpha > 0$ be chosen such that $\|U\|_\infty < \alpha/f(\alpha)$. Using the continuity of function f, there is some $\varepsilon > 0$ such that $0 \leq U \leq \alpha/f(\alpha+\varepsilon)$. We note first that $v_1 \equiv \varepsilon$ is a sub-solution. Now let $v_2(x) = \alpha c^{-1}U(x) + \varepsilon$ where $c = \sup_{\mathbb{R}^N} U$. Hence

$$-\Delta v_2 = \alpha c^{-1}\rho(x) \geq \rho(x)f(\alpha+\varepsilon) \geq \rho(x)f(v_2) \geq \varphi(x,v_2)$$

in $\mathbb{R}^N$, so v_2 is a super-solution of (14). Therefore there exists a solution u of equation (14) such that $v_2 \geq u \geq \varepsilon$. Moreover, we have

$$\lim_{r\to\infty} \fint_{B_r} u(x)d\sigma = \varepsilon.$$

As ε can be chosen arbitrarily small, we get infinitely many bounded positive entire solutions for equation (14). ∎

Inversely, we have a partial necessary condition as follows.

Theorem 5 *Let ρ, f be as in Theorem 4. If $-\Delta u = \rho(x)f(u)$ has a entire solution which is bounded from below and above by positive constants, then the solution of equation (4) exists and satisfies (11).*

Proof. Suppose that a positive entire solution u exists for $-\Delta u = \rho(x)f(u)$. If the r.h.s. of (11) is infinite, by the remark under Theorem 1, the claim is obviously true. So we suppose that the integral is finite and define a function w by

$$w(x) = \int_0^{u(x)} \frac{dt}{f(t)}, \quad \forall\, x \in \mathbb{R}^N.$$

We get

$$-\Delta w = -\frac{\Delta u}{f(u)} + \frac{f'(u)}{f^2(u)}|\nabla u|^2 \geq -\frac{\Delta u}{f(u)} = \rho(x).$$

Since $w > 0$, it is a super-solution of (4), we will get a solution U of (4), verifying $0 \leq U \leq w$, the proof is completed. ∎

Remark Theorem 4 and 5 generalize some results of [14] for the special case $f(u) = e^u$. Furthermore, if the equation (14) has only ground state solutions (so which are not bounded below by positive constant), we can not always have a bounded entire solution for (2). For example, by Theorem 3 in [17], the equation

$$-\Delta u = \frac{u^p}{1+r^2} \quad \text{in} \quad \mathbb{R}^N \quad (N \geq 3)$$

possesses infinitely many ground state solutions for any $p \in (1, \infty)$, while the corresponding linear equation with $\rho(x) = (1+r^2)^{-1}$ does not have any entire bounded solution.

4 Other applications

Here, we will prove the existence of solutions for some semilinear elliptic equations by using the solution for the associated linear equation of type (2) or (3).

4.1 A precise existence and uniqueness result

Theorem 6 *Let $f(u)$ be a nondecreasing, locally Lipschitz function in $[0, \infty)$ such that $f \geq 0$ and $f(0) = 0$. Let ρ be a nonnegative L^∞_{loc} function in $\mathbb{R}^N$ such that $-\Delta U = \rho$ has a ground state solution, then each bounded solution u of $\Delta u = \rho(x)f(u)$ has a limit as $|x|$ tends to infinity, and for any $\alpha > 0$, the equation $\Delta u = \rho(x)f(u)$ has a unique positive solution in $\mathbb{R}^N$ such that $\lim_{|x|\to\infty} u(x) = \alpha$.*

Proof. If u is a bounded solution of $\Delta u = \rho(x)f(u)$. Using $\sup_{\mathbb{R}^N} f(u) \times U$ as supersolution, we get a ground state solution v of $-\Delta v = \rho(x)f(u)$. The sum $w = u + v$ is then a bounded harmonic function in $\mathbb{R}^N$, so constant. Thus $\lim_{|x|\to\infty} u(x)$ exists.

For any $\alpha > 0$, define $g(u) = f(|u|)sgn(u)$, then g is locally Lipschitz and nondecreasing in $\mathbb{R}$. We consider at first the equation $\Delta u = \rho(x)g(u)$. Let U be the positive solution of (3), for any $\alpha > 0$, since $\rho(x)g(\alpha) \geq 0$, $u_1 \equiv \alpha$ is a super-solution. Let $u_2 = \alpha - f(\alpha)U$, then $u_2 \leq \alpha$ and

$$\Delta u_2 = -f(\alpha)\Delta U = \rho(x)g(\alpha) \geq \rho(x)g(u_2),$$

which means that u_2 is a sub-solution. Hence we get a solution u satisfying $\Delta u = \rho(x)g(u)$ in $\mathbb{R}^N$ and $u_1 \geq u \geq u_2$, so $\lim_{|x|\to\infty} u(x) = \alpha$. Since u is bounded, using the properties of g, we have $-\Delta u + c(x)u = 0$ with $c(x) \geq 0$ and $c \in L^\infty_{loc}(\mathbb{R}^N)$, hence the strong maximum principle implies that $u > 0$ in $\mathbb{R}^N$. Thus u verifies $\Delta u = \rho(x)f(u)$ in $\mathbb{R}^N$.

The uniqueness of u can be proved easily by classical method as follows. Let u and v be two solutions satisfying $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = \alpha$. For any $a > 0$, we take $(u-v-a)_+$ as a test function, which has compact support. Since f is nondecreasing, we get easily $(u-v-a)_+ = 0$. As a is arbitrary, then $u \leq v$ in $\mathbb{R}^N$, and the inverse inequality is also true, hence $u \equiv v$ in $\mathbb{R}^N$. ∎

Remark This result generalizes Theorem II (i) or Theorem 3.7 in [3]. In this case, a complete understanding of the ordered, or layered structure of all solutions lies now in the study of unbounded solutions. If the primitive F of f verifies the condition (22) below (as for $f(u) = e^u$ or $f(u) = u^p$ with $p > 1$), the existence of large solution on bounded domain (see [1, 8]) ensures the existence of a maximal positive solution u_∞, and under suitable conditions on ρ, we can get the uniqueness of unbounded solution. But in general case (i.e. without more precise assumptions on the asymptotic behavior of f near $+\infty$), the existence and/or the uniqueness of unbounded solutions are quite delicate.

4.2 Ground state solution

Until now, our results give entire bounded positive solutions which are bounded from below by a positive constant. In fact it was proved in Theorem 1.4 of [13] that there exists some cases such that the corresponding equation does not possess any bounded positive solution which tends to 0 at ∞. In contrast with this fact, we give here a result for existence of ground state solution, which generalizes the results in [11, 19].

Theorem 7 *Let f be a positive, nonincreasing and continuous function defined on $(0, \infty)$. Let $\rho \geq 0$ such that $\rho \in L^\infty_{loc}(\mathbb{R}^N)$. Then the equation*

$$-\Delta u = \rho(x)f(u) \quad in \quad \mathbb{R}^N, \quad \lim_{|x|\to+\infty} u(x) = 0. \tag{15}$$

admits a solution if and only if the linear problem (3) has a solution U. If a solution of (15) exists, it is unique.

Proof. Let U be the ground state solution of (3) and $c = \sup_{\mathbb{R}^N} U$. Using the monotonicity, we have $\lim_{t\to+\infty} f(t)/t = 0$, so

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x \frac{t}{f(t)}dt = +\infty.$$

Therefore, we obtain some $x_0 > 0$ such that

$$cx_0 \leq \int_0^{x_0} \frac{t}{f(t)}dt \tag{16}$$

We define now a function v by

$$U(x) = \frac{1}{x_0}\int_0^{v(x)} \frac{t}{f(t)}dt, \quad \forall\, x \in \mathbb{R}^N.$$

Then $v(x) > 0$ and is bounded from above, since $v(x) \leq x_0$ in $\mathbb{R}^N$, and we claim that v is a super-solution of (15). In fact, by the monotonicity of $t/f(t)$, we have

$$-\rho(x) = \Delta U = \frac{|\nabla v|^2}{x_0}\left(\frac{t}{f(t)}\right)'_{|t=v} + \frac{v}{x_0 f(v)}\Delta v \geq \frac{v}{x_0 f(v)}\Delta v,$$

so $-\Delta v \geq x_0\rho(x)f(v)/v \geq \rho(x)f(v)$. Since $\lim_{|x|\to+\infty} U(x) = 0$ implies that $\lim_{|x|\to+\infty} v(x) = 0$ and 0 is a sub-solution, we get then a solution for (15). The uniqueness is proved as for Theorem 6.

Inversely, if a solution u of (15) exists, we have $f(u) \geq f(\max_{\mathbb{R}^N} u) = a > 0$, so u/a is a super-solution for (3), the proof is done. ∎

4.3 Existence of bounded solutions on an exterior domain

In [20], Zhao studied the existence of solutions to the problem

$$\begin{cases} -\Delta u &= \rho(x) f(u) \quad \text{in } \Omega \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega \\ \lim_{|x|\to+\infty} u &= \lambda \end{cases} \tag{17}$$

where Ω is an unbounded domain in $\mathbb{R}^N$ $(N \geq 3)$, with compact Lipschitz boundary and $f(u)$ is a continuous function in $(0, b)$ for some $b \in (0, +\infty]$ and satisfying

$$\lim_{u\to 0^+} \frac{f(u)}{u} = 0.$$

By using a Brownian path integration method and potential theory, he proved that if $\rho(x)$ is a measurable function in some Kato classes, then the equation (17) has a solution. More precisely, he supposed that

$$\lim_{\substack{m(A)\to 0 \\ A\,\subset\,\Omega}} \left[\sup_{x\in\Omega} \int_A \frac{|\rho(y)|}{|y-x|^{N-2}} dy\right] = 0 \quad \text{and} \quad \lim_{M\to\infty} \left[\sup_{x\in\Omega} \int_{|y|>M,\ y\in\Omega} \frac{|\rho(y)|}{|y-x|^{N-2}} dy\right] = 0. \tag{18}$$

Then he proved that under the condition (18), there exists bounded solutions of (17) for any $\lambda \in (0, b_0]$ where $b_0 \in (0, b)$.

It is not difficult to see that under the condition (18), $\rho(x) \in L^1_{loc}(\Omega)$ and if we extend ρ by 0 in $\mathbb{R}^N \setminus \Omega$, then the function

$$U(x) = \int_{\mathbb{R}^N} \frac{\rho(y)}{(N-2)\omega_{N-1}|x-y|^{N-2}} dy \tag{19}$$

is well defined, bounded and satisfies $-\Delta U = \rho$ in Ω and $\lim_{|x|\to+\infty} U = 0$. Using harmonic functions (since $N \geq 3$), we can get a unique solution for

$$-\Delta V = \rho(x) \quad \text{in } \Omega, \quad V|_{\partial\Omega} = 0 \quad \text{and} \quad \lim_{|x|\to+\infty} V(x) = 1. \tag{20}$$

Therefore again by the barrier method, we can generalize Zhao's result as follows

Theorem 8 *Suppose that f is a positive continuous function defined in $(0, \infty)$, satisfies the condition (6). Then the equation (17) is solvable for poitive λ small or large enough.*

4.4 Existence of large solutions

Now we consider the following problem $(N \geq 3)$

$$\Delta u = \rho(x) f(u) \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = \infty. \tag{21}$$

Such a solution is called a large solution. Suppose that f is a nondecreasing, locally Lipschitz function defined on $[0, +\infty)$ such that $f(t) > 0$ in $(0, \infty)$. Suppose moreover that its primitive $F(t) = \displaystyle\int_0^t f(s)ds$ satisfies the standard condition:

$$\int_1^\infty \frac{1}{\sqrt{F(t)}} dt < \infty. \tag{22}$$

Then we have the following

Theorem 9 *Let ρ be a positive continuous function such that the linear problem (3) is solvable. Then equation (21) admits a positive solution.*

Our approach is very classical. For the convenience of readers, we give the sketch of proof here.

Proof. By Bandle and Marcus's result [1] (cf. e.g. [8]), using the condition (22), we know that for any $k \in \mathbb{N}^*$, there exists a positive solution v_k of equation

$$\Delta v_k = \rho(x) f(v_k) \quad \text{in } B_k, \quad \lim_{|x|\to k} v_k = \infty. \tag{23}$$

According to the maximum principle, it is clear that $v_k \geq v_{k+1}$ in B_k. Therefore $v = \lim_{k\to\infty} v_k$ exists and $\Delta v = \rho(x) f(v)$ in $\mathbb{R}^N$. For estimating v, define

$$w_k(x) = \int_{v_k(x)}^{\infty} \frac{1}{f(s)} ds \quad \text{in } B_k, \quad \forall\, k \in \mathbb{N}^*.$$

By (22) and the monotonicity of f, we see that w_k is well defined. A simple calculus shows that $-\Delta\omega_k \leq \rho$ in B_k and $\omega_k = 0$ on ∂B_k, which yields $\omega_k(x) \leq U(x)$ on B_k by the maximum principle, where U is the solution of (3). Thus

$$\int_{v(x)}^{\infty} \frac{1}{f(s)} ds \leq U(x) \quad \forall\, x \text{ in } \mathbb{R}^N.$$

Thus, v is positive in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} U(x) = 0$ implies that $\lim_{|x|\to\infty} v(x) = \infty$. ■

4.5 Bounded positive entire solutions for systems

In what follows we consider the following semilinear elliptic system:

$$\begin{cases} \Delta u &= \varphi(x, u, v) \\ \Delta v &= \psi(x, u, v) \end{cases} \quad \text{in} \quad \mathbb{R}^N \ (N \geq 3). \tag{24}$$

For simplicity, we suppose that $\varphi(x, t, s)$, $\psi(x, t, s)$ are locally Hölder continuous in x and locally Lipschitz continuous in (t, s) on $\mathbb{R}^N \times (0, \infty) \times (0, \infty)$ and there exist positive, nondecreasing functions f, g in $(0, \infty)$, such that $0 \leq \varphi(x, t, s) \leq \rho_1(x) f(s)$ and $0 \leq \psi(x, t, s) \leq \rho_2(x) g(t)$ on $\mathbb{R}^N \times (0, \infty) \times (0, \infty)$. Then we have the following

Theorem 10 *Assume that $\varphi_v(x, u, v) \geq 0$ and $\psi_u(x, u, v) \geq 0$ on $\mathbb{R}^N \times (0, \infty) \times (0, \infty)$. Suppose also that one of the conditions in (6) is fulfilled by f and g simultaneously. If $\rho_i \in L^1_{loc}(\mathbb{R}^N)$ satisfies (2) for $i = 1, 2$, then the system (24) has infinitely many bounded positive solutions.*

By the lower-upper solutions method for systems (see for instance [6] or [18]), we know that for proving the existence of a solution of (24), it suffices to prove the existence of positive vector functions $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ which are upper and lower solution of (24) respectively, i.e. they satisfy $\bar{u} \geq \underline{u}$, $\bar{v} \geq \underline{v}$,

$$\begin{cases} \Delta \bar{u} &\leq \varphi(x, \bar{u}, \underline{v}) \\ \Delta \bar{v} &\leq \psi(x, \underline{u}, \bar{v}) \end{cases} \quad \text{and} \quad \begin{cases} \Delta \underline{u} &\geq \varphi(x, \underline{u}, \bar{v}) \\ \Delta \underline{v} &\geq \psi(x, \bar{u}, \underline{v}) \end{cases} \quad \text{in } \mathbb{R}^N. \tag{25}$$

Proof. Let U, V be positive bounded entire solutions satisfying $\Delta U = \rho_1$ and $\Delta V = \rho_2$ respectively. Without loss of generality, we can suppose that there exist two positive constants C and C' such that

$$C \geq U \geq C' \quad \text{and} \quad C \geq V \geq C' \quad \text{in} \quad \mathbb{R}^N.$$

Suppose now that f and g satisfy the first condition in (6), then there exists a positive constant a, small enough, such that $g(aC) + f(aC) \leq a$, hence if we take $(\bar{u}, \bar{v}) = (aC, aC)$ and $(\underline{u}, \underline{v}) = (aU, aV)$, the first system in (25) is trivially verified, and the following holds

$$\Delta \underline{u} = \Delta(aU) = a\rho_1 \geq \rho_1 f(aC) \geq \varphi(x, aU, aC) = \varphi(x, \underline{u}, \bar{v}) \quad \text{in} \quad \mathbb{R}^N. \tag{26}$$

Similarly, we have $\Delta \underline{v} \geq \psi(x, \bar{u}, \underline{v})$ in $\mathbb{R}^N$. Thus we obtain the existence of a solution (u, v) for (24), which satisfies $aC \geq u \geq aU$ and $aC \geq u \geq aV$. Since we can choose $a > 0$ arbitrarily small, there exist then infinitely many couple of positive solutions.

The similar argument shows that if f and g satisfy the second condition in (6), then it suffices to choose $a > 0$ large enough to obtain our conclusion. ■

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