Classification of Harish-Chandra modules over the Schrödinger-Virasoro algebra

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Background and Definitions

[H1] M. Henkel, *Schrödinger invariance and strongly anisotropic critical systems*, J. Stat. Phys., **75** (1994), 1023-1029.
[H2] M. Henkel, *Phenomenology of local scale invariance: from conformal invariance to dynamical scaling*, Nucl. Phys. B, **641** (2002), 405-410.

The *Schrödinger-Virasoro algebra* $\mathfrak{sv}$ introduced in [H1, H2], in the context of non-equilibrium statistical physics as a by-product of the computation of n-point functions that are covariant under the action of the Schrödinger group, is the infinite-dimensional Lie algebra with $\mathbb{C}$-basis
$\{L(m), I(n), G(r), C \,|\, m, n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}\}$ and Lie brackets,

$$\mathfrak{sv} = \bigoplus_{m \in \mathbb{Z}} L(m) \oplus \bigoplus_{n \in \mathbb{Z}} I(n) \oplus \bigoplus_{r \in \frac{1}{2}+\mathbb{Z}} G(r) \oplus \mathbb{C}C \qquad (1)$$

which satisfy the following commutation relations:

$$\begin{aligned}
&[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}C,\\
&[L(m), I(n)] = -nI(m+n),\\
&[L(m), G(r)] = (\frac{m}{2}-r)G(m+r),\\
&[G(r), G(s)] = (r-s)G(r+s),\\
&[I(m), I(n)] = [I(m), G(r)] = 0,\\
&[\mathfrak{sv}, C] = 0.
\end{aligned}$$

- C. Roger, J. Unterberger, The Schrödinger-Virasoro Lie group and algebra: from geometry to representation theory, preprint (arXiv:cond-mat/0601050), (2006).
- J. Unterberger, *On vertex algebra representations of the Schr?dinger-Virasoro Lie algebra* , arXiv:cond-mat/0703214v2
- J. Li; Y.Su, Representations of the Schrödinger-Virasoro algebras, J. Math. Phys., 49(5)(2008), 053512, 14 pp.

A weight $\mathcal{L}$-module V is called a highest (resp. lowest) weight module with highest weight (resp. lowest weight) $\lambda \in \mathbb{C}$, if there exists a nonzero weight vector $v \in V_\lambda$ such that

- V is generated by v as $\mathcal{L}$-module;
- $\mathcal{L}_+ v = 0$ (resp. $\mathcal{L}_- v = 0$).

Remark. For a highest (lowest) vector v we always suppose that $I(0)v = c_0 v$ for some $c_0 \in \mathbb{C}$ although the action of $I(0)$ is not semisimple.

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Obviously, if M is an irreducible weight $\mathcal{L}$-module, then there exists $\lambda \in \mathbb{C}$ such that $\mathrm{Supp}(M) \subset \lambda + \mathbb{Z}$. So M is a $\mathbb{Z}$-graded module.

If, in addition, all weight spaces M_λ of an irreducible weight $\mathcal{L}$-module M are finite dimensional, the module is called a *Harish-Chandra module*. Clearly a highest (lowest) weight module is a Harish-Chandra module.

Moreover if $\dim V_\lambda \leq 1$, the nontrivial Harish-Chandra module V is called the intermediate series.

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Two important references

- Kaplansky, I.; Santharoubane, L. J., *Harish-Chandra modules over the Virasoro algebra*, Infinite-dimensional groups with applications (Berkeley, Calif., 1984), 217–231, Math. Sci. Res. Inst. Publ., 4, Springer, New York, 1985.

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Kaplansky-Santharoubane gave a classification of Vir-modules of the intermediate series.

- $\mathcal{A}_{a,\,b} = \sum_{i\in\mathbb{Z}} \mathbb{C}v_i$: $L(m)v_i = (a+i+bm)v_{m+i}$;
- $\mathcal{A}(a) = \sum_{i\in\mathbb{Z}} \mathbb{C}v_i$: $L(m)v_i = (i+m)v_{m+i}$ if $i \neq 0$, $L(m)v_0 = m(m+a)v_m$;
- $\mathcal{B}(a) = \sum_{i\in\mathbb{Z}} \mathbb{C}v_i$: $L(m)v_i = iv_{m+i}$ if $i \neq -m$, $L(m)v_{-m} = -m(m+a)v_m$, for some $a, b \in \mathbb{C}$.

- When $a \notin \mathbb{Z}$ or $b \neq 0, 1$, it is well-known that the module $\mathcal{A}_{a,\, b}$ is simple.

- In the opposite case the module contains two simple subquotients namely the trivial module and $\mathbb{C}[t, t^{-1}]/\mathbb{C}$. Denote the nontrivial simple subquotients of $\mathcal{A}_{a,\, b}$, $\mathcal{A}(a)$, $\mathcal{B}(a)$ by $\mathcal{A}'_{a,\, b}$, $\mathcal{A}(a)'$, $\mathcal{B}(a)'$ respectively.

The following theorem was posted as a conjecture by Kac in 1983, and was completely proved by Mathieu in 1992.

Theorem

(Mathieu, 1992) Any Harish-Chandra module over the Virasoro algebra is a highest weight module, a lowest weight module or a module of the intermediate series.

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Highest weight modules

Fix a linear form $\lambda : \mathfrak{sv}_0 \to \mathbb{C}$. We define in a standard way the Verma module over $\mathfrak{sv}$:

$$M(\lambda) = \mathrm{Ind}_{\mathfrak{sv}_0 \oplus \mathfrak{sv}_+}^{\mathfrak{sv}} \mathbb{C}_\lambda = U(\mathfrak{sv}) \otimes_{U(\mathfrak{sv}_+ \oplus \mathfrak{sv}_0)} \mathbb{C}_\lambda (\simeq U(\mathfrak{sv}_-)),$$

where $\mathbb{C}_\lambda = \mathbb{C}v$(with $v_\lambda \neq 0$) is one-dimensional $(\mathfrak{sv}_0 \oplus \mathfrak{sv}_+)$-module such that

$$\mathfrak{sv}_+ v = 0, \quad hv = \lambda(h) v_\lambda, \quad \text{for} \quad h \in \mathfrak{sv}_0.$$

Given $c, h, \ell \in \mathbb{C}$, we assume that

$$\lambda(C) = c, \quad \lambda(L(0)) = h, \quad \lambda(I(0)) = \ell.$$

We also denote $M(\lambda)$ by $M(c, h, \ell)$.

Theorem

The Verma module $M(c, h, \ell)$ *is irreducible if and only if* $\ell \neq 0$.

Theorem

Let c be any complex number. Then there exists a unique $\mathbb{Z}/2$-graded vertex operator algebra structure $(V_{\mathfrak{sv}}(c,0,0), Y, \omega, \mathbf{1})$ on $V_{\mathfrak{sv}}(c,0,0)$ such that $\mathbf{1} := 1 \in \mathbb{C}$ is the vacuum vector, with $\omega := L(-2)\mathbf{1}$ is a conformal vector, and such that $Y(L(-2)\mathbf{1}, x) = L(x)$, $Y(I(-1)\mathbf{1}, x) = I(x)$, $Y(G(-\frac{3}{2})\mathbf{1}, x) = G(x)$. The vertex operator map for this vertex operator algebra structure is given by

$$\begin{aligned} & Y(L(m_1)\cdots L(m_p)I(n_1)\cdots I(n_q)G(r_1)\cdots G(r_t)\mathbf{1}, x) \\ = \; & {}^{\circ}_{\circ}\, \partial^{(-m_1-2)}L(x)_{m_1+1}\cdots\partial^{(-n_p-2)}L(x)_{m_p+1}\partial^{(-n_1-1)}I(x)_{n_1}\cdots\partial^{(-n} \\ & \partial^{(-r_1-\frac{3}{2})}G(x)_{r_1+\frac{1}{2}}\cdots\partial^{(-r_t-\frac{3}{2})}G(x)_{n_t+\frac{1}{2}}1\, {}^{\circ}_{\circ}\, . \end{aligned}$$

for $p,q,t \geq 0$, $m_1,\ldots,m_p, n_1,\ldots,n_q \in \mathbb{Z}$ and $r_1,\ldots,r_t \in \mathbb{Z}+\frac{1}{2}$, where

$$\partial^{(n)} = \frac{1}{n!}\left(\frac{d}{dx}\right)^n$$

${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ *is the normal ordering, and 1 is the identity operator.*

Remark It is clear that $V_{\mathfrak{sv}}(c,0,0)$ is not a simple vertex operator algebra. Let I_c be the sum of all the proper submodules of $V_{\mathfrak{sv}}(c,0,0)$. Then the irreducible quotient

$$V_{\mathfrak{sv}}(c,0,0)/I_c \simeq L_{Vir}(c,0),$$

where Vir is the Virasoro algebra and $L_{Vir}(c,0)$ is the associated simple vertex operator algebra of central charge c.

Theorem

(1) If M is a restricted $\mathfrak{sv}$-module with central charge c, then M is a weak $V_{\mathfrak{sv}}(c,0,0)$-module with

$$Y_M(L(-2)\mathbf{1},x) = L(M)(x),\ Y_M(I(-1)\mathbf{1},x) = I_M(x),\ Y_M(G(-\frac{3}{2})\mathbf{1},x)$$

In particular, any quotient module of $M(c,h,\ell)$ is an ordinary module for $V_{\mathfrak{sv}}(c,0,0)$.
(2) Any irreducible admissible $V_{\mathfrak{sv}}(c,0,0)$-module is ordinary and $\{L_{\mathfrak{sv}}(c,h,\ell) | c,h,\ell \in \mathbb{C}\}$ gives a complete list of irreducible $V_{\mathfrak{sv}}(c,0,0)$-modules up to isomorphism.

IHarish-Chandra module over the Schrödinger-Virasoro algebra

For convenience, throughout the following two sections, we replace $I(m+1), G(m+\frac{1}{2})$ by $I(m), G(m)$, respectively, in the definition of the Schrödinger-Virasoro algebra $\mathfrak{sv}$. Then $\mathfrak{sv}$ is defined to be a Lie algebra with $\mathbb{C}$-basis $\{L(m), I(m), G(m), C \mid m \in \mathbb{Z}\}$ subject to the following Lie brackets:

$$
\begin{aligned}
&[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}C, \\
&[L(m), I(n)] = -(n+1)I(m+n), \\
&[L(m), G(r)] = (\frac{m-1}{2} - r)G(m+r), \\
&[G(r), G(s)] = (r-s)I(r+s), \\
&[I(m), I(n)] = [I(m), G(r)] = 0, \\
&[\mathfrak{sv}, C] = 0.
\end{aligned}
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Irreducible(indecomposable) modules with weight multiplicity one

Denoted $\mathcal{A}_{a,\, b,\, 0,0}$, $\mathcal{A}(a, 0, 0)$, $\mathcal{B}(a, 0, 0)$ by the $\mathfrak{sv}$-module induced from the Vir-module $\mathcal{A}_{a,\, b}$, $\mathcal{A}(a)$, $\mathcal{B}(a)$ with the trivial actions of $I(n)$ and $G(n)$ for any $n \in \mathbb{Z}$, respectively.

Theorem

Suppose that V is a nontrivial irreducible weight $\mathfrak{sv}$-module with weight multiplicity one. Then we have $V \simeq \mathcal{A}'_{a,\, b,\, 0,\, 0}$ for some $a, b \in \mathbb{C}$. Meanwhile, the above three kinds of modules are all indecomposable weight $\mathfrak{sv}$-modules with weight multiplicity one.

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Remark. The paper [LS] lists all Harish-Chandra $\mathfrak{sv}$-modules of the intermediate series(see Theorem 1.3 in [LS]). However they are all not irreducible. In fact, they are some special indecomposable weight $\mathfrak{sv}$-modules with weight multiplicity two.

Uniformly bounded irreducible weight modules

Now, we assume that V is a uniformly bounded nontrivial irreducible weight module over $\mathcal{L}$.

So there exists $\alpha \in \mathbb{C}$ such that $\text{Supp}(V) \subset \alpha + \mathbb{Z}$.

From representation theory of Vir, we have $C = 0$ and $\dim V_{\alpha+n} = p$ for all $\alpha + n \neq 0$. If $\alpha \in \mathbb{Z}$, we also assume that $\alpha = 0$.

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Consider V as a Vir-module. We have a Vir-submodule filtration

$$0 = W^{(0)} \subset W^{(1)} \subset W^{(2)} \subset \cdots \subset W^{(p)} = V,$$

where $W^{(1)}, \cdots, W^{(p)}$ are Vir-submodules of V, and the quotient modules $W^{(i)}/W^{(i-1)}$ have weight multiplicity one for all nonzero weights.

Choose $v_n^1, \cdots, v_n^p \in V_{\alpha+n}$ such that the images of $v_n^i + W^{(i-1)}$ form a basis of $(W^{(i)}/W^{(i-1)})_{\alpha+n}$ for all $\alpha + n \neq 0$. We may suppose that

$$L(i)(v_n^1, \cdots, v_n^p) = (L(i)v_n^1, \cdots, L(i)v_n^p) = (v_{n+i}^1, \cdots, v_{n+i}^p)A_{i,n},$$

where $A_{i,n}$ are upper triangular $p \times p$ matrices, and $A_{i,n}(j,j) = \alpha + n + i\beta_j$.

Denote

$$I(i)(v_n^1, \cdots, v_n^p) = (v_{n+i}^1, \cdots, v_{n+i}^p)F_{i,n}, \qquad (4.1)$$

where $F_{i,n}$ are $p \times p$ matrices.

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where $W^{(1)}, \cdots, W^{(p)}$ are Vir-submodules of V, and the quotient modules $W^{(i)}/W^{(i-1)}$ have weight multiplicity one for all nonzero weights.

Choose $v_n^1, \cdots, v_n^p \in V_{\alpha+n}$ such that the images of $v_n^i + W^{(i-1)}$ form a basis of $(W^{(i)}/W^{(i-1)})_{\alpha+n}$ for all $\alpha + n \neq 0$. We may suppose that

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where $F_{i,n}$ are $p \times p$ matrices.

The Lie brackets give

$$A_{i,j+n}A_{j,n} - A_{j,i+n}A_{i,n} = (j-i)A_{i+j,n}, \tag{4.2}$$

$$A_{i,j+n}F_{j,n} - F_{j,i+n}A_{i,n} = (j+\frac{1-i}{2})F_{i+j,n}, \tag{4.3}$$

$$F_{i,j+n}F_{j,n} - F_{j,i+n}F_{i,n} = (j-i)G_{i+j,n}, \tag{4.4}$$

where the $I(p)$ is the $p \times p$ unit matrix and the last three formulas have the restriction $(\alpha+n)(\alpha+n+i)(\alpha+n+j)(\alpha+n+i+j) \neq 0$. We shall denote the (i,j)-entry of a matrix A by $A(i,j)$.

Classification of irreducible weight modules over $\mathcal{L}$ with finite-dimensional weight spaces

Theorem

If V is a nontrivial irreducible uniformly bounded weight module over $\mathfrak{sv}$, then V is isomorphic to $\mathcal{A}'_{\alpha,\beta,0,\,0}$ for some $\alpha, \beta \in \mathbb{C}$.

Theorem Let V be an irreducible weight module over $\mathcal{L}$ with all weight spaces finite-dimensional. If V is not uniformly bounded, then V is either a highest weight module or a lowest weight module.

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Theorem Let V be an irreducible weight module over $\mathcal{L}$ with all weight spaces finite-dimensional. If V is not uniformly bounded, then V is either a highest weight module or a lowest weight module.

Combining the above two theorems, we obtain the main result of this paper:

Theorem

If V is a nontrivial irreducible weight module over $\mathcal{L}$ with finite dimensional weight spaces, then V is isomorphic to $\mathcal{A}'_{\alpha,\beta,0,\,0}$ for some $\alpha, \beta \in \mathbb{C}$, or a highest or lowest weight module.

Thank you !

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