Arithmetic on self-similar sets

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Abstract

Let K be the attractor of the following IFS

$$\{f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda\},$$

where $f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset$, and $I = [0,1]$ is the convex hull of K. Let $K * K = \{x * y : x, y \in K\}$, where $* = +, -, \cdot$ or $\div$ (if $* = \div$, then $y \neq 0$). We prove that the following conditions are equivalent:

(1) For any $u \in [0,1]$, there are some $x, y \in K$ such that $u = x \cdot y$;

(2) For any $u \in [0,1]$, there are some $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in K$ such that

$$u = x_1 + x_2 = x_3 - x_4 = x_5 \cdot x_6 = x_7 \div x_8;$$

(3) $c \geq (1-\lambda)^2$.

Suppose that f is a continuous function defined on an open set $U \subset \mathbb{R}^2$. Denote the image

$$f_U(K, K) = \{f(x, y) : (x, y) \in (K \times K) \cap U\}.$$

If $\partial_x f$, $\partial_y f$ are continuous on U, and there is a point $(x_0, y_0) \in (K \times K) \cap U$ such that one of the following conditions is satisfied,

$$\max\left\{\frac{1-c-\lambda}{\lambda}, \frac{1-\lambda}{1-c}\right\} < \left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right| < \frac{1}{1-c-\lambda},$$

or

$$\max\left\{\frac{1-c-\lambda}{\lambda}, \frac{1-\lambda}{1-c}\right\} < \left|\frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}}\right| < \frac{1}{1-c-\lambda},$$

then $f_U(K, K)$ has a non-empty interior. As an application, we let $c = \lambda = \dfrac{1}{3}$ and if

$$f(x, y) = x^\alpha y^\beta (\alpha\beta \neq 0),\ x^\alpha \pm y^\alpha (\alpha \neq 0)\ \sin(x)\cos(y),\ \text{or}\ x\sin(xy)$$

then $f_U(C, C)$ contains a non-empty interior, where C is the middle-third Cantor set. Similar results are available for inhomogeneous self-similar sets or Moran sets.