Summer Seminar on Mathematical Theory of Shock Waves and Hyperbolic Conservation Laws

(3)

Structure and Regularity of Weak Solutions

Center for Partial Differential Equations
School of Mathematical Sciences
Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice
East China Normal University

July 10–23, 2019
Shanghai • China

Foreword

This summer seminar is the third program we hold on the Mathematical Theory of Shock Waves and Hyperbolic Conservation Laws. The previous two were "Workshop on measure solutions of hyperbolic conservation laws" held at Shanghai Normal University, Dec. 1-2, 2018, and "Multi-dimensional Riemann Problems: Progresses and Challenges", held at East China Normal University, March 30-31, 2019. (See page iii and page iv for photos of these two workshops.)

It is our great pleasure that Dr. Elio Marconi at Basel University (Switzerland), and Dr. Stefano Modena at University of Leipzig (Germany) will give us two mini-courses on the structure and regularity theory of weak solutions of hyperbolic conservation laws and related partial differential equations. The mini-courses are "(Non)uniqueness for the Transport Equation and Convex Integration" (Modena),"Regularity estimates for scalar conservation laws in one space dimension" (Marconi). Dr. Lei Yu will give a mini-course titled "Entropy Solutions of Hyperbolic Conservation Laws", which is scheduled to be rather elementary to help students to follow Dr. Modena and Dr. Marconi's lectures easier.

We will also invite many professors to present academic talks during the seminar. Confirmed speakers include Prof. Philippe G. LeFloch (Sorbonne University), Prof. Yachun Li (Shanghai Jiaotong University), Prof. Jianli Liu (Shanghai University), Prof. Peng Qu (Fudan University), Dr. Tianyi Wang (Wuhan University of Science and Technology), Dr. Yong Wang (Chinese Academy of Science), Dr. Qinghua Xiao (Wuhan Institute of Physics and Mathematics), Prof. Yongqian Zhang (Fudan University). Some local graduate students will also report their research work.

This seminar is supported by National Natural Science Foundation of China under Grant No. 11871218, Center for Partial Differential Equations, School of Mathematical Sciences, East China Normal University, and Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice. We are grateful to all the supports received from our institutes and colleagues. We thank particularly Dr. Xia Huang and graduate student Junlei Gao for their great help.

Wish all the participants of this program enjoy it and have a good time in ECNU!

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“双曲守恒律方程测度值解研讨会”部分与会者合影，2018年12月1日—2日，上海师范大
Department of Mathematics
Shanghai Normal University
• 84 teachers
• 22 professors


激波与双曲守恒律方程数学理论研讨会

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Part 1 Schedule

Important notice: Venue for lectures and talks on 10 July and 11 July is lecture room 221, the Third Teaching Building at Minhang Campus (闵行校区三教221). From 12 July to 23 July, the venue is lecture room 102 in Mathematics Building (闵行数学楼102).

10 July (Wednesday)

9:00 — 10:30 Yu (Lecture 1)

13:30 — 15:00 Yu (Lecture 2)

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11 July (Thursday)

9:00 — 10:30 Yu (Lecture 3)

13:30 — 15:00 Yu (Lecture 4)

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12 July (Friday)

9:00 — 10:30 Yu (Lecture 5)

11:00 — 12:00 Peng Qu (Time-periodic solutions to quasilinear hyperbolic systems with time-periodic boundary conditions)

13:30 — 15:00 Yu (Lecture 6)

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13 July (Saturday)

9:00 — 10:30 Yu (Lecture 7)

11:00 — 12:00 Yong Wang (Global existence of finite energy weak solution to the Compressible Euler Equations with spherical Symmetry and Large Initial Data)

13:30 — 15:00 Yu (Lecture 8)

15:30 — 16:30 Yongqian Zhang (Weakly Nonlinear Geometric Optics Expansion for Hyperbolic Systems of Conservation Laws)

14 July (Sunday)

9:00 — 10:30 Modena (Lecture 1)

13:30 — 14:30 Tianyi Wang (Compensated Compactness and Gas Dynamic Equations (I))

15:00 — 16:00 Tianyi Wang (Compensated Compactness and Gas Dynamic Equations (II))

15 July (Monday)

9:00 — 10:30 Modena (Lecture 2)

14:00 — 15:00 P. G. LeFloch (Global dynamics of massive matter from geometric analysis to scientific computation)

15:30 — 16:30 Yunjuan Jin (Measure solutions of hyperbolic conservation laws and limiting hypersonic flow passing bodies)

16 July (Tuesday)

9:00 — 10:30 Modena (Lecture 3)

11:00 — 12:00 Qinghua Xiao (The Riemann problem of relativistic Euler system with Synge energy)

13:30 — 15:00 Modena (Lecture 4)

15:30 — 16:30 Junlei Gao (Subsonic Rayleigh Flow)

17 July (Wednesday)

Break

18 July (Thursday)

9:00 — 10:30 Modena (Lecture 5)

11:00 — 12:00 Jianli Liu (A class of large solutions to the supercritical surface quasi-geostrophic equation)

13:30 — 15:00 Marconi (Lecture 1)

19 July (Friday)

9:00 — 10:30 Marconi (Lecture 2)

13:30 — 15:00 Marconi (Lecture 3)

20 July (Saturday)

9:00 — 10:30 Marconi (Lecture 4)

13:30 — 15:00 Marconi (Lecture 5)

21 July (Sunday)

9:00 — 10:30 Marconi (Lecture 6)

22 July (Monday)

9:00 — 10:30 Marconi (Lecture 7)

13:30 — 15:00 Marconi (Lecture 8)

23 July (Tuesday)

9:00 — 10:30 Marconi (Lecture 9)

11:00 — 12:00 Yachun Li (Local Well-posedness of non-isentropic Euler equations with physical vacuum)

13:30 — 15:00 Marconi (Lecture 10)

Part 2　Entropy Solutions of Hyperbolic Conservation Laws

Lei Yu (余磊)
(同济大学)

本课程（共八节课）将对双曲守恒律方程的熵解理论作一个初步介绍，可作为Marconi课程的先修和补充。本课程包含如下内容：

1. 有界变差函数的相关理论；
2. 双曲守恒律方程（组）的黎曼问题；
3. 单个守恒律方程的熵解；
4. 守恒律系统的熵解；
5. 波强度的Lagrange表示的初步介绍（有关Marconi的课程）；
6. 若干前沿问题介绍。

主要参考书目：

[1] C. M. Dafermos. Hyperbolic conservation laws in continuum physics. Grundlehren der mathematischen Wissenschaften. Springer, USA, third edition edition, 2010.

[2] A. Bressan. Hyperbolic Systems of Conservation Laws: The Onedimensional Cauchy Problem. Oxford lecture series in mathematics and its applications. Oxford University Press, USA, 2000.

[3] L. Ambrosio, N. Fusco, and D. Pallara. Functions of bounded variation and free discontinuity problems. Oxford Clarendon Press, 2000.

Part 3 (Non)uniqueness for the Transport Equation and Convex Integration

STEFANO MODENA
(Mathematisches Institut, Universität Leipzig, D-04109 Leipzig, Germany)

Short abstract: One of the main questions in the theory of the linear transport equation is whether uniqueness of weak solutions holds in the case the given vector field is not smooth. In this series of lectures, I will show that even for incompressible, Sobolev (thus quite “well-behaved”) vector fields, uniqueness of solutions can drastically fail. This result can be seen as a counterpart to DiPerna and Lions’ well-posedness theorem (joint with G. Sattig and L. Székelyhidi).

Long abstract: I will consider the Cauchy problem associated to the linear transport equation

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho &= 0, \\ \rho|_{t=0} &= \rho^0, \end{aligned} \tag{3.1}$$

with unknown density ρ and given (divergenge-free) vector field u.

In particular I will discuss the well-posedness of (3.1) in the case of <u>rough</u> vector fields (say, less regular than Lipschitz continuous). I will first present an overview of the main results about uniqueness and non-uniqueness of solutions to (3.1) present in the literature (e.g. [6], [1], [2], [5]). I will then focus on some recent achievements, obtained in collaboration with G. Sattig and L. Székelyhidi [8, 9, 10], concerning the non-uniqueness of weak solutions in the class of densities $\rho \in L^\infty_t L^p_x$, for vector fields in $u \in L^\infty_t W^{1,\tilde{p}}_x \cap L^\infty_t L^{p'}_x$, if

$$\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d}, \tag{3.2}$$

where $p, \tilde{p} \in [1, \infty)$, d is the dimension of the physical space and p' is the dual Hölder exponent to p, $1/p + 1/p' = 1$. This can be seen as a partial converse to the

celebrated theorem of DiPerna and Lions [6] where uniqueness is shown under the condition

$$\frac{1}{p} + \frac{1}{\tilde{p}} \leq 1.$$

The proof of our result is based on a convex integration scheme, developed for Hölder continuous solutions to the incompressible Euler equations in connection with Onsager's conjecture (see, for instance [7, 3]). Unlike that case, here we need to construct vector fields $u \in C_t W_x^{1,\tilde{p}}$, i.e. with bounds on one full derivative. The key idea to get such better regularity (inspired by the intermittency added to the convex integration schemes by T. Buckmaster and V. Vicol in [4]) is to play not only with fast oscillating fields, as usual in convex integration schemes, but also with highly concentrated fields.

Prerequisites: Just undergraduate courses (analysis, measure theory, functional analysis, partial differential equations, basic theory of Sobolev spaces) are needed. I do not expect any previous knowledge of convex integration or of the theory of weak solutions to the transport equation.

Part 4 Regularity estimates for scalar conservation laws in one space dimension

Elio Marconi
(University of Basel)

We consider the Cauchy problem for the scalar conservation law in one space dimension

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,\cdot) = u_0, \end{cases}$$

where the flux function $f : \mathbb{R} \to \mathbb{R}$ is smooth and $u_0 \in L^\infty(\mathbb{R})$. We study the regularizing effect induced on the entropy solution u by the nonlinearity of f. The paradigm is the celebrated Oleinik one side Lipschitz estimate: if $f'' \geq c > 0$, then for every $t > 0$ and every $y < x$ it holds

$$u(t,x) - u(t,y) \geq \frac{x-y}{ct}.$$

This implies that if the flux is uniformly convex, then bounded entropy solutions have locally bounded variation.

In this course we discuss a similar approach under various nonlinearity assumptions on the flux function f. In particular we obtain a substitute for the Oleinik estimate and we introduce the appropriate generalization of the space of functions with bounded variation in order to describe the regularity of the entropy solution u. A key point to achieve this goal is to understand the structure of the characteristics. A part of the course will be devoted to introduce the notion of Lagrangian representation, which is an extension to the non-smooth setting of the classical method of characteristics.

Preliminaries:

1. Standard theory of (1d) scalar conservation laws: in particular I will only briefly recall the following topics.

a). Admissibility conditions for weak solutions;

b). Kruzkov theorem;

c). Wave-front tracking algorithm.

All these topics can be found in Section 4.5 and Chapter 6 of the book by A. Bressan: "Hyperbolic systems of conservation laws" .

2. Some properties of functions with bounded variation (See Section 2.4-2.5 of the book by Bressan).

3. The method of characteristics for first order PDEs is central (see for example Section 3.2 in the book of Evans "Partial differential equations".

4. It will recall only briefly the case of uniformly convex fluxes, it will be helpful to know some basic facts (see Sections 3.4.2 and 3.4.3 in the book by Evans).

5. It will be useful but not necessary to know the basic notions of the initial-boundary value problem for the scalar conservation law. What is in my paper "Regularity estimates for scalar conservation laws in one space dimension" is sufficient for the aim of this course. See Section 6.9 in Dafermos' book "Hyperbolic conservation laws in continuum physics" (4th ed.) if you want more details.

Part 5　Titles and Abstracts of Talks

Subsonic Rayleigh Flow

Junlei Gao (高俊磊)
(East China Normal University)

Measure solutions of hyperbolic conservation laws and limiting hypersonic flow passing bodies

Yunjuan Jin (金云娟)
(East China Normal University)

Global dynamics of massive matter from geometric analysis to scientific computation

Philippe G. LeFloch
(Sorbonne University)

I will present recent mathematical advances on the nonlinear dynamical stability of massive matter fields such as compressive fluids in presence of gravitation effects. Recent progress concerns the following problems:

– Spherically symmetric fluid flows, for instance outside of a Schwarzschild black hole with random boundary forcing

– Formulation and convergence of the finite volume method for conservation laws on a spacetime with boundary

– Global nonlinear stability of Minkowski spacetime for the Einstein equations in presence of massive fields

Local Well-posedness of non-isentropic Euler equations with physical vacuum

Yachun Li (李亚纯)
(Shanghai Jiaotong University)

We consider the local well-posedness of the one-dimensional non-isentropic Euler equations with moving physical vacuum boundary. The physical vacuum singularity requires the sound speed to be scaled as the square root of the distance to the vacuum boundary. The main difficulty lies in the fact that the system of hyperbolic conservation laws becomes characteristic and degenerate at the vacuum boundary. To overcome this difficulty, our proof is based on an approximation of the Euler equations by a degenerate parabolic regularization obtained from a specific choice of a degenerate artificial viscosity term. Then we construct solutions to this degenerate parabolic problem and establish uniform estimates that are independent of the artificial viscosity parameter. Solutions to the compressible Euler equations are obtained as the limit of the vanishing artificial viscosity. This is a joint work with Yongcai Geng, Dehua Wang and Runzhang Xu.

A class of large solutions to the supercritical surface quasi-geostrophic equation

Jianli Liu (刘见礼)
(Shanghai University)

Whether or not classical solutions to the surface quasi-geostrophic (SQG) equation can develop finite time singularities remains an outstanding open problem. This paper constructs a class of large global-in-time classical solutions to the SQG equation with supercritical dissipation. The construction process presented here implies that any solution of the supercritical SQG equation must be globally regular if its initial data is sufficiently close to a function (measured in a Sobolev norm) whose Fourier transform is supported in a suitable region away from the origin.

Time-periodic solutions to quasilinear hyperbolic systems with time-periodic boundary conditions

Peng Qu (曲鹏)
(Fudan University)

For quasilinear hyperbolic systems with time-periodic boundary conditions which satisfy the dissipative structure requirement, the existence, uniqueness and stability of the time-periodic classical solutions are discussed. Moreover, the feedback boundary control with dissipative structure can stabilize the system to the time-periodic solutions, when the time periodic boundary conditions possess more regularity. This presentation is based on a joint work with Prof. Ta-Tsien Li.

Global existence of finite energy weak solution to the Compressible Euler Equations with spherical Symmetry and Large Initial Data

Yong Wang (王勇)
(Chinese Academy of Science)

For far field density $\bar{\rho} > 0$, various evidences indicate that the spherically symmetric solutions of the compressible Euler equations may blow up near the origin at certain time. In this paper, we established the global existence of finite energy weak solution by vanishing viscosity limit of weak solutions of the compressible Navier-Stokes equations with spherical symmetry and large initial data in $\mathbb{R}^N (N \geq 2)$ and $\bar{\rho} > 0$. This indicates that concentration is not formed in the vanishing physical viscosity limit, even though the density may blow up at certain time.

Compensated Compactness and Gas Dynamic Equations (I,II)

Tianyi Wang (王天怡)
(Wuhan University of Science and Technology)

Compensated Compactness is a powerful tool in nonlinear partial differential equation in the last half century. In this talk, we will introduce the basic ideas and theoretical framework of the method. Latter, the milestone application in the gas dynamic equations and some latest progress will be presented.

The Riemann problem of relativistic Euler system with Synge energy

Qinghua Xiao (肖清华)
(Wuhan Institute of Physics and Mathematics, Chinese Academy of Science)

We study the Riemann problem of relativistic Euler system for rarefied monatomic and diatomic gases when the constitutive equation for the energy is the Synge equation that is the only one compatible with the relativistic kinetic theory. The Synge equation is involved with modified Bessel functions of the second kind and this makes the relativistic Euler system quite complex. Based on delicate estimates of the modified Bessel functions of the second kind, we provide a detailed investigation of basic hyperbolic properties and the structure of elementary waves, especially for the structure of shock waves and in this way, the mathematical theory of the Riemann problem for these relativistic Euler system, which is analogous to the corresponding theory of the classical ones, is rigorously provided.

Weakly Nonlinear Geometric Optics Expansion for Hyperbolic Systems of Conservation Laws

Yongqian Zhang (张永前)
(Fudan University)

We establish an L^1-estimate to validate the weakly nonlinear geometric optics for entropy solutions of nonlinear hyperbolic systems of conservation laws with arbitrary initial data of small bounded variation. This implies that the simpler geometric optics expansion function can be employed to study the properties of general entropy solutions to hyperbolic systems of conservation laws. Our analysis involves techniques which rely on the structure of the approximate equations, besides the properties of the wave-front tracking algorithm and the standard semigroup estimates. (This is a joint work with Prof. Guiqiang Chen and Wei Xiang (Oxford).)

References

[1] AMBROSIO, L. Transport equation and Cauchy problem for BV vector fields. Invent. math. 158, 2 (2004), 227–260.

[2] BIANCHINI, S., AND BONICATTO, P. A uniqueness result for the decomposition of vector fields in Rd. SISSA (2017).

[3] BUCKMASTER, T., DE LELLIS, C., SZÉKELYHIDI JR, L., AND VICOL, V. Onsager's conjecture for admissible weak solutions. arXiv (2017).

[4] BUCKMASTER, T., AND VICOL, V. Nonuniqueness of weak solutions to the Navier-Stokes equation. Annals of Mathematics (2019).

[5] DEPAUW, N. Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan. C. R. Math. Acad. Sci. Paris 337, 4 (2003), 249–252.

[6] DIPERNA, R. J., AND LIONS, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. Invent. math. 98, 3 (1989), 511–547.

[7] ISETT, P. A Proof of Onsager's Conjecture. arXiv (2016).

[8] MODENA, S., AND SATTIG, G. Convex integration solutions to the transport equation with full dimensional concentration. arXiv:1902.08521 (2019).

[9] MODENA, S., AND SZÉKELYHIDI JR, L. Non-renormalized solutions to the continuity equation. arXiv:1806.09145 (2018).

[10] MODENA, S., AND SZÉKELYHIDI JR, L. Non-uniqueness for the transport equation with Sobolev vector fields. Annals of PDE 4, 2 (2018), 18.