A Characterization of Vertex Operator Algebra $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$

Chongying Dong and Cuipo Jiang

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Abstract

The vertex operator algebra $V^0 = L(\frac{1}{2},0) \otimes L(\frac{1}{2},0)$ is characterized in [Zhang-Dong] as a unique simple rational, C_2-cofinite vertex operator algebra with $c = \tilde{c} = 1$, weight one subspace being zero and weight two subspace being 2-dimensional. In the present work we strengthen this result by allowing the dimension of weight two subspace to be greater than or equal to 2. This proves the conjecture given in [Zhang-Dong].

Background

The importance of the vertex operator algebra $V^0 = L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$ was first noticed in [Dong-Mason-Zhu] (also see [Miyamoto2], [Dong-Griess-Hoehn]) for the study of the moonshine vertex operator algebra $V^\natural$ [Frenkel- Lepowsky -Meurman].

It was proved in [Dong-Mason-Zhu] that the fixed point vertex operator subalgebra V_L^+ under the involution induced from the -1 isometry of L is isomorphic to V^0, if L is a rank one lattice generated by a vector whose squared length is 4 and $V^\natural$ contains $L(\frac{1}{2},0)^{\otimes 48}$.

This led to the theory of code vertex operator algebras [Miyamoto1]-[Miyamoto3] and framed vertex operator algebras [Dong-Griess-Hoehn]. For example, a new construction of the moonshine vertex operator algebra $V^\natural$ is given in [Miyamoto4] by using the theory of code and framed vertex operator algebras. Furthermore, the recent progress in [Dong-Griess-Lam] and [Lam-Yamauchi] on proving the uniqueness of $V^\natural$ depends largely on the theory of framed VOAs and code VOAs.

Background

The characterization of $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$ is a necessary step in the classification of rational vertex operator algebras with $c=1$. It is a well known conjecture ([Dijkgraaf-E.Verlinde-H.Verlinde], [Kiritsis]) that any simple rational vertex operator algebra with $c=1$ is either V_L, V_L^+ or $V_{L_{A_1}}^G$, where L is a rank one positive definite even lattice, L_{A_1} is the root lattice of type A_1 and G is a subgroup of $SO(3)$ isomorphic to A_4, S_4 or A_5. A characterization of V_L for an arbitrary positive definite even lattice is obtained in [Dong-Mason-1].

Although there were some progress at the q-character level on the classification of rational vertex operator algebras with $c=1$ in the physics literature [Kiritsis], there is still a long way to prove the conjecture completely by a lack of characterization of V_L^+. It is desirable that the characterization of $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$ may help to understand V_L^+ in general.

Basic Concepts

A vertex algebra is a vector space V equipped with a linear map

$$\begin{array}{rcl} Y(\cdot,x): V & \longrightarrow & (\mathrm{End}\, V)[[x,x^{-1}]] \\ v & \mapsto & Y(v,x)=\sum_{n\in\mathbb{Z}} v_n x^{-n-1} \ (\text{where } v_n\in \mathrm{End}\, V), \end{array}$$

and a distinguished element $\mathbf{1}\in V$ (the vacuum), satisfying the following conditions for $u,v\in V$:

Basic Concepts

- $$u_n v = 0 \text{ for } n >> 0$$
- $$Y(\mathbf{1}, x) = \mathrm{id};$$
- $$Y(v,x)\mathbf{1} \in V[[x]] \text{ and } \lim_{x \to 0} Y(v,x)\mathbf{1} = v$$
- and the Jacobi identity holds:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u,x_1)Y(v,x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)$$

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A vertex operator algebra over $\mathbb{C}$ is a $\mathbb{Z}$-graded vector space:

$$V = \bigoplus_{n \in \mathbb{Z}} V_n; \text{ for } v \in V_n, \ n = \text{wt } v;$$

- such that

$$\dim V_n < \infty \text{ for } n \in \mathbb{Z}, \quad V_n = 0 \text{ for } n << 0,$$

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Basic Concepts

- $$[L(m),L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{n+m,0}c$$

  for $m,n \in \mathbb{Z}$, where

  $$L(n) = \omega_{n+1} \text{ for } n \in \mathbb{Z}, \text{ i.e., } Y(\omega,x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$$

- and

  $$c \in \mathbb{C} \text{ (central charge)};$$

  $$L_0 v = nv = (\text{wt } v)v \text{ for } n \in \mathbb{Z} \text{ and } v \in V_n;$$

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Basic Concepts

A V module is a weak V module which carries a $\mathbb{C}$-grading $M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$, such that:

- $dim(M_{\lambda}) < \infty$,
- $M_{\lambda+n} = 0$ for fixed λ and $n << 0$,
- $L(0)w = \lambda w = \mathrm{wt}(w)w$ for $w \in M_{\lambda}$ where $L(0)$ is the component operator of $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

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A vertex operator algebra V is called of CFT type if

$$V = \bigoplus_{n=0}^{\infty} V_n$$

and $V_0 = \mathbb{C}\mathbf{1}$. V is said to be of strongly CFT type if its adjoint module V' is isomorphic to itself.

Basic Concepts

- A vertex operator algebra is called rational if its module category is semisimple.
- A vertex operator algebra is called C_2-cofinite if $C_2(V)$ has finite codimension, where $C_2(V) = \langle u_{-2}v | u, v \in V \rangle$.

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Theorem 1 ([Zhu], [Dong-Li-Mason-2])

If V is a rational vertex operator algebra, then V has finitely many irreducible modules up to isomorphism.

- Suppose that V is a rational vertex operator algebra and let $M^1, ..., M^k$ be the irreducible modules such that

$$M^i = \oplus_{n \geq 0} M^i_{\lambda_i + n}$$

where $\lambda_i \in \mathbb{Q}$ [Dong-Li-Mason-3], $M^i_{\lambda_i} \neq 0$ and each $M^i_{\lambda_i+n}$ is finite dimensional.

- Let λ_{min} be the minimum of λ_i's. The effective central charge $\tilde{c}$ is defined to be

$$\tilde{c} = c - 24\lambda_{min}.$$

- For each M^i, define the q-character of M^i by

$$\mathrm{ch}_q M^i = q^{-c/24} \sum_{n \geq 0} (\dim M^i_{\lambda_i+n}) q^{n+\lambda_i}.$$

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- If V is rational and C_2-cofinite, then $\mathrm{ch}_q M^i$ converges to a holomorphic function on the upper half plane [Zhu].
- Using the modular invariance result from [Zhu] and results on vector valued modular forms from [Knopp-Mason] we have (see [Dong-Mason-1])
- Let V be rational and C_2-cofinite. For each i, the coefficients of $Z_i(q){=}\eta(q)^{\tilde{c}}\mathrm{ch}_q M^i$ satisfy the polynomial growth condition, where

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Theorem 2 ([Kiritsis])

Let V be a rational CFT type vertex operator algebra such that $c = 1$ and $\sum Z_i(q)\overline{Z_i(q)}$ is modular invariant, then the q-character of V is equal to the character of one of the following vertex operator algebras V_L, V_L^+ and $V_{\mathbb{Z}\alpha}^G$, where L is any positive definite even lattice of rank 1, V_L^+ is the fixed points of the automorphism of V lifted from the -1 isometry of L, and $\mathbb{Z}\alpha$ is the root lattice of type A_1 such that (α, α) = 2 and G is a finite subgroup of $SO(3)$ isomorphic to A_4, S_4 or A_5.

Conjecture ([Dijkgraaf-E.Verlinde-H.Verlinde],[Kiritsis], [Zhang-Dong])

V_L, V_L^+ and $V_{\mathbb{Z}\alpha}^G$ should give a complete list of simple and rational vertex operator algebras with $c = \tilde{c} = 1$.

Definition

A vertex operator algebra V is called strongly rational if it satisfies the following conditions:

1. V is of strong CFT type.
2. V is C_2-cofinite.
3. V is rational.

Theorem 3 [Dong-Mason]

Let V be a strongly rational vertex operator algebra. Then the Lie algebra V_1 is reductive. Moreover, $l \leq \tilde{c}$, where l is the Lie rank of V_1, i.e., the dimension of a maximal abelian subalgebra of V_1.

Theorem 4 [Dong-Mason]

Let V be a strongly rational vertex operator algebra. Then the following are equivalent:

(i) $\tilde{c} = l = c$.

(ii) There is a positive-definite even lattice L such that V is isomorphic to V_L.

By Theorem 3 and 4, to classify simple rational vertex operator algebras with $\tilde{c} = c = 1$, one needs only to consider the case that that $V_1 = 0$.

Theorem 4 ([Zhang-Dong])

If V is a simple, rational and C_2-cofinite vertex operator algebra of the moonshine type such that $c = \tilde{c} = 1$ and $\dim V_2 = 2$, then V is isomorphic to $L(1/2,0) \otimes L(1/2,0)$.

- Note that for a vertex operator algebra V, if its weight one subspace V_1 is 0, then its weight two subspace is a commutative (non associative) algebra. Since the weight two subspace V_2 in [Zhang-Dong] is assumed to be 2-dimensional, it is necessarily a commutative associative algebra.
- The above theorem was based on the study of vertex operator algebra $W(2,2)$ and the growth of the graded dimensions of rational vertex operator algebras.

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- But if we assume $\dim V_2 \geq 2$, then V_2 is not an associative algebra and the situation is much more complicated. The method used in [Zhang-Dong] is not available anymore.
- The key point in the following work is to apply the $A(V)$ theory to compute the fusion rules for the Virasoro algebra with $c = 1$ to deal with the case that $\dim V_2 \geq 2$.
- Although the fusion rules for the Virasoro algebra with $c = 1$ have been investigated from different point of views [Rehern-Tuneke], [Xu], we give a complete proof of the fusion rules based on the $A(V)$-theory developed in [Zhu], [Frenkel-Zhu] and [Li2]. We certainly believe that the fusion rules computed in this paper will play important roles in the future classification of rational vertex operator algebras with $c = 1$.

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$A(V)$-theory and fusion rules

- Let V be a vertex operator algebra. An associative algebra $A(V)$ has been introduced and studied in [Zhu]. It turns out that $A(V)$ is very powerful and useful in representation theory for vertex operator algebras.
- As a vector space, $A(V)$ is a quotient space of V by $O(V)$, where $O(V)$ denotes the linear span of elements

$$u \circ v = \mathrm{Res}_z \left(Y(u,z) \frac{(z+1)^{\mathrm{wt}\, u}}{z^2} v \right) = \sum_{i \geq 0} \binom{\mathrm{wt}\, u}{i} u_{i-2} v \qquad (1)$$

for $u, v \in V$ with u being homogeneous.

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$A(V)$-theory and fusion rules

- Product in $A(V)$ is induced from the multiplication

$$u * v = \mathrm{Res}_z \left(Y(u,z) \frac{(z+1)^{\mathrm{wt}\, u}}{z} v \right) = \sum_{i \geq 0} \binom{\mathrm{wt}\, u}{i} u_{i-1} v \qquad (2)$$

  for $u, v \in V$.

- $A(V) = V/O(V)$ is an associative algebra with identity $\mathbf{1} + O(V)$ and with $\omega + O(V)$ being in the center of $A(V)$.

- The most important result about $A(V)$ is that for any V-module $M = \oplus_{n \geq 0} M(n)$ with $M(0) \neq 0$, $M(0)$ is an $A(V)$-module such that $v + O(V)$ acts as $o(v)$ where $o(v) = v_{\mathrm{wt} v - 1}$ for homogeneous v.

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$A(V)$-theory and fusion rules

For a V-module W, define $O(W) \subset W$ to be the linear span of elements of type

$$\mathrm{Res}_z(Y(v,z)\frac{(z+1)^{\mathrm{wt}\,v}}{z^2}w) = \sum_{i\geq 0}\binom{\mathrm{wt}\,v}{i}v_{i-2}w$$

for homogeneous $v \in V$ and $w \in W$. Let $A(W) = W/O(W)$. Then $A(W)$ has an $A(V)$-bimodule structure [Frenkel-Zhu] induced by the following bilinear operations $V \times W \to W$ and $W \times V \to W$: for $w \in W$ and homogeneous $v \in V$,

$$v * w = \mathrm{Res}_z(Y(v,z)\frac{(z+1)^{\mathrm{wt}\,v}}{z}w) = \sum_{i\geq 0}\binom{\mathrm{wt}\,v}{i}v_{i-1}w,$$

$$w * v = \mathrm{Res}_z(Y(v,z)\frac{(z+1)^{\mathrm{wt}\,v-1}}{z}w) = \sum_{i\geq 0}\binom{\mathrm{wt}\,v-1}{i}v_{i-1}w.$$

Proposition 1 ([Frenkel-Zhu])

If W is a module for a vertex operator algebra V and M is a submodule of W, then the image $\bar{M}$ of M in $A(W)$ is a sub $A(V)$-bimodule of $A(W)$, and the quotient $A(W)/\bar{M}$ is isomorphic to the $A(V)$-bimodule $A(W/M)$ associated to the quotient V-module W/M.

$A(V)$-theory and fusion rules

- In order to give the next proposition, we recall some basic facts about the highest weight modules for the Virasoro algebra.
- Let $c,h \in \mathbb{C}$ and $V(c,h)$ be the corresponding highest weight module for the Virasoro algebra with central charge c and highest weight h.
- We set $\overline{V}(c,0) = V(c,0)/U(Vir)L_{-1}v$, where v is a highest weight vector with highest weight 0. Then $\overline{V}(c,0)$ is a vertex operator algebra.
- We denote the irreducible quotient of $V(c,h)$ by $L(c,h)$.

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Proposition 2 ([Frenkel-Zhu])

- The associative algebra $A(\bar{V}(c,0))$ is isomorphic to the polynomial algebra $\mathbb{C}[x]$, with the isomorphism being given by $x^n \in \mathbb{C}[x] \mapsto [(L_{-2}+L_{-1})^n\mathbf{1}]$, where $[a] = a + O(\bar{V}(c,0))$ for $a \in \bar{V}(c,0)$.
- The $A(\bar{V}(c,0))$-bimodule $A(V(c,h))$ is $\mathbb{C}[x,y]$ with x and y acting on the left and right as multiplications by x and y respectively. The isomorphism from $\mathbb{C}[x,y]$ to $A(V(c,h))$ is given by $x^m y^n \mapsto [(L_{-2}+2L_{-1}+L_0)^m(L_{-2}+L_{-1})^n\mathbf{1}_h]$, where $\mathbf{1}_h$ is a fixed nonzero highest weight vector of $V(c,h)$.

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- Let W^i $(i = 1,2,3)$ be V-modules. We denote by $I_V \begin{pmatrix} W^3 \\ W^1 \, W^2 \end{pmatrix}$ the vector space of all intertwining operators of type $\begin{pmatrix} W^3 \\ W^1 \, W^2 \end{pmatrix}$.
- For a V-module W, let W' denote the graded dual of W. Then W' is also a V-module [Frenkel-Huang-Lepowsky].
- It is well known that fusion rules have the following symmetry (see [Frenkel-Huang-Lepowsky]).

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Proposition 3 ([Frenkel-Huang-Lepowsky])

Let W^i $(i = 1, 2, 3)$ be V-modules. Then

$$\dim I_V \begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix} = \dim I_V \begin{pmatrix} W^3 \\ W^2\, W^1 \end{pmatrix},$$

$$\dim I_V \begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix} = \dim I_V \begin{pmatrix} (W^2)' \\ W^1\, (W^3)' \end{pmatrix}.$$

We next turn our attention to the fusion rules for the vertex operator algebra $L(1,0)$. The following theorem is the foundation in our computation of the fusion rules.

Theorem 5 ([Dong-Jiang])

Let r be a positive integer. Then

$$A(L(1,r^2)) = \mathbb{C}[x,y]/\overline{I},$$

where

$$\overline{I} = < (x-y)\prod_{i=1}^{r}[(x-y)^2 - 2i^2(x+y) + i^4] >$$

is a two-sided ideal of $\mathbb{C}[x,y]$ generated by

$$f(x,y) = (x-y)\prod_{i=1}^{r}[(x-y)^2 - 2i^2(x+y) + i^4].$$

Sketch of the proof

Step 1

- Since $\overline{V}(1,0) = L(1,0)$, by Proposition 2, the associative algebra $A(L(1,0))$ is $\mathbb{C}[x]$ and the $A(L(1,0))$-bimodule $A(V(1,r^2))$ is isomorphic to $\mathbb{C}[x,y]$ with x and y acting on the left and right as multiplications by x and y respectively.
- By Proposition 1, as an $A(L(1,0))$-bimodule,

$$A(L(1,r^2)) \cong \mathbb{C}[x,y]/\overline{I},$$

where $\overline{I}$ is the image in $A(V(1,r^2))$ of the maximal proper submodule I of $V(1,r^2)$.

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where $\overline{I}$ is the image in $A(V(1,r^2))$ of the maximal proper submodule I of $V(1,r^2)$.

Since I is generated by a non-zero element $v^{(r+1)}$ in $V(1,r^2)$ such that

$$L(0)v^{(r+1)} = (r+1)^2 v^{(r+1)}, \;\; L(k)v^{(r+1)} = 0, \;\; 0 < k \in \mathbb{Z}_+,$$

it follows that $\overline{I}$ is generated by a polynomial $f(x,y)$ in $\mathbf{C}[x,y]$ with degree $s \leq 2r+1$.

Step 2

The main idea in step 2 is to use the vertex operator algebra V_L associated to the rank one even positive definite lattice $L = \mathbb{Z}\alpha$ with $(\alpha, \alpha) = 2$ [Frenkel-Lepowsky-Meurman] to prove that

$$I\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\,L(1,n^2) \end{pmatrix} \neq 0,$$

for all $m, n, k \in \mathbb{Z}_+$ such that $|m - n| \leq k \leq n + m$.

Proof of Theorem 5

Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, and $\hat{\mathfrak{h}}_{\mathbb{Z}}$ the corresponding Heisenberg algebra. Denote by $M(1) = \mathbb{C}[\alpha(-n)|n > 0]$ the associated irreducible induced module for $\hat{\mathfrak{h}}_{\mathbb{Z}}$ such that the canonical central element of $\hat{\mathfrak{h}}_{\mathbb{Z}}$ acts as 1. Let $\mathbb{C}[L]$ be the group algebra of L with a basis e^{γ} for $\gamma \in L$. Let $\beta \in \mathfrak{h}$ be such that $(\beta, \beta) = 1$.

Proof of Theorem 5

It is known that $V_L = M(1) \otimes \mathbb{C}[L]$ is a simple rational vertex operator algebra with $\mathbf{1} = 1 \otimes e^0$ and $\omega = \frac{1}{2}\beta(-1)^2\mathbf{1}$ [Borcherds], [Frenkel-Lepowsky-Meurman], [Dong], [Dong-Li-Mason-1]. The subalgebra generated by ω of V_L is isomorphic to $L(1,0)$ and

$$M(1) = \bigoplus_{p \geq 0} L(1, p^2)$$

$$V_L = \bigoplus_{m \geq 0} (2m+1) L(1, m^2) \tag{3}$$

as modules for the Virasoro algebra (cf. [Dong-Griess]).

Proof of Theorem 5

It is well-known that V_L is isomorphic to the fundamental representation $L(\Lambda_0)$ for the affine Kac-Moody algebra $A_1^{(1)}$ [Frenkel-Kac]. Note that the weight one subspace $(V_L)_1$ of V_L forms a Lie algebra $\mathfrak{g}$ isomorphic to $sl(2,\mathbb{C})$ where the Lie bracket in $(V_L)_1$ is defined as $[u,v] = u_0 v$ and u_0 is the component operator of $Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$. $\mathfrak{g}$ acts on V_L via v_0 for $v \in (V_L)_1$. The $\mathfrak{g}$-invariant elements $V_L^{\mathfrak{g}} = \{v \in V_L | \mathfrak{g} \cdot v = 0\}$ form a simple vertex operator algebra and is isomorphic to $L(1,0)$ (see [DG]).

Proof of Theorem 5

Let W_m be the unique $m+1$-dimensional highest weight module for $\mathfrak{g}$ with highest weight $m \in \mathbb{Z}_{\geq 0}$. Let $V_L^{W_m}$ be the sum of irreducible $\mathfrak{g}$-submodules of V_L isomorphic to W_m, and $(V_L)_{W_m}$ the space of highest weight vectors in $V_L^{W_m}$. Then by [DG], as a $(V_L^{\mathfrak{g}}, \mathfrak{g})$-module V_L has decomposition

$$V_L = \bigoplus_{m \geq 0} V_L^{W_{2m}} = \bigoplus_{m \geq 0} (V_L)_{W_{2m}} \otimes W_{2m} \tag{4}$$

and $(V_L)_{W_{2m}}$ is an irreducible module for $V_L^{\mathfrak{g}}$.

Proof of Theorem 5

Moreover, $(V_L)_{W_{2k}}$ and $(V_L)_{W_{2m}}$ are isomorphic if and only if $k = m$. By [DG], $(V_L)_{W_{2m}}$ is isomorphic to $L(1, m^2)$ as $L(1,0)$-module. For $m, n \in \mathbb{Z}_+$, $m \geq n$, let

$$W_{2m,2n} = span\{u_j v | u \in W_{2m}, v \in W_{2n}, j \in \mathbb{Z}\}.$$

Then $W_{2m,2n}$ is a $\mathfrak{g}$-module. Let $u \in W_{2m}$ and $v \in W_{2n}$ such that

$$\alpha(0)u = (2m - 2i)u, \alpha(0)v = (2n - 2j)v,$$

for some $0 \leq i \leq 2m, 0 \leq j \leq 2n$ where $\alpha(0) = (\alpha(-1)\mathbf{1})_0$ is the component operator of $\alpha(z) = Y(\alpha(-1)\mathbf{1}, z) = \sum_{k \in \mathbb{Z}} \alpha(k) z^{-k-1}$.

Proof of Theorem 5

Then

$$\alpha(0)u_p v = (\alpha(0)u)_p v + u_p \alpha(0)v = (2m+2n-2i-2j)u_p v,$$

for all $p \in \mathbb{Z}$. This means that $W_{2m,2n}$ is a sum of irreducible $\mathfrak{g}$-modules in $\{W_{2k} | 0 \leq k \leq m+n\}$. On the other hand, we have the following well-known tensor product decomposition:

$$W_{2m} \otimes W_{2n} = W_{2(m-n)} \oplus W_{2(m-n)+2} \oplus \cdots \oplus W_{2(m+n)-2} \oplus W_{2(m+n)}. \quad (5)$$

By Lemma 2.2 of [Dong-Mason-2], for small enough integer p, the map $\psi_p : W_{2m} \otimes W_{2n} \to W_{2m,2n}$ defined by $\psi_p : u \otimes v \mapsto \sum_{i=p}^{\infty} u_i v, u \in W_{2m}, v \in W_{2n}$ is injective. Therefore in the decomposition of $W_{2m,2n}$ into irreducible $\mathfrak{g}$-modules, each W_{2k} appears for $m-n \le k \le m+n$. Denote by $U_{m,n}$ the $L(1,0)$-submodule of V_L generated by $W_{2m,2n}$. Then by (4), we have

$$U_{m,n} \supseteq \bigoplus_{m-n \le k \le m+n} (V_L)_{W_{2k}} \otimes W_{2k}.$$

This proves that

$$I\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\,L(1,n^2) \end{pmatrix} \neq 0,$$

for all $m,n,k \in \mathbb{Z}_+$ such that $|m-n| \leq k \leq n+m$.

Step 3 **To prove that $f(x,y)$ is just the desired polynomial.**

Let $m = r$, then we have $f(n^2, k^2) = 0$, for all $n, k \in \mathbb{Z}_+$ satisfying $|r - n| \leq k \leq n + r$. Thus for $n \in \mathbb{Z}_+$ with $n - r \geq 0$, we have

Proof of Theorem 5

$$\begin{bmatrix} 1 & (n-r)^2 & (n-r)^4 & \cdots & (n-r)^{2s} \\ 1 & (n-r+1)^2 & (n-r+1)^4 & \cdots & (n-r+1)^{2s} \\ 1 & (n-r+2)^2 & (n-r+2)^4 & \cdots & (n-r+2)^{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (n+r)^2 & (n+r)^4 & \cdots & (n+r)^{2s} \end{bmatrix} \begin{bmatrix} a_0(n^2) \\ a_1(n^2) \\ a_2(n^2) \\ \vdots \\ a_s(n^2) \end{bmatrix} = 0$$

Proof of Theorem 5

If $s \leq 2r$, then for each $n \in \mathbb{Z}_+$ such that $n \geq r$, the coefficient matrix of (49) contains a $(s+1) \times (s+1)$-minor which is a non-singular Vandermonde determinant, it follows that (49) has only zero solution. This implies that $a_i(x) = 0$ for all i, a contradiction. So we have

$$s = 2r + 1.$$

We may assume that $a_{2r+1}(x) = 1$. Then we have

Proof of Theorem 5

$$A_{(n)} \begin{bmatrix} a_0(n^2) \\ a_1(n^2) \\ a_2(n^2) \\ \vdots \\ a_{2r}(n^2) \end{bmatrix} = \begin{bmatrix} -(n-r)^{2(2r+1)} \\ -(n-r+1)^{2(2r+1)} \\ -(n-r+2)^{2(2r+1)} \\ \vdots \\ -(n+r)^{2(2r+1)} \end{bmatrix}, \tag{6}$$

where

$$A_{(n)} = \begin{bmatrix} 1 & (n-r)^2 & (n-r)^4 & \cdots & (n-r)^{4r} \\ 1 & (n-r+1)^2 & (n-r+1)^4 & \cdots & (n-r+1)^{4r} \\ 1 & (n-r+2)^2 & (n-r+2)^4 & \cdots & (n-r+2)^{4r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (n+r)^2 & (n+r)^4 & \cdots & (n+r)^{4r} \end{bmatrix}.$$

This shows that (6) has a unique solution for each $n \in \mathbb{Z}_+$ such that $n \geq r$. Since $a_i(x)$, $i = 0, 1, ..., 2r+1$ are polynomials in x with degrees at most $2r+1$, it follows that $f(x,y)$ is uniquely determined (up to a non-zero scalar) by the condition that $f(n^2, k^2) = 0$ for all $n, k \in \mathbb{Z}_+$ such that $|n-r| \leq k \leq n+r$. Let

$$f_i(x,y) = (x-y)^2 - 2i^2(x+y) + i^4, \ i = 1, 2, \cdots, r.$$

Then

Proof of Theorem 5

$$
\begin{aligned}
& f_i(n^2,(n+i)^2) \\
& = (n^2-(n+i)^2)^2 - 2i^2(n^2+(n+i)^2) + 4i^4 \\
& = (n^2-(n+i)^2-i^2)^2 - 4i^2(n+i)^2 \\
& = [n^2-(n+i)^2-i^2+2i(n+i)][n^2-(n+i)^2-i^2-2i(n+i)] \\
& = 0.
\end{aligned}
$$

Similarly, we have

$$f_i(n^2,(n-i)^2) = 0.$$

Proof of Theorem 5

This proves that the polynomial

$$(x-y)\prod_{i=1}^{r}[(x-y)^2-2i^2(x+y)+i^4]$$

satisfies the above condition. So we have

$$f(x,y)=(x-y)\prod_{i=1}^{r}[(x-y)^2-2i^2(x+y)+i^4],$$

as expected. □

Theorem 6 ([Dong-Jiang])

We have

- $$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\,L(1,n^2) \end{pmatrix} = 1,\ \ k\in\mathbb{Z}_+,\ |n-m|\le k\le n+m,$$

- $$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\,L(1,n^2) \end{pmatrix} = 0,\ \ k\in\mathbb{Z}_+,\ k<|n-m| \text{ or } k>n+m,$$

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Theorem 6 ([Dong-Jiang])

We have

- For $n \in \mathbb{Z}_+$ such that $n \neq p^2$, for all $p \in \mathbb{Z}_+$, we have

$$\dim I_{L(1,0)} \begin{pmatrix} L(1,n) \\ L(1,m^2)\, L(1,n) \end{pmatrix} = 1,$$

- $$\dim I_{L(1,0)} \begin{pmatrix} L(1,k) \\ L(1,m^2)\, L(1,n) \end{pmatrix} = 0,$$

  for $k \in \mathbb{Z}_+$ such that $k \neq n$.

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In order to obtain our main result, we need to prove that there is no nilpotent element element x in V_2 such that $x^2=0$.

Suppose that V_2 has a nilpotent element x such that $x^2=0$, then through several lemmas we deduce that there is a nonzero intertwining operator of type

$$\begin{pmatrix} L(1,n+5) \\ L(1,4),L(1,2) \end{pmatrix},$$

which is in contradiction with Theorem 6. By this result we show that

Uniqueness of $L(1/2,0)\otimes L(1/2,0)$

Theorem

([Dong-Jiang]) *If V is a simple, rational and C_2-cofinite vertex operator algebra such that $\dim V_2 \geq 2$, $V_1 = 0$, $c = \tilde{c} = 1$, and V is generated by highest vectors of the Virasoro algebra, then $\dim V_2 = 2$ and V is isomorphic to $L(1/2,0)\otimes L(1/2,0)$.*

Thank You !

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