

Support varieties for Lie algebras of Cartan type

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1. Introduction: Back grounds and Motivation

- Cohomological support varieties for restricted Lie algebras have become an important aspect of the study of modular representations of Lie algebras. With the aid of support variety theory, Premet managed to prove an important conjecture raised by Kac-Weisfeiler (cf. [Pr1]).

[Pr1] Premet A., *Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture*, *Invent. Math.* 121 (1995), 79-117.

In his series of papers, Premet deeply studied the support varieties for a reductive Lie algebra $\mathfrak{g} = \text{Lie}(G)$, where G is a connected simply connected semisimple algebraic group. He proved a remarkable result that for any $\chi \in \mathfrak{g}^*$ and any module in the χ -reduced module category, the associated support variety is contained in the centralizer of χ (cf. [Pr2], [Pr3]).

[Pr2] Premet A., *Support varieties of non-restricted modules over Lie algebras of reductive groups*, J. London Math. Soc. 55(2) (1997), 236-250.

[Pr3] Premet A., *Complexity of Lie algebra representations and nilpotent elements of the stabilizers of linear form*, Math. Z. 228 No. 2 (1998), 255-282.

- **Question:** what the structures of support varieties for Cartan type Lie algebras are?

For restricted representations and generalized restricted representations, Lin-Nakano and Shu obtained some good description in [LN] and [Shu] respectively.

[LN] Lin Z. and Nakano D., *Algebraic group actions in the cohomology theory of Lie algebras of Cartan type*, J. Algebra 179 (1996), 852-888.

[Shu] Shu Bin, *The realizations of primitive p -envelopes and the support varieties for graded Cartan type Lie algebras*, Comm. Algebra 25 No. 10 (1997), 3209-3223.

For the Witt algebra, Feldvoss-Nakano gave some thorough study in [FN].

Below, we will focus our concern on the study of support varieties of non-restricted modules for any restricted Cartan type Lie algebras (also extended to the graded cases).

[FN] Feldvoss J. and Nakano D., *Representation theory of the Witt algebra*, J. Algebra 203 (1998), 447-469.

2. Preliminaries and Main results

Throughout, the base field F is always assumed to be algebraically closed with characteristic $p > 2$. All modules (vector spaces, varieties) are defined over F .

- Quick review of the four classes of Cartan type Lie algebras

Fix a positive integer m and an m -tuple $\mathbf{n} = (n_1, \dots, n_m)$ of positive integers. Denote by $A(m; \mathbf{n})$ the index set $\{\alpha = (\alpha_1, \dots, \alpha_m) \mid 0 \leq \alpha_i \leq p^{n_i} - 1, i = 1, 2, \dots, m\}$, denote $(p^{n_1} - 1, p^{n_2} - 1, \dots, p^{n_m} - 1)$ by τ for brevity.

Associated with (m, \mathbf{n}) , one has a divided power algebra $\mathfrak{A}(m; \mathbf{n})$. The generalized Jacobson-Witt algebra $W(m; \mathbf{n})$ is the subalgebra of derivation algebra of $\mathfrak{A}(m; \mathbf{n})$ consisting of those so-called “special” derivations.

One can get other three series of subalgebras in $W(m; \mathbf{n})$, which are called (graded) Cartan type Lie algebras of series S , H , and K respectively, arising from the three exterior differentials $\omega_S, \omega_H, \omega_K$ (cf. [KS]).

For $L = X(m; \mathbf{n})$, $X \in \{W, S, H, K\}$ we have a graded structure $L = \sum_i^{\oplus} L_{[i]}$. Recall that under the mapping $\phi : W(m; \mathbf{n})_{[0]} \rightarrow \mathfrak{gl}(m)$, $x_i D_j \mapsto E_{ij}$, one have the basic isomorphisms: $S(m; \mathbf{n})_{[0]} \cong \mathfrak{sl}(m)$, and $H(m; \mathbf{n})_{[0]} \cong \mathfrak{sp}(m)$. Furthermore, we have $K(2r + 1; \mathbf{n})_{[0]} \cong \mathfrak{sp}(2r) \oplus FI$, where is a one-dimensional central extension of $\mathfrak{sp}(2r)$.

[KS] Kostrikin A. I. and Šafarevič I. R., *Graded Lie algebras of finite characteristic*, Math. USSR Izv. **3** (1969), 237-304.

Summing up, we have $L = \sum_i L_{[i]}$ with $L_{[0]}$ being a reductive Lie algebra $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}^-$, where \mathfrak{h} is the canonical maximal torus, and \mathfrak{n}^\pm is the sum of canonical positive (resp. negative) root spaces.

The filtration associated with the graded structure $L = \sum_i L_{[i]}$ will be denoted by $L_i = \sum_{j \geq i} L_{[j]}$. Let

$$N^+ := \mathfrak{n} \bigoplus_{i>0} L_{[i]}, N^- := \mathfrak{n}^- \bigoplus_{i<0} L_{[i]} \text{ and}$$

$$B^+ := \mathfrak{h} \bigoplus N^+, B^- := \mathfrak{h} \bigoplus N^-. \quad B^+ \text{ is often simply written as } B.$$

- Automorphisms of graded Lie algebras of Cartan type

Let $L = X(m; \mathbf{n})$, $X \in \{W, S, H, K\}$. Set $G_X := \text{Aut}(L)$. For $i \geq 0$, set $G_i = \{\varphi \in \text{Aut}(L) \mid (\varphi - \text{id})L_j \subseteq L_{i+j}\}$. Then from [W], $G_X = G_0 = G_X^* \rtimes G_1$, where $G_X^* = \{\varphi \in G_X \mid \varphi(L_{[i]}) = L_{[i]}, \forall i\}$. Note that G_X is a connected algebraic group for $X \in \{W, S, H, K\}$.

[W] Wilson R.L., *Automorphisms of graded Lie algebras of Cartan type*, Comm. Algebra 3 no. 7 (1975), 591-613.

As in [Shu], choose the canonical maximal torus T_W of the general linear algebraic group $GL(m; F)$, i.e.

$$T_W = \{t_{a_1, a_2, \dots, a_m} = \text{diag}(a_1, a_2, \dots, a_m) \mid a_i \in F^*, 1 \leq i \leq m\}.$$

[Shu] Shu Bin, *The realizations of primitive p -envelopes and the support varieties for graded Cartan type Lie algebras*, Comm. Algebra 25 No. 10 (1997), 3209-3223.

Set

$$T_X = \begin{cases} \{t \in T_W \mid t = \text{diag}(a_1, a_2, \dots, a_m), \prod_{i=1}^m a_i = 1\}, & \text{if } X = S, \\ \{t \in T_W \mid t = \text{diag}(a_1, a_2, \dots, a_{2r}), a_j a_{j+r} = 1, \\ 1 \leq j \leq r\}, & \text{if } X = H, \\ \{t \in T_W \mid t = \text{diag}(a_1, a_2, \dots, a_{2r}, 1), a_j a_{j+r} = 1, \\ 1 \leq j \leq r\} \times \{\text{diag}(a, a, \dots, a, a^2) \mid a \in F^*\}, & \text{if } X = K. \end{cases}$$

Then it can be easily verified that T_W, T_S, T_H, T_K are respectively the maximal tori of $GL(m, F)$, $SL(m, F)$, $SP(2r, F)$ and $CSP(2r, F)$

The following proposition is due to Shu.

Proposition 2.1 ([Shu, Proposition 1.4])

Keep notations as above, Let $L = X(m; \mathbf{n})$, $X \in \{W, S, H, K\}$, then $\text{Lie}(T_X) = \mathfrak{h}$, the canonical maximal torus of $L_{[0]}$.

[Shu] Shu Bin, *The realizations of primitive p -envelopes and the support varieties for graded Cartan type Lie algebras*, Comm. Algebra 25 No. 10 (1997), 3209-3223.

- Cohomological support varieties

Let V be a vector space over F . Denote $V^{(-1)}$ the F -space with underlying abelian group V and with scalars $k \in F$ acting through multiplication by k^p . For a restricted Lie algebra \mathfrak{g} , set

$$H(\mathfrak{g}) := \bigoplus_{i \geq 0} \text{Ext}_{u(\mathfrak{g})}^{2i}(F, F)^{(-1)}$$

which is a commutative, associative graded F -algebra.

Moreover, $H(\mathfrak{g})$ is Noetherian. The cohomological variety $|\mathfrak{g}|$ of \mathfrak{g} is the affine algebraic variety associated to the commutative Noetherian F -algebra $H(\mathfrak{g})$. More precisely, $|\mathfrak{g}|$ has coordinate algebra equal to $H(\mathfrak{g})$ modulo its radical.

For any $\varphi \in \mathfrak{g}^*$, there exists a split Lie algebra extension $F \oplus \mathfrak{g}$ with $[p]$ -mapping satisfying $(k, x)^{[p]} = (\varphi(x)^p, x^{[p]})$ defined by Hochschild (cf. [Hoc]). Then we obtain a natural algebra homomorphism $\Phi^* : S(\mathfrak{g}^*) \longrightarrow H(\mathfrak{g})$ extending the Hochschild map $\mathfrak{g}^* \longrightarrow \text{Ext}_{u(\mathfrak{g})}^2(F, F)^{(-1)}$. Φ^* induces the following morphism of algebraic varieties:

$$\Phi : |\mathfrak{g}| \longrightarrow \mathfrak{g} \cong \text{Spec } S(\mathfrak{g}^*)$$

which is finite by [FP1, Proposition 2.4].

[Hoc] Hochschild G., *Cohomology of restricted Lie algebras*, Amer. J. Math. 76 (1954), 555-580.

[FP1] Friedlander E. and Parshall B., *Geometry of p -unipotent Lie algebras*, J. Algebra 109 (1987), 25-45.

The following Lemma is due to J.C. Jantzen.

Lemma 2.2 ([Jan])

Let \mathfrak{g} be a restricted Lie algebra, Φ is defined as above. Then $\Phi(|\mathfrak{g}|) = \mathcal{N}_p(\mathfrak{g})$, where $\mathcal{N}_p(\mathfrak{g}) := \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$.

[Jan] Jantzen J.C., *Kohomologie von p -Lie algebren und nilpotente Elemente*, Abh. Math. Sem. Univ. Hamburg 56 (1986), 191-219.

Let $\chi \in \mathfrak{g}^*$, M be a χ -reduced \mathfrak{g} -module. By [FP2, Proposition 5.1], the graded algebra $H(\mathfrak{g})$ acts on the graded vector space

$$H_\chi(M) := \bigoplus_{i \geq 0} \text{Ext}_{U_\chi(\mathfrak{g})}^i(M, M)^{(-1)}.$$

Define $|\mathfrak{g}|_M$ to be the closed subvariety of $|\mathfrak{g}|$ given as the support of the $H(\mathfrak{g})$ -module $H_\chi(M)$ (cf. [FP2]). As Φ is finite, $\Phi(|\mathfrak{g}|_M)$ is a Zariski closed conical subset of $\mathcal{N}_p(\mathfrak{g})$. Set $\mathcal{V}_\mathfrak{g}(M) = \Phi(|\mathfrak{g}|_M)$ which is called the support variety of M .

[FP2] Friedlander E. and Parshall B., *Modular representation theory of Lie algebras*, Amer. J. Math. 110 (1988), 1055-1094.

The following Proposition presents the basic properties of support varieties first obtained by Friedlander and Parshall.

Proposition 2.3 ([FP2])

Let \mathfrak{g} be a restricted Lie algebra, $\chi \in \mathfrak{g}^*$. M, N are χ -reduced \mathfrak{g} -modules. $\mathfrak{h} \subseteq \mathfrak{g}$ is a restricted subalgebra of \mathfrak{g} . Then

- (1) $\mathcal{V}_{\mathfrak{g}}(M) = \{x \in \mathcal{N}_p(\mathfrak{g}) \mid M|_{U_{\chi}(Fx)} \text{ is not projective}\} \cup \{0\}$.
- (2) $\mathcal{V}_{\mathfrak{g}}(M) = 0$ if and only if M is a projective $U_{\chi}(\mathfrak{g})$ -module if and only if $M^* \otimes M$ is a projective $u(\mathfrak{g})$ -module.
- (3) $\mathcal{V}_{\mathfrak{g}}(M \oplus N) = \mathcal{V}_{\mathfrak{g}}(M) \cup \mathcal{V}_{\mathfrak{g}}(N)$.
- (4) $\mathcal{V}_{\mathfrak{h}}(M) = \mathfrak{h} \cap \mathcal{V}_{\mathfrak{g}}(M)$.

[FP2] Friedlander E. and Parshall B., *Modular representation theory of Lie algebras*, Amer. J. Math. 110 (1988), 1055-1094.

- The main results

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, be a restricted Lie algebra of Cartan type. We need the following definition analogous to the case of classical Lie algebras.

Definition 2.4

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, $\chi \in L^*$. χ is called semisimple (resp. regular semisimple) if $\chi|_{L_{[i]}} = 0$ for all $i \neq 0$, and $\chi|_{L_{[0]}}$ is semisimple (resp. regular semisimple) in the sense of classical Lie algebras, under the equivalence of $L_{[0]}$ with $\mathfrak{gl}(m)$, $\mathfrak{sl}(m)$, $\mathfrak{sp}(m)$ or $\mathfrak{sp}(m-1) \oplus FI$ respectively for $X = W, S, H, K$.

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, $\chi \in L^*$ be semisimple. Set $\mathcal{X}(\chi) = \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p, \forall h \in \mathfrak{h}\}$. Take any $\lambda \in \mathcal{X}(\chi)$, one can define a one dimensional \mathfrak{h} -module $\mathbf{1}_\lambda$ via $h \cdot v_\lambda = \lambda(h)v_\lambda, \forall h \in \mathfrak{h}$, where v_λ is a basis of $\mathbf{1}_\lambda$. Then the one dimensional \mathfrak{h} -module $\mathbf{1}_\lambda$ can be extended to a B -module with the trivial action of N^+ . Define $Z(\lambda) = U_\chi(L) \otimes_{U_\chi(B)} \mathbf{1}_\lambda$. Our main results is the following on the support varieties for a class of L -modules with semisimple characters.

Theorem 2.5

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$ and $L \neq W(1; \mathbf{1})$, $\chi \in L^*$ be semisimple. For any $\lambda \in \mathcal{X}(\chi)$,
 $\mathcal{V}_L(Z(\lambda)) = \mathcal{V}_{N^+}(Z(\lambda)) = \{x \in N^+ \mid x^{[p]} = 0\}$.

Theorem 2.6

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, $\chi \in L^*$ be regular semisimple. Then for any finite dimensional $U_\chi(L)$ -module M ,
 $\mathcal{V}_L(M) \subseteq \{x \in N^+ \mid x^{[p]} = 0\}$.

3. The proof of the main results (Sketch)

In this section, we give the proof of the main results presented in the previous section. For that we will make some necessary preparation in the first two subsections.

- **Actions of algebraic groups on χ -reduced enveloping algebras**

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, $\chi \in \mathfrak{g}^*$. The χ -reduced enveloping algebra $U_\chi(\mathfrak{g})$ is by definition the quotient of the universal enveloping algebra $U(\mathfrak{g})$ by the ideal I_χ generated by those central elements $x^p - x^{[p]} - \chi(x)^p$ for all $x \in \mathfrak{g}$. When $\chi = 0$, $U_0(\mathfrak{g})$ is often called the restricted enveloping algebra of \mathfrak{g} , and is simply denoted by $u(\mathfrak{g})$.

For a restricted Lie algebra \mathfrak{g} , consider the p -automorphism group $G = \text{Aut}_p(\mathfrak{g})$ consisting of all restricted automorphisms of \mathfrak{g} . Then it can be easily shown that G is an algebraic group. Let $\chi \in \mathfrak{g}^*$, for any $\sigma \in G$, define $\sigma \cdot \chi \in \mathfrak{g}^*$ such that $(\sigma \cdot \chi)(x) = \chi(\sigma^{-1}x)$, $\forall x \in \mathfrak{g}$. This defines a rational action of G on \mathfrak{g}^* . Set $G_\chi := \{\sigma \in G \mid \sigma \cdot \chi = \chi\}$, the centralizer of χ in G . It is obvious that G_χ is a closed subgroup of G . The following Proposition implies that the action of G on \mathfrak{g} induces a rational representation of G_χ on $U_\chi(\mathfrak{g})$.

Proposition 3.1

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, G and G_χ be as above. Then $U(\mathfrak{g})$ is a rational G -module. Furthermore, $U_\chi(\mathfrak{g})$ is a rational G_χ -module such that the action of G_χ on \mathfrak{g} is the restriction of G to G_χ .

Remark 3.2

If $(\mathfrak{g}, [p])$ is a centerless restricted Lie algebra, then $\text{Aut}_p(\mathfrak{g}) = \text{Aut}(\mathfrak{g})$.

Let G_χ^0 denote the connected component of G_χ containing the unit. The following result implies that any irreducible $U_\chi(\mathfrak{g})$ -module and projective $U_\chi(\mathfrak{g})$ -module can be given rational G_χ^0 -module structures.

Proposition 3.3

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, G, G_χ and G_χ^0 be as above. Then any irreducible or projective $U_\chi(\mathfrak{g})$ -module is a rational G_χ^0 -module.

- The T_X -module structure on $Z(\lambda)$

Lemma 3.4

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, $\chi \in L^*$ be semisimple. Let $G = \text{Aut}(L)$. Then $T_X \subseteq G_\chi$.

Proposition 3.5

For any $\lambda \in X(\chi)$, $Z(\lambda)$ has a T_X -module structure.

Remark 3.6

With the T_X -module structure on $U_\chi(L)$ and $Z(\lambda)$, one can easily see that $H(L)$, $H_\chi(Z(\lambda))$ have rational T_X -module structures, and Φ (see Section 2 for definition) is T_X -equivariant. So $\mathcal{V}_L(Z(\lambda))$ is a T_X -variety.

Proposition 3.7

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$ and $L \neq W(1; \mathbf{1})$. Let $\chi \in L^*$ be semisimple. For any given $n \in N^+$, there exists $y \in U_\chi(B^-)$ such that $y \otimes v_\lambda$ is a nonzero element in $Z(\lambda)$, while $n(y \otimes v_\lambda) = 0$.

Proposition 3.8

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$ and $L \neq W(1; \mathbf{1})$, $\chi \in L^*$ be semisimple. For any given $n \in N^+$, $Z(\lambda)$ is not projective as a $U_\chi(n)$ -module.

by combining Proposition 2.3, Proposition 3.8 and a result of Lin-Nakano [LN, Lemma 3.1], we complete the proof of Theorem 2.5.

The proof of Theorem 2.6 can be achieved by Theorem 2.5 and the fact that $\mathcal{V}_L(M_2) \subseteq \mathcal{V}_L(M_1) \cup \mathcal{V}_L(M_3)$ for an exact sequence of L -modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0.$$

[LN] Lin Z. and Nakano D., *Algebraic group actions in the cohomology theory of Lie algebras of Cartan type*, J. Algebra 179 (1996), 852-888.

For the case that $L = W(1; \mathbf{1})$, the support variety of $Z(\lambda)$ for $\lambda \in \mathcal{X}(\chi)$ was obtained by Feldvoss J. and Nakano D. in [FN]. For completeness, we cite it in the following Theorem.

Theorem 3.9 [FN, Theorem 4.4]

Let $L = W(1; \mathbf{1})$, $0 \neq \chi \in L^*$ be semisimple. Then for any $\lambda \in \mathcal{X}(\chi)$, $\mathcal{V}_L(Z(\lambda)) = L_2$.

[FN] Feldvoss J. and Nakano D., *Representation theory of the Witt algebra*, J. Algebra 203 (1998), 447-469.

4. More general results

Let $L = \bigoplus_{i=-r}^s L_{[i]}$ be a graded restricted Lie algebra with admissible graded structure and restricted structure, i.e.

$(L_{[i]})^{[p]} \subseteq L_{[pi]}, \forall i$. Set $L_i = \bigoplus_{j=i}^s L_{[j]}, \forall -r \leq i \leq s$. Let $\chi \in L^*$.

Definition 4.1

An L -module M with character χ is called a baby Verma module if there exists an L_0 -submodule M_0 of M such that $M \cong U_{\chi}(L) \otimes_{U_{\chi}(L_0)} M_0$.

Definition 4.2

Let $L = \bigoplus_{i=-r}^s L_{[i]}$ be a graded Lie algebra, $\chi \in L^*$. Define the height $\text{ht}(\chi)$ of χ by $\text{ht}(\chi) := \min\{i \mid \chi|_{L_i} = 0, -r \leq i \leq s\}$ if $\chi|_{L_{[s]}} = 0$ and $\text{ht}(\chi) := s + 1$ if $\chi|_{L_{[s]}} \neq 0$.

The following result can be proved using an elementary way, which can also be obtained by applying [FN, Proposition 4.1].

Theorem 4.3

Let $L = \bigoplus_{i=-r}^s L_{[i]}$ be a graded restricted Lie algebra with admissible graded structure and restricted structure. M is a baby Verma module of L with character χ . Then $\mathcal{V}_L(M) \subseteq \mathcal{N}_p(L_0)$.

[FN] Feldvoss J. and Nakano D., *Representation theory of the Witt algebra*, J. Algebra 203 (1998), 447-469.

In the following sequel, we always assume $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, $\chi \in L^*$. Then we have the following direct corollaries.

Corollary 4.4

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, M be a baby Verma module of L with character χ . Then $\mathcal{V}_L(M) \subseteq \mathcal{N}_p(L_0)$.

Corollary 4.5

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$. $\chi \in L^*$ with $\text{ht}(\chi) < p - 1 - \delta_{XK}$. Then for any non-exceptional irreducible $U_\chi(L)$ -module M , $\mathcal{V}_L(M) \subseteq \mathcal{N}_p(L_0)$.

For the case of the exceptional L -modules with character χ such that $\text{ht}(\chi) = 0$, we have the following result.

Theorem 4.6

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$. $\chi \in L^*$ with $\text{ht}(\chi) = 0$. Let $\bar{\mathcal{V}}_i$ be an irreducible $U_\chi(L)$ -module with exceptional highest weight ϖ_i . Then $\mathcal{V}_L(\bar{\mathcal{V}}_i) \subseteq \mathcal{N}_p(L_0)$.

As a consequence of Corollary 4.5 and Theorem 4.6, we have the following corollary.

Corollary 4.7

Let $L = X(m; \mathbf{1})$, $\chi \in L^*$ such that $0 \leq \text{ht}(\chi) < p - 1 - \delta_{XK}$, then for any finite dimensional $U_\chi(L)$ -module M , we have $\mathcal{V}_L(M) \subseteq \mathcal{N}_p(L_0)$.

We conclude this section with the following Conjecture and Remarks.

Conjecture

Let $L = X(m; \mathbf{1})$, $X \in \{W, S, H, K\}$, $\chi \in L^*$ such that $\text{ht}(\chi)$ is neither minimal nor maximal, then for any finite dimensional $U_\chi(L)$ -module M , $\mathcal{V}_L(M) \subseteq \mathcal{N}_p(L_0)$.

Remark 4.8

The above Conjecture holds for the Witt algebra $W(1; \mathbf{1})$ by [FN]. And furthermore, by Theorem 2.6, the Conjecture holds for χ being regular semisimple.

Remark 4.9

Our results can be extend to the case of generalized restricted Lie algebras of Cartan type parallel to the arguments in [LN] and [Shu].

[FN] Feldvoss J. and Nakano D., *Representation theory of the Witt algebra*, J. Algebra 203 (1998), 447-469.

[LN] Lin Z. and Nakano D., *Algebraic group actions in the cohomology theory of Lie algebras of Cartan type*, J. Algebra 179 (1996), 852-888.

[Shu] Shu Bin, *The realizations of primitive p -envelopes and the support varieties for graded Cartan type Lie algebras*, Comm. Algebra 25 No. 10 (1997), 3209-3223.

Thank you very much!

