

# Atypicality, Complexity and Module Varieties for Classical Lie Superalgebras

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## References

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## Motivation

*Let us first consider blocks of the Category  $\mathcal{O}$  (or relative Category  $\mathcal{O}$ ) for complex semisimple Lie algebras:*

- these are highest weight categories with finitely many simple modules,*
- the projective resolutions of modules have finite length,*
- the cohomology only lives in finitely many degrees.*

*Furthermore, if one takes the endomorphism algebra of a progenerator this gives rise to a finite-dimensional quasi hereditary algebra (which is Morita equivalent to  $\mathcal{O}$ ).*

## Motivation

*On the other hand, if one considers the category  $\mathcal{F}$  of finite-dimensional modules for a classical Lie superalgebra (i.e.,  $\mathfrak{g} = \mathfrak{g}(m|n)$ ) which are completely reducible over  $\mathfrak{g}_0$  then*

- *this is also a highest weight category (as observed by Brundan),*
- *there are infinitely many simple modules,*
- *$\mathcal{F}$  is self-injective (i.e., projective is equivalent to injective) [BKN3],*
- *projective resolutions have infinite length, and the terms can grow in dimension.*

*Thus, the cohomology can also grow in dimension so one is motivated to study these objects with ideas and tools from modular representation theory.*

## Motivation

*It is useful to think about modeling our approach on finite group schemes or finite-dimensional cocommutative Hopf algebras. Let  $A$  be a finite dimensional cocommutative Hopf algebra over an algebraically closed field  $k$ .*

- $A$  is a Frobenius algebra (an  $A$ -module is projective if and only if it is injective).*
- every finite dimensional  $A$ -module admits a minimal projective resolution whose terms have dimensions which increase at a polynomial rate of growth.*

## Motivation

*Observe that if  $A$  is a finite-dimensional algebra which is Frobenius and quasi-hereditary then  $A$  is semisimple.*

*Therefore, in order to have both features (i.e., Frobenius and highest weight category) and have a non-trivial theory, we must have infinitely many simple modules in our module category.*

Throughout let  $k = \mathbb{C}$ . Let  $\mathfrak{g}$  be a *Lie superalgebra* which is a  $\mathbb{Z}_2$ -graded vector space

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with a bracket operation  $[\ , \ ] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  which preserves the  $\mathbb{Z}_2$ -grading and satisfies graded versions of the usual Lie bracket axioms.

### Definition

A finite dimensional Lie superalgebra  $\mathfrak{g}$  is called *classical* if there is a connected reductive algebraic group  $G_{\bar{0}}$  such that  $\text{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$  and an action of  $G_{\bar{0}}$  on  $\mathfrak{g}_{\bar{1}}$  which differentiates to the adjoint action of  $\mathfrak{g}_{\bar{0}}$  on  $\mathfrak{g}_{\bar{1}}$ .

# The Lie Superalgebra $\mathfrak{gl}(m|n)$

## Example

The underlying vector space for  $\mathfrak{gl}(m|n)$  is the set of  $m \times n$  matrices over  $\mathbb{C}$ . We have  $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ , where  $\mathfrak{g}_{\bar{0}}$  consists of matrices of the form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Moreover,  $\mathfrak{g}_{\bar{1}}$  consists of matrices

$$\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.$$

The supercommutator is given by

$$[E_{i,j}, E_{k,l}] = E_{i,j}E_{k,l} - (-1)^{\bar{E}_{i,j}\bar{E}_{k,l}} E_{k,l}E_{i,j}.$$

Let  $\mathcal{F}$  be the full subcategory of  $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)}$  (relative category) where the objects are the finite dimensional supermodules which are completely reducible as  $\mathfrak{g}_0$ -supermodules.

$U(\mathfrak{g})$  universal enveloping algebra of  $\mathfrak{g}$

$S^\bullet(\mathfrak{g}_1^*)$  symmetric algebra on the dual of  $\mathfrak{g}_1$ .

## Definition

A *basic classical* Lie superalgebra is a classical Lie superalgebra with a nondegenerate invariant supersymmetric even bilinear form.

For  $\mathfrak{g}$  basic classical:

$\Delta$  the set of roots;  $\Delta_0$  the set of even roots;  $\Delta_1$  the set of odd roots.

$$\rho = \frac{1}{2}(\sum_{\alpha \in \Delta_0^+} \alpha - \sum_{\alpha \in \Delta_1^+} \alpha).$$

# Defect and Atypicality

## Definition

(Kac-Wakimoto) Let  $\mathfrak{g}$  be a basic classical Lie algebra. The *defect* of  $\mathfrak{g}$ , denoted by  $\text{defect}(\mathfrak{g})$ , is the dimension of the maximal isotropic subspace in the  $\mathbb{R}$ -span of  $\Delta$ .

## Example

- $\text{defect}(\mathfrak{gl}(m|n)) = \text{defect}(\mathfrak{sl}(m|n)) = \min(m, n)$ ,
- $\text{defect}(D(2, 1; \alpha)) = \text{defect}(G(3)) = 1$ .

## Definition

Let  $\mathfrak{g}$  be a basic classical Lie algebra and  $\lambda \in \mathfrak{t}^*$  be a weight. The *atypicality* of  $\lambda$ , denoted by  $\text{atp}(\lambda)$ , is the maximal number of linearly independent mutually orthogonal, positive isotropic roots  $\alpha \in \Delta^+$  such that  $(\lambda + \rho, \alpha) = 0$ .

## Example

Let  $\mathfrak{g} = \mathfrak{gl}(1|1)$ . The simple modules in the principal block are one dimensional and indexed by  $L(\lambda | -\lambda)$  where  $\lambda \in \mathbb{Z}$ . The projective cover  $P(\lambda | -\lambda)$  of  $L(\lambda | -\lambda)$  is four dimensional.

The minimal projective resolution of the trivial module  $L(0 | 0)$  is given by

$$\dots \rightarrow P(1 | -1) \oplus P(-1 | 1) \rightarrow P(0 | 0) \rightarrow L(0 | 0) \rightarrow 0.$$

Therefore,  $\dim P_n = 4(n+1)$  and  $c_{\mathcal{F}}(L(0 | 0)) = 2$  (rate of growth of the minimal proj. resolution).

In fact, one can easily show that  $c_{\mathcal{F}}(L(\lambda | -\lambda)) = 2$  for all  $\lambda \in \mathbb{Z}$ . The atypicality of every simple module in the principal block is one and equal to  $\dim H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ .

# Relative Cohomology and Detecting Subalgebras

Let  $H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, M)$  be the relative Lie algebra cohomology of the pair  $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$  which is obtained from the complex

$$C^\bullet = \text{Hom}_{\mathfrak{g}_{\bar{0}}}(\Lambda_{super}^\bullet(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M).$$

## Theorem

Let  $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$  be as above. Then

$$\text{Ext}_{\mathcal{F}}^\bullet(\mathbb{C}, \mathbb{C}) \cong H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong (\Lambda_{super}^\bullet(\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*)^{G_{\bar{0}}} \cong \mathcal{S}^\bullet(\mathfrak{g}_1^*)^{G_{\bar{0}}}.$$

Note that the cohomology ring is finitely generated because  $G_{\bar{0}}$  is reductive.

## Theorem

Let  $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$  be a classical (simple) Lie superalgebra. Then there exists a subalgebra  $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$  isomorphic to  $\oplus_{\mathfrak{q}}(1)$  and a finite reflection group  $W$  such that restriction map

$$H^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \rightarrow H^{\bullet}(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W.$$

is an isomorphism.

## Theorem

Let  $\mathfrak{g}$  be a basic classical simple Lie superalgebra. Moreover, assume that  $\mathfrak{g} \not\cong \mathfrak{psl}(n|n)$ . Then the following are equal:

- (a)  $\text{defect}(\mathfrak{g})$
- (b)  $\dim H^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ ;
- (c)  $\dim H^{\bullet}(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$ .
- (d)  $\dim \mathfrak{e}_{\bar{1}}$ .

# Support Varieties and Atypicality

Using the finite generation of cohomology [BKN1] we can define the following support varieties for modules over  $\mathfrak{g}$  and  $\mathfrak{e}$ :

$$\mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}(M) = \text{Maxspec}(\mathbf{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})/J_{\mathfrak{g}}(M \otimes M^*)) \subseteq \mathfrak{g}_{\bar{1}}/G_{\bar{0}}$$

$$\mathcal{V}_{(\mathfrak{e}, \mathfrak{e}_{\bar{0}})}(N) = \text{Maxspec}(\mathbf{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})/J_{\mathfrak{e}}(N \otimes N^*)) \subseteq \mathfrak{e}_{\bar{1}}$$

where  $J_{\mathfrak{g}}(M \otimes M^*)$  is the annihilator of the cohomology ring on  $\mathbf{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M \otimes M^*)$ . The ideal  $J_{\mathfrak{e}}(-)$  is similarly defined.

Let  $M$  be a supermodule for  $\mathfrak{g}$ . The restriction map in cohomology induces a map  $res^*$  on these varieties with

$$\mathcal{V}_{(\mathfrak{e}, \mathfrak{e}_{\bar{0}})}(M)/W \hookrightarrow \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}(M).$$

# Atypicality Conjecture

## Conjecture

*Let  $\mathfrak{g}$  be a basic classical Lie superalgebra. Then*

$$\text{atyp}(\lambda) = \dim \mathcal{V}_{(\epsilon, \epsilon_{\bar{0}})}(L(\lambda)).$$

Note that if this conjecture is true then one can extend the definition of atypicality to all modules in  $\mathcal{F}$ .

## Theorem

Let  $\mathfrak{g} = \mathfrak{gl}(m|n)$ ,  $r = \text{defect}(\mathfrak{g})$ , and  $L(\lambda)$  be a simple  $\mathfrak{g}$ -module of atypicality  $k$ . Then

$$(a) \quad \text{res}^* \left( \mathcal{V}_{(\epsilon, \epsilon_{\bar{0}})}(L(\lambda)) \right) = \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}(L(\lambda)) \cong \mathbb{A}^k.$$

$$(b) \quad \mathcal{V}_{(\epsilon, \epsilon_{\bar{0}})}(L(\lambda)) = W \cdot \tilde{\mathbf{e}}_{\bar{1}}.$$

where  $\dim \tilde{\mathbf{e}}_{\bar{1}} = k$ . In particular,  $\mathcal{V}_{(\epsilon, \epsilon_{\bar{0}})}(L(\lambda))$  is the union of  $\binom{r}{k}$   $k$ -dimensional subspaces.

# Complexity for modules in $\mathcal{F}$

## Definition

Let  $\mathcal{V} = \{V_t : t \in \mathbb{N}\} = \{V_\bullet\}$  be a sequence of finite dimensional  $\mathbb{C}$ -vector spaces. The *rate of growth* of  $\mathcal{V}$ ,  $r(\mathcal{V})$ , is the smallest positive integer  $c$  such that  $\dim_k V_t \leq C \cdot t^{c-1}$  for some constant  $C > 0$ . If no such integer exists then  $\mathcal{V}$  has infinite rate of growth.

## Definition

Let  $M \in \mathcal{F}$  and  $P_\bullet \twoheadrightarrow M$  be a minimal projective resolution for  $M$ . Following Alperin (1977), we define the *complexity of  $M$*  to be  $r(\{P_n : n = 0, 1, 2, \dots\})$ .

## Theorem

Let  $M$  be an object of  $\mathcal{F}$ . Then

- (i)  $c_{\mathcal{F}}(M) = 0$  if and only if  $M$  is projective;
- (ii)  $c_{\mathcal{F}}(M) \leq \dim \mathfrak{g}_{\bar{1}}$ .

Part (i) follows by the self injectivity of  $\mathcal{F}$ . For part (ii) we construct an explicit projective resolution for the trivial module

$$\dots \xrightarrow{\partial_3} D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0 \xrightarrow{\partial_0} \mathbb{C} \rightarrow 0$$

where

$$D_p := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} \Lambda_{\text{sup}}^p(\mathfrak{g}/\mathfrak{g}_0)$$

for  $p \geq 0$ . Note  $r(\{D_p : p = 0, 1, 2, \dots\}) = \text{Kr. dim } S^\bullet(\mathfrak{g}_{\bar{1}})$ . The result follows by tensoring this resolution by  $M$ .

## Theorem

Let  $M \in \mathcal{F}$  and let  $P_\bullet \twoheadrightarrow M$  be a minimal projective resolution. Then

$$c_{\mathcal{F}}(M) := r(P_\bullet) = r\left(\mathrm{Ext}_{(\mathfrak{g}, \mathfrak{g}_0)}^\bullet(M, \bigoplus S^{\dim P(S)})\right)$$

where the sum is over all simple modules  $S$  in  $\mathcal{F}$ , and  $P(S)$  is the projective cover of  $S$ .

# Type I Lie superalgebras

Let  $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  (compatible  $\mathbb{Z}$ -grading) be a Type I classical Lie superalgebra. The Lie superalgebras  $\mathfrak{gl}(m|n)$  and the simple Lie superalgebras of types  $A(m, n)$  and  $C(n)$  are Type I.

Let

$$\mathfrak{p}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{p}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}.$$

Fix a Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}_0$  and Borel subalgebra  $\mathfrak{b}_0 \subseteq \mathfrak{g}_0$  so that  $\mathfrak{h} \subseteq \mathfrak{b}_0$ .

## Kac and dual Kac modules

Let  $X^+$  denote the parameterizing set of highest weights for the simple finite dimensional  $\mathfrak{g}_0$ -modules with respect to the pair  $(\mathfrak{h}, \mathfrak{b}_0)$ . For  $\lambda \in X^+$ , let  $L_0(\lambda)$  denote the simple  $\mathfrak{g}_0$ -module of highest weight  $\lambda$ . View  $L_0(\lambda)$  as a simple  $\mathfrak{p}^\pm$ -supermodule via inflation. Set

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} L_0(\lambda) \quad \text{and} \quad K^-(\lambda) = \text{Hom}_{U(\mathfrak{p}^-)}(U(\mathfrak{g}), L_0(\lambda))$$

be the *Kac supermodule* and the *dual Kac supermodule*, respectively

The category  $\mathcal{F}$  for Type I Lie superalgebras can be regarded as a highest weight category (as defined in Cline-Parshall-Scott using the Kac and dual Kac modules). This category is also self-injective.

## Support varieties for $\mathfrak{g}_{\pm 1}$

Observe that  $\mathfrak{g}_{\pm 1}$  is an abelian Lie superalgebra, thus

$$R = H^\bullet(\mathfrak{g}_{\pm 1}, \mathbb{C}) = H^\bullet(\mathfrak{g}_{\pm 1}, \{0\}; \mathbb{C}) \cong S(\mathfrak{g}_{\pm 1}^*)$$

Let  $\mathcal{C}$  be the category of finite dimensional  $\mathfrak{g}_{\pm 1}$ -supermodules. If  $M \in \mathcal{C}$ , then one can define the  $\mathfrak{g}_{\pm 1}$  *support variety* of  $M$ . Set

$$I_M = \{r \in R \mid r.m = 0 \text{ for all } m \in \text{Ext}_{\mathcal{C}}^\bullet(M, M)\}$$

and then the support variety of  $M$  is

$$\begin{aligned} \mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) &= \text{MaxSpec}(R/I_M) \\ &\cong \{x \in \mathfrak{g}_{\pm 1} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}. \end{aligned}$$

Here  $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$  is canonically isomorphic to the “rank variety”, and  $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$  detects  $\mathfrak{g}_{\pm 1}$  projectivity.

# Criteria for Kac and dual Kac filtrations

## Theorem

*Let  $\mathfrak{g}$  be a Type I classical Lie superalgebra, and  $M$  be a  $\mathfrak{g}$ -supermodule. Then the following are equivalent.*

- (a)  *$M$  has a Kac filtration.*
- (b)  *$\text{Ext}_{\mathcal{F}}^1(M, K^-(\mu)) = 0$  for all  $\mu \in X^+(T)$ .*
- (c)  *$\text{Ext}_{\mathfrak{g}_{-1}}^1(\mathbb{C}, M) = 0$ .*
- (d)  *$\mathcal{V}_{\mathfrak{g}_{-1}}(M) = 0$ .*

# Criteria for Kac and dual Kac filtrations

## Theorem

Let  $\mathfrak{g}$  be a Type I classical Lie superalgebra, and  $M$  be a  $\mathfrak{g}$ -supermodule. Then the following are equivalent.

- (a)  $M$  has a dual Kac filtration.
- (b)  $\text{Ext}_{\mathcal{F}}^1(K^+(\mu), M) = 0$  for all  $\mu \in X^+(T)$ .
- (c)  $\text{Ext}_{\mathfrak{g}_{+1}}^1(\mathbb{C}, M) = 0$ .
- (d)  $\mathcal{V}_{\mathfrak{g}_{+1}}(M) = \{0\}$ .

## Definition

A module in  $\mathcal{F}$  is *tilting* if and only if it has both a Kac and a dual Kac filtration.

# Criteria for Projectivity

## Theorem

Let  $\mathfrak{g}$  be a Type I classical Lie superalgebra and  $M \in \mathcal{F}$ . Then

- (a)  $M$  is projective if and only if  $\mathcal{V}_{\mathfrak{g}_1}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}(M) = \{0\}$ .
- (b) Suppose that  $\mathfrak{g}$  admits a strong duality  $\tau$ . If  $M^\tau \cong M$  then  $M$  is projective if and only if  $\mathcal{V}_{\mathfrak{g}_1}(M) = \{0\}$ .

## Corollary

Let  $\mathfrak{g}$  be a Type I classical Lie superalgebra. A module  $M \in \mathcal{F}$  is projective if and only if  $M$  is tilting.

# Connections with Duflo-Serganova Associated Varieties

Consider the subvariety of  $\mathfrak{g}_{\bar{1}}$  given by

$$\mathcal{X} = \left\{ x \in \mathfrak{g}_{\bar{1}} \mid x^2 = [x, x]/2 = 0 \right\}.$$

For  $M \in \mathcal{F}$ , Duflo and Serganova defined the *associated variety*

$$\begin{aligned} \mathcal{X}_M &= \{x \in \mathcal{X} \mid \text{Ker}(x) / \text{Im}(x) \neq 0\} \\ &= \{x \in \mathcal{X} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}. \end{aligned}$$

Here  $x$  is considered as an operator from  $M \rightarrow M$  by  $x(m) = x.m$  with  $x^2 = 0$ .

Note that

$$\mathcal{V}_{\mathfrak{g}_{-1}}(M) \cup \mathcal{V}_{\mathfrak{g}_1}(M) \subseteq \mathcal{X}_M \quad (1)$$

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \mathcal{X}_M \cap \mathfrak{g}_{\pm 1}. \quad (2)$$

We can now recover a theorem of Duflo-Serganova:

### Theorem

*Let  $\mathfrak{g}$  be a classical Type I Lie superalgebra with  $M$  in  $\mathcal{F}$ . Then  $\mathcal{X}_M = 0$  if and only if  $M$  is projective.*

# Refinements of the Projectivity Test

## Theorem

Let  $\mathfrak{g}$  be a Type I classical Lie superalgebra and  $M \in \mathcal{F}$ . Let  $\{x_i \mid i \in I\}$  be a set of orbit representatives for the minimal orbits<sup>a</sup> of the action of  $G_{\bar{0}}$  on  $\mathfrak{g}_1$  and  $\{y_j \mid j \in J\}$  be a set of orbit representatives for the minimal orbits of the action of  $G_{\bar{0}}$  on  $\mathfrak{g}_{-1}$ .

- (a) Then  $M$  is projective in  $\mathcal{F}$  if and only if  $M$  is projective on restriction to  $U(\langle x_i \rangle)$  for all  $i \in I$  and to  $U(\langle y_j \rangle)$  for all  $j \in J$ .
- (b) Furthermore, assume that  $\mathfrak{g}$  admits a strong duality  $\tau$  and  $M \cong M^\tau$ . Then  $M$  is a projective in  $\mathcal{F}$  if and only if  $M$  is projective on restriction to  $U(\langle x_i \rangle)$  for all  $i \in I$ .

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<sup>a</sup>By minimal orbit, we mean minimal non-zero orbit with respect to the partial order on orbits given by containment in closures.

## Further Questions

- (1) If  $P(S)$  is a projective cover for a simple module  $S$  in  $\mathcal{F}$  then  $P(S) \cong I(T)$  where  $I(T)$  is the injective hull of a simple module  $T$  in  $\mathcal{F}$ . Can one find a general formula (like the one for finite-dimensional Hopf algebras) which describes the relationship between  $S$  and  $T$ ?
- (2) Given a simple module  $S$  in  $\mathcal{F}$ , can one give a concrete realization of  $\text{End}_{\mathcal{F}}(P(S))$ . An answer to this question may contribute to a solution of (1).
- (3) One fundamental and elusive question is whether one can construct a “support variety” for a given module  $M$  in  $\mathcal{F}$  whose dimension is equal to the complexity of  $M$ .