Cartan invariants in the category of restricted representation for $\mathfrak{gl}(m|n)$

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Lie superalgebra

- $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$: $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$-graded algebra, the superbracket $[\ ,]$ is defined by
  - $\mathbb{Z}_2$-gradation: $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, (i, j \in \mathbb{Z}_2)$;
  - graded-anti symmetry : $[X_i, X_j] = -(-1)^{\overline{X_i}\,\overline{X_j}}[X_j, X_i]$;
  - generalized Jacobi identity

$$(-1)^{\overline{X_i}\,\overline{X_l}}[X_i, [X_j, X_l]] + (-1)^{\overline{X_j}\,\overline{X_i}}[X_j, [X_l, X_i]]$$
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- basic classical Lie superalgebra Kac
- $$\mathfrak{gl}(m|n), \mathfrak{sl}(m|n), \underbrace{C(n+1); B(m|n), D(m|n)}_{\mathfrak{osp}(l|2n)}, F(4), G(3), D(2,1;\alpha).$$

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The general linear Lie superalgebra

- $\mathfrak{g} = \mathrm{Lie}(\mathrm{GL}(m|n)) = \mathfrak{gl}(m|n)$
  is generated by all the $(m+n) \times (m+n)$-matrix with the form
- $e_{ij}, i, j \in \{1, \cdots, m, \cdots, m+n\}$ the homogeneous basis.

$$[e_{ij}, e_{ls}] = \delta_{jl} e_{is} - (-1)^{(\bar{i}+\bar{j})(\bar{l}+\bar{s})} \delta_{is} e_{lj}.$$

the degree of e_{ij} is $\bar{i} + \bar{j}$, $i = 1, \ldots, m$. where $\bar{i} = \bar{0}$, if $i = 1, \ldots, m$. $\bar{i} = \bar{1}$, if $i = m+1, \ldots, m+n$.

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- the standard simple root system
  $\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \varepsilon_{m+1}, \dots, \varepsilon_{m+n-1} - \varepsilon_{m+n}\}$.
  where $\varepsilon_m - \varepsilon_{m+1}$ is a simple odd root.
- the positive root system $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where

$$\Delta_0^+ = \{\varepsilon_i - \varepsilon_j \mid \bar{i} + \bar{j} = \bar{0}, i < j\},$$

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- The distinguished $\mathbb{Z}$-gradation of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_{-1} \bigoplus \mathfrak{g}_{0} \bigoplus \mathfrak{g}_{1}.$$

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where $\mathfrak{g}_{-1} = \bigoplus_{\alpha \in \Delta_1^-} \mathfrak{g}_{\alpha}$, $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$, $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Delta_1^+} \mathfrak{g}_{\alpha}$.

- the restricted enveloping algebra

$$u(\mathfrak{g}) = U(\mathfrak{g})/J, \quad J = \langle\, x^p - x^{[p]}, \text{ for all } x \in \mathfrak{g}_{\bar{0}} \rangle$$

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$(u(\mathfrak{g}), T)$-$\mathfrak{mod}$ category

- Let T be the maximal torus of $\mathrm{GL}(m|n)$ consisting of diagonal matrices which are denoted by $\mathrm{Diag}(t_1, \ldots, t_{m+n}), t_i \in k^{\times}$.
- the character group

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Definition

$(u(\mathfrak{g}), T)$-$\mathfrak{mod}$ category is defined as such a category whose objects are $k-$superspaces endowed with both $u(\mathfrak{g})$-module and rational T-module structure satisfying the compatible conditions for $V \in (u(\mathfrak{g}), T)$-$\mathfrak{mod}$,

1. the action of $u(\mathfrak{h})$ coincides with the action of $\mathrm{Lie}(T)$ induced from T.
2. For $a \in u(\mathfrak{g})$, $t \in T$, and $v \in V$: $t(av) = (Adt(a))tv$.

The morphisms of $(u(\mathfrak{g}), T)$-$\mathfrak{mod}$ is defined to be linear maps of k-superspaces acting as both $u(\mathfrak{g})$-$\mathfrak{mod}$ homomorphisms, and rational T-module homomorphisms.

Remark

Let u be a Hopf subalgebra of $u(\mathfrak{g})$ containing $u(\mathfrak{h})$. We can define a category of (u, T)-$\mathfrak{mod}$ by the same way as above, of which each objects are simply called a $\widehat{\mathrm{u}}$-module.

Notations and Properties

- restricted baby Kac-module

$$\widehat{K}^{+}(\lambda) := u(\mathfrak{g}) \bigotimes_{u(\mathfrak{g}_0+\mathfrak{g}_1)} \widehat{L}^{0}(\lambda)$$

- Set $\mathfrak{g}^{\pm} = \mathfrak{g}_0 + \mathfrak{g}_{\pm 1}$. $\mathfrak{b}^{\pm} = \mathfrak{b}_0^{\pm} + \mathfrak{g}_{\pm 1}$.
  (1) Denote by $\widehat{Q}^{0}(\lambda)$(resp. $\widehat{Q}(\lambda)$) the projective cover of simple $\mathfrak{g}_0$(resp. $\mathfrak{g}$)-module $\widehat{L}^{0}(\lambda)$(resp. $L(\lambda)$).
  (2) $\widehat{K}_{Q}^{\pm}(\lambda) := u(\mathfrak{g}) \bigotimes_{u(\mathfrak{g}^{\pm})} \widehat{Q}^{0}(\lambda)$, with trivial $\mathfrak{g}_{+1}$-action on $\widehat{Q}^{0}(\lambda)$ for $\widehat{K}_{Q}^{+}(\lambda)$, and with trivial $\mathfrak{g}_{-1}$-action on $\widehat{Q}^{0}(\lambda)$ for $\widehat{K}_{Q}^{-}(\lambda)$.

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  2. (2) $\widehat{K}_Q^{\pm}(\lambda) := u(\mathfrak{g}) \bigotimes_{u(\mathfrak{g}^{\pm})} \widehat{Q}^{0}(\lambda)$, with trivial $\mathfrak{g}_{+1}$-action on $\widehat{Q}^{0}(\lambda)$ for $\widehat{K}_Q^{+}(\lambda)$, and with trivial $\mathfrak{g}_{-1}$-action on $\widehat{Q}^{0}(\lambda)$ for $\widehat{K}_Q^{-}(\lambda)$.

(3) The iso-classes of restricted represented $\mathfrak{g}^{\pm}$-modules coincides withe the one of restricted irreducible $\mathfrak{g}_0$-modules represented by $\{L^0(\lambda) \mid \lambda \in X(T)/pX(T)\}$ with $L^0(\lambda)$ being endowed with trivial $\mathfrak{g}_{+1}$-action (resp. $\mathfrak{g}_{-1}$-action) when regarded as $\mathfrak{g}^{+}$-modules(resp. $\mathfrak{g}^{-}$-modules).

(4) Recall that the baby Verma module for restricted Lie algebra $\mathfrak{g}_0$:$\widehat{V}^0_{\pm}(\lambda) = u(\mathfrak{g}_0) \bigotimes_{u(\mathfrak{b}_0^{\pm})} k_\lambda$, $\lambda \in X(T)$. We have generalized Verma modules $\widehat{V}^0_{\mathfrak{g}^{\pm}}(\lambda) = u(\mathfrak{g}^{\pm}) \bigotimes_{u(\mathfrak{b}^{\pm})} k_\lambda$, and $\widehat{V}^{\pm}(\lambda) = u(\mathfrak{g}) \bigotimes_{u(\mathfrak{b}^{\pm})} k_\lambda$. For those Verma modules, the following facts will be useful later.

(5) $\widehat{Q}^0(\lambda)$ can be regarded $u(\mathfrak{g}^{\pm})$-modules. and is filtrable by $\widehat{V}^0_{\pm}(\mu), \mu \in X(T)$. Hence $\widehat{K}^{\pm}_{Q}(\lambda)$ can be filtrable by $\widehat{V}^{\pm}(\mu), \mu \in X(T)$.

- Assume V is a module in the $(u(\mathfrak{g}), T)$ module category. Denote by $[V]$ the character formula of V.

Lemma

(1) *The one to one correspondence* σ *on* $X(T) : \mu \longmapsto \mu + 2(p-1)\rho_0 + 2\rho_1$ *such that*

$$[\widehat{V}^-(\mu)] = [\widehat{V}^+(\sigma(\mu))].$$

(2) $[\widehat{V}^0_{\mathfrak{g}^\pm}(\lambda) : \widehat{L}^0(\mu)] = [\widehat{V}^0_{\mathfrak{g}^\pm}(\lambda) : \widehat{L}^0(\mu)]$,
$[K^\pm_Q(\lambda) : \widehat{V}^\pm(\mu)] = [Q^0_\pm(\lambda) : \widehat{V}^0_{\mathfrak{g}^\pm}(\mu)]$.

(3) $[\widehat{V}^+(\mu) : \widehat{L}(\mu)] = \sum_{\gamma \in X(T)} [\widehat{V}^0(\lambda) : \widehat{L}^0(\gamma)][K^+(\gamma) : \widehat{L}(\mu)]$.

Definition

Let $V \in (u(\mathfrak{g}), T)\text{-}\mathfrak{mod}$. Call V has a $\widehat{K}^+$-filtration (resp. $\widehat{K}_Q^+$-filtration) if there is a submodule filtration of V:

$$0 = V_r \subset V_{r-1} \subset \cdots \subset V_1 \subset V_0 = V \tag{1}$$

such that each subquotient $V_i/V_{i+1} \cong \widehat{K}^+(\lambda_i)$ (resp. $\widehat{K}_Q^+(\lambda_i)$) for some $\lambda_i \in X(T), i = 1, \cdots, r$.

- $[V : \widehat{K}^+(\lambda_i)]$
  the times of occurrence of $\widehat{K}^+(\lambda_i)$ in the subquotients of the above filtration.

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Proposition

1. *Each projective module in* $(u(\mathfrak{g}), T)$-$\mathfrak{mod}$ *has* $\widehat{K}_Q^{\pm}$*-filtration, and also has* $\widehat{K}^{\pm}$*-filtration .*
2. $[\widehat{Q}(\lambda) : \widehat{K}_Q^{\pm}(\mu)] = [\widehat{K}^{\mp}(\mu) : \widehat{L}(\lambda)]$, *and* $[\widehat{Q}(\lambda) : \widehat{K}^{\pm}(\mu)] = [\widehat{K}_Q^{\pm}(\mu) : \widehat{L}(\lambda)]$.

Cartan invariants

Lemma

For $\lambda, \mu \in X(T)$, we have

$$[\widehat{Q}(\lambda) : \widehat{L}(\mu)] = \sum_{\gamma \in X(T)} [\widehat{K}^{+}(\gamma)) : \widehat{L}(\lambda)][\widehat{K}^{+}(\gamma)) : \widehat{L}(\mu)]$$

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Proposition

$$[\widehat{Q}(\lambda) : \widehat{L}(\mu)] = \sum_{\gamma, \gamma_1, \gamma_2 \in X(T)} [\widehat{K}^{+}(\gamma_1) : \widehat{L}(\lambda)][\widehat{Q}^{0}(\gamma_1) : \widehat{V}^{0}_{+}(\gamma - 2\rho_1)] [\widehat{V}^{0}(\gamma) : \widehat{L}^{0}(\gamma_2)][K^{+}(\gamma_2) : \widehat{L}(\mu)].$$

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Proposition

$$\begin{aligned}[\widehat{Q}(\lambda) : \widehat{L}(\mu)] \quad = & \sum_{\gamma, \gamma_1, \gamma_2 \in X(T)} [\widehat{K}^+(\gamma_1) : \widehat{L}(\lambda)][\widehat{Q}^0(\gamma_1) : \widehat{V}^0_+(\gamma - 2\rho_1)] \\ & [\widehat{V}^0(\gamma) : \widehat{L}^0(\gamma_2)][K^+(\gamma_2) : \widehat{L}(\mu)].\end{aligned}$$

Main Result

By the above argument, we can get the following

Theorem

Denote the Cartan invariants by $c_{\lambda\mu} = [\widehat{Q}(\lambda) : \widehat{L}(\mu)], \lambda, \mu \in X(T)$. *Then the Cartan invariants can be given via the following matrix formula for* $C = (c_{\lambda\mu})_{\lambda,\mu\in X(T)}$:

$$C = AB_1B_2A^t$$

where $A = ([\widehat{K},(\alpha) : \widehat{L}(\beta)])^t_{\alpha,\beta\in X(T)}$,
$B_1 = ([\widehat{Q}^0(\alpha) : \widehat{V}^0(\beta - 2\rho_1)])^t_{\alpha,\beta\in X(T)}$,
$B_2 = ([\widehat{Q}^0(\alpha) : \widehat{V}^0(\beta)])^t_{\alpha,\beta\in X(T)}$.

[Kac] V.G. Kac,
Classification of simple Lie superalgebra, Func. Anal. Appl.9(1975),263-265.

Thank you!