Modular Representations of Lie Superalgebras

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Motivation and Goal

Background:

1. Representation theory of Lie superalgebras over $\mathbb{C}$
2. Modular representations of restricted Lie algebras in prime characteristic

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1. Representation theory of Lie superalgebras over $\mathbb{C}$
   e.g.
   - Kac, Serganova, Brundan: irreducible characters for $\mathfrak{gl}(m|n)$
   - Cheng-Wang (and Zhang): connection to parabolic category $\mathcal{O}$
   - Cheng-Lam-Wang: super-duality and complete solution to the finite-dimensional irreducible character for orthosymplectic Lie superalgebras
2. Modular representations of restricted Lie algebras in prime characteristic

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   - Kac-Weisfeiler; Friedlander-Parshall; Premet, Skryabin; Jantzen; Lusztig; Bezrukavnikov, Mirkovic, Rumynin.

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Develop systematically modular representations of Lie superalgebras.

Goal of this talk: For the basic classical Lie superalgebras in characteristic p, we

- Give a new proof of Super KW Conjecture (formulated and proved by Wang-Z).
- Give a semisimplicity criterion for the reduced enveloping superalgebras with semisimple p-characters.

Outline

1. Basic Properties of Restricted Lie Superalgebras
2. Formulation of the Super KW Property
3. Proof of Super KW for Basic Classical Lie Superalgebras
4. Semisimplicity criterion for semisimple p-characters
5. Example of $\mathfrak{osp}(1|2)$ (time permitting)

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Notation and Definition of Lie Superalgebras

1. Let K be an algebraically closed field with char. $p > 2$.
2. A superspace is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$.
3. Denote $\underline{\dim}\, V = \dim V_{\bar{0}} | \dim V_{\bar{1}}$.

Definition

A Lie superalgebra is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ together with $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that,

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ *for* $\alpha, \beta \in \mathbb{Z}_2$;
2. $[x, y] = -(-1)^{|x||y|}[y, x]$, *where* $|.|$ *denotes the* $\mathbb{Z}_2$*-parity.*
3. $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

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Restricted Lie Superalgebras

Definition

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ *is* restricted *if*

1. $\mathfrak{g}_{\bar{0}}$ *is a restricted Lie algebra with a pth power map* $^{[p]} : \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}$,
2. $\mathfrak{g}_{\bar{1}}$ *is a restricted* $\mathfrak{g}_{\bar{0}}$*-module by the adjoint action.*

1. Associative (super)algebras with supercommutators, e.g. $\mathfrak{gl}(V)$. The pth power map is in the usual sense of associative algebra.
2. Simple Lie (super)algebras with Chevalley generators. The pth power map is specified by

$$e_i^{[p]} = 0 = f_i^{[p]}, \ h_i^{[p]} = h_i$$

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- Key property of a restricted Lie (super)algebra $\mathfrak{g}$:
  $\forall x \in \mathfrak{g}_{\bar{0}}$, $x^p - x^{[p]}$ is central in enveloping algebra $U(\mathfrak{g})$.

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Reduced Enveloping Superalgebras

Basic definitions/properties:

- The *p*-center $\mathbb{Z}_p(\mathfrak{g}) := K\langle x^p - x^{[p]} \mid x \in \mathfrak{g}_{\bar{0}} \rangle \subset U(\mathfrak{g})$, is a polynomial algebra in $\dim \mathfrak{g}_{\bar{0}}$ variables.

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- For $\chi \in \mathfrak{g}_{\bar{0}}^*$, define the ideal $I_\chi = \langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_{\bar{0}} \rangle$.
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- $\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}_{\bar{0}}} 2^{\dim \mathfrak{g}_{\bar{1}}}$.
- Every simple $\mathfrak{g}$-module admits a p-character χ, hence it becomes a $U_\chi(\mathfrak{g})$-module.

Super KW Property: Formulation

- Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and regard $\chi \in \mathfrak{g}^*$ by setting $\chi(\mathfrak{g}_{\bar{1}}) = 0$.
- Denote the centralizer $\mathfrak{g}_\chi = \mathfrak{g}_{\chi,\bar{0}} + \mathfrak{g}_{\chi,\bar{1}}$,
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Conjecture (Super KW Property/Conjecture)

The dimension of every $U_\chi(\mathfrak{g})$-module is divisible by

$$\begin{cases} p^{\frac{d_0}{2}} 2^{\frac{d_1}{2}} & \text{if } d_1 \text{ is even,} \\ p^{\frac{d_0}{2}} 2^{\frac{d_1+1}{2}} & \text{if } d_1 \text{ is odd.} \end{cases}$$

(cross characteristic phenomenon: p and 2!)

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Remarks (Premet's theorem)

The (original) KW Conjecture holds (with $d_1 = 0$ above) for Lie algebras of reductive algebraic groups under mild assumptions on p.

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Question: For which Lie superalgebras does the Super KW Property hold?

Super KW Property: Formulation

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Introducing a new notation

$$\lfloor \frac{d_1}{2} \rfloor = \begin{cases} \frac{d_1}{2} & \text{if } d_1 \text{ is even,} \\ \frac{d_1+1}{2} & \text{if } d_1 \text{ is odd.} \end{cases}$$

Basic Classical Lie Superalgebras

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A full list (over $\mathbb{C}$):

- Four series of type A, B, C, D (which make sense for $p > 2$): $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$,$\mathfrak{osp}(2|2n)$, $\mathfrak{osp}(2m|2n)$;
- Three exceptional ones (which make sense for $p > 3$): $D(2,1;\alpha)$, $F(4)$, and $G(3)$.

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Basic classical Lie superalgebras are restricted.

Baby Verma modules for basic classical Lie superalgebras

- $\mathfrak{g}$ admits a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \text{and let } \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+.$$

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$$Z_\chi(\lambda) := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} K_\lambda,$$

where K_λ is the 1-diml $U_\chi(\mathfrak{b})$-module upon which $\mathfrak{h}$ acts via λ.

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  where K_λ is the 1-diml $U_\chi(\mathfrak{b})$-module upon which $\mathfrak{h}$ acts via λ.
- Each f-diml simple $U_\chi(\mathfrak{g})$-module is a quotient of some baby Verma. But baby Verma modules in general do not have unique simple quotients.

Super KW Property for Basic Classical Lie Superalgebras: Nilpotent p-Character

- Non-deg. bilinear form on $\mathfrak{g} \Rightarrow$
  the notion of nilpotent p-characters: $\chi \leftrightarrow e$, with $e \in \mathfrak{g}_{\bar{0}}$ nilpotent.
- Recall $d_0|d_1 =$ codimension of $\mathfrak{g}_\chi$.

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Theorem (The case of nilpotent p-character)

Let $\mathfrak{g}$ be a basic classical Lie superalgebra and χ a nilpotent p-character. The dimension of every $U_\chi(\mathfrak{g})$-module is divisible by

$$p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$$

Proof in case of Nilpotent p-Character

Modifying the approach of Premet-Skryabin, *bypassing support variety*.

Proof in case of Nilpotent p-Character

- For a subset X of $\mathfrak{g}_{\bar{0}}^*$, let $I(X) = \{f \in K[\mathfrak{g}_{\bar{0}}^*] |\ f(X) = 0\}$. This is a naturally filtered ideal of $K[\mathfrak{g}_{\bar{0}}^*]$.

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- For a pair of non-negative integers $(d_0|d_1)$ with d_0 even, define,

$$\mathscr{X}_{d_0,d_1} := \{\xi \in \mathfrak{g}_{\bar{0}}^* | \ \exists U_\xi(\mathfrak{g})\text{-module } V \text{ with } p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor} \nmid \dim V\}$$
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A key observation is

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- It turns out that

$$\mathbb{P}(\mathbb{K}\mathscr{X}_{d_0,d_1}) \subseteq \mathscr{X}'_{d_0,d_1} \cap \mathbb{P}(\mathfrak{g}_{\bar{0}}^*) = \mathbb{P}(\mathscr{Y}_{d_0,d_1}).$$

So $\mathbb{K}\mathscr{X}_{d_0,d_1} \subseteq \mathscr{Y}_{d_0,d_1}$.

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Recall that $\chi \in \mathbb{K}G.\chi$ when χ is nilpotent.
The Super KW property for nilpotent p-characters follows from the following proposition

Proposition

Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be nilpotent and let

$$s_\alpha(\chi) = \max_{\xi \in \mathbb{K}G.\chi} codim \mathfrak{g}_{\xi,\alpha}, \quad \alpha \in \mathbb{Z}_2.$$

Then each finite-dimensional $U_\chi(\mathfrak{g})$-module has dimension divisible by $p^{\frac{s_0}{2}} 2^{\lfloor \frac{s_1}{2} \rfloor}$.

Proof.

Suppose NOT. Then $\chi \in \mathscr{X}_{s_0,s_1}$. So $G.\chi \subseteq \mathscr{X}_{s_0,s_1}$.
Thus $\mathbb{K}G.\chi \subseteq \mathbb{K}\mathscr{X}_{s_0,s_1} \subseteq \mathscr{Y}_{s_0,s_1}$ by $(\star)$.
Then codim$\mathfrak{g}_\xi \prec (s_0|s_1)$ $\forall\, \xi \in \mathbb{K}G.\chi$. $\rightarrow\leftarrow$ the choice of $(s_0|s_1)$. □

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Semisimplicity criterion for semisimple *p*-characters

Let χ be semisimple with $\chi(\mathfrak{n}_{\bar{0}}^{\pm}) = 0$.

- Let $\Delta^{+} = \Delta_{\bar{0}}^{+}\,(\text{even}) \cup \Delta_{\bar{1}}^{+}\,(\text{odd})$ be the positive roots of $\mathfrak{g}$. Let ρ be the half sum of positive roots.

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A baby Verma module $Z_\chi(\lambda)$ for $\lambda \in \Lambda_\chi$ is irreducible iff

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1. 3 types of simple roots $\rightsquigarrow$ repn. th. of 3 rank-1 Lie superalgebras.
2. non-conjugate root systems $\rightsquigarrow$ odd reflections.

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Remark

Chaowen Zhang has also obtained the above formula in his recent preprint on arXiv.

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Corollary

The algebra $U_\chi(\mathfrak{g})$ is a semisimple algebra iff χ is regular semisimple, i.e. $\chi(H_\alpha) \neq 0$ for each coroot vector H_α.

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Surprise: no projective simple (=Steinberg) module.

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There are three coadjoint orbits of $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_2$:

- regular semisimple: $\chi \leftrightarrow h$
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- The algebra $U_\chi(\mathfrak{g})$ is semisimple.
- Every baby Verma module is simple.
- Every simple module is of type **M** and of dimension $2p$.

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1. Every baby Verma $U_\chi(\mathfrak{g})$-module, of dimension $2p$, is simple. Among them,
2. $\frac{p-3}{2}$ non-isomorphic simple modules of type **M**, and
3. one simple module of type **Q**.

The End

Thank you.