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Generalized twisted modules associated to general automorphisms of a vertex operator algebra

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Abstract

We introduce a notion of strongly $\mathbb{C}^{\times}$-graded, or equivalently, $\mathbb{C}/\mathbb{Z}$-graded generalized g-twisted V-module associated to an automorphism g, not necessarily of finite order, of a vertex operator algebra. We also introduce a notion of strongly $\mathbb{C}$-graded generalized g-twisted V-module if V admits an additional $\mathbb{C}$-grading compatible with g. Let $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ be a vertex operator algebra such that $V_{(0)} = \mathbb{C}\mathbf{1}$ and $V_{(n)} = 0$ for $n < 0$ and let u be an element of V of weight 1 such that $L(1)u = 0$. Then the exponential of $2\pi\sqrt{-1}\ \mathrm{Res}_x Y(u, x)$ is an automorphism g_u of V. In this case, a strongly $\mathbb{C}$-graded generalized g_u-twisted V-module is constructed from a strongly $\mathbb{C}$-graded generalized V-module with a compatible action of g_u by modifying the vertex operator map for the generalized V-module using the exponential of the negative-power part of the vertex operator $Y(u, x)$. In particular, we give examples of such generalized twisted modules associated to the exponentials of some screening operators on certain vertex operator algebras related to the triplet W-algebras. An important feature is that we have to work with generalized (twisted) V-modules which are doubly graded by the group $\mathbb{C}/\mathbb{Z}$ or $\mathbb{C}$ and by generalized eigenspaces (not just eigenspaces) for $L(0)$, and the twisted vertex operators in general involve the logarithm of the formal variable.

1 Introduction

The present paper is the first in a series of papers developing systematically a general theory of twisted modules for vertex operator algebras. This theory is in fact equivalent to the study of orbifold theories in conformal field theory. Though twisted modules associated to finite-order automorphisms of a vertex operator algebra have been introduced and studied (see below for more discussions and references), curiously, twisted modules associated to infinite-order automorphisms of a vertex operator algebras have not been previously formulated and studied mathematically. To develop our general theory of twisted representations of vertex operator algebras, it is necessary to first find the correct category of twisted modules associated to both finite- and infinite-order automorphisms of a vertex operator algebra, and to study and construct such twisted modules. In this paper, we introduce a notion of strongly $\mathbb{C}^{\times}$-, or equivalently, $\mathbb{C}/\mathbb{Z}$-graded generalized g-twisted module associated to an automorphism of a vertex operator algebra and more generally, a notion of strongly $\mathbb{C}$-graded generalized g-twisted V-module when V has an additional $\mathbb{C}$-grading compatible with g. We also construct examples. The category of such generalized twisted modules will be the category of twisted modules that we shall study in this series of papers.

Twisted modules for vertex operator algebras were introduced in the construction of the moonshine module vertex operator algebra $V^{\natural}$ by I. Frenkel, J. Lepowsky and A. Meurman [FLM1], [FLM2] and [FLM3]. Given a vertex operator algebra V and an automorphism g of V of finite order, the notion of g-twisted V-module is a natural but very subtle generalization of the notion of V-module such that the action of g is incorporated. The main axiom in the definition of the notion of g-twisted V-module is the twisted Jacobi identity of the type proved in [Le1], [FLM2], [Le2] and [FLM3]. In fact, $V^{\natural}$ was the first example of an orbifold theory, before orbifold theories were studied in physics.

Orbifold theories are important examples of conformal field theories obtained from known conformal field theories and their automorphisms (see [DHVW1], [DHVW2], [H], [DFMS], [HV], [NSV], [M], [DGH], [DVVV], [DGM], as well as, for example [KS], [FKS], [Ba1], [Ba2], [BHS], [dBHO], [HO], [GHHO], [Ba3] and [HH]). Mathematically, the study of orbifold theories is equivalent to the theory of twisted representations of vertex operator algebras, in the sense that any result or conjecture on orbifold conformal field theories can be formulated precisely as a result or conjecture in the theory of twisted representations of vertex operator algebras.

There have been a number of papers constructing and studying twisted modules associated to finite-order automorphisms of vertex operator algebras (see for example, [Le1], [FLM2], [Le2], [FLM3], [D], [DL], [DonLM1], [DonLM2], [Li], [BDM], [DoyLM1], [DoyLM2], [BHL], and the references in these papers, especially in the last three papers). However, as far as the author knows, the precise notion of g-twisted V-module for an infinite-order automorphism g has not been previously formulated, and such twisted modules have not been previously constructed and studied.

The main difficulty in formulating such a general notion of g-twisted V-module is that the right-hand side

$$\frac{1}{k}\sum_{j\in \mathbb{Z}/k\mathbb{Z}} x_2^{-1}\delta\left(\eta^j \frac{(x_1-x_0)^{1/k}}{x_2^{1/k}}\right) Y^g(Y(g^j u, x_0)v, x_2)$$

of the twisted Jacobi identity (see [Le1], [FLM2], [Le2] and [FLM3]; cf. [DL]) and the form of the expression

$$(x_2+x_0)^N Y^g(Y(u, x_0)v, x_2)$$

needed in the weak associativity relation (see [Li]) cannot be generalized to the general case. In the present paper, we first reformulate the notion of g-twisted V-module in the case of finite order g using a duality property (incorporating both commutativity and associativity) formulated in terms of complex variables and then generalize this equivalent definition to the general case. The precise generalization involves more subtle issues which fortunately have been treated in the logarithmic tensor product theory developed by Lepowsky, Zhang and the author in [HLZ1] and [HLZ2]. First, in the general case, we have to consider a twisted generalization of the notion of generalized V-module in the sense of [HLZ1] and [HLZ2] instead of the notion of V-module. Second, the twisted vertex operators in general involve the logarithm of the formal variable. Third, we have to consider an additional grading by $\mathbb{C}^\times$ (equivalently by $\mathbb{C}/\mathbb{Z}$) or by $\mathbb{C}$ on such a twisted generalization to introduce a notion of strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized g-twisted V-module in the spirit of the the notion of strongly $\tilde{A}$-graded generalized V-module in [HLZ1] and [HLZ2] for a suitable abelian group $\tilde{A}$. It is important to note that in the general case, all these three new ingredients are necessary. In particular, many of the general results obtained in [HLZ1] and [HLZ2] are needed for the study of strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized g-twisted V-modules introduced and constructed in the present paper.

Even when g is a finite-order automorphism of V, how to explicitly construct a nontrivial g-twisted V-module is still an open problem in general.

The problem of constructing strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized g-twisted V-module for a general automorphism of V is certainly more difficult. However, when we consider certain special types of automorphisms of V, we might still be able to construct such generalized twisted V-modules associated to these automorphisms even though the general problem is not solved; indeed, in the finite-order case, the construction of certain twisted modules for lattice vertex operator algebras was given by Frenkel, Lepowsky and Meurman as one of the many hard steps in the construction of the moonshine module in the work [FLM3] and has been used to solve many other significant problems.

Given $u \in V_{(1)}$ such that $L(1)u = 0$, we show that $g_u = e^{2\pi\sqrt{-1}\,\mathrm{Res}_x Y(u,x)}$ is an automorphism of V. In general, its order might not be finite. In the present paper, generalizing a construction by Li [Li] in the case that $\mathrm{Res}_x Y(u,x)$ acts on V semisimply and has eigenvalues belonging to $\frac{1}{k}\mathbb{Z}$, we construct a strongly $\mathbb{C}$-graded generalized g_u-twisted V-module for $u \in V_{(1)}$ from a strongly $\mathbb{C}$-graded generalized V-module with an action of g_u such that the $\mathbb{C}$-grading is given by the generalized eigenspaces of the action of $\mathrm{Res}_x Y(u,x)$. We modify the vertex operator map of the generalized V-module using the exponentials of the negative-power part of the vertex operator $Y(u,x)$. We study such exponentials for u not necessarily of weight 1 and apply the result to the case of $u \in V_{(1)}$ later in the construction. Our results show the importance of studying exponentials of vertex operators.

As explicit examples, we apply this construction to certain vertex operator algebras constructed from certain lattice vertex operator algebras and automorphisms obtained by exponentiating suitable screening operators. The triplet W-algebras (see [FGST] and [AM]) are in fact subalgebras of the fixed-point subalgebras of the vertex operator algebras under the group generated these automorphisms. In particular, our results show that in this special case, the intertwining operators for suitable fixed-point subalgebras given by Adamović and Milas in Theorem 9.1 in [AM] are in fact twisted vertex operators. This connection between suitable intertwining operators and twisted vertex operators shows that the triplet logarithmic conformal field theories are closely related to orbifold theories. Note that up to now, the triplet W-algebras are the main konwn examples of vertex operator algebras for logarithmic conformal field theories. We expect that our theory will provide an orbifold-theoretic approach to the triplet logarithmic conformal field theories and will give more interesting examples of logarithmic conformal field theories.

The present paper is organized as follows: In Section 2, we reformulate the notion of g-twisted V-module in the case of finite-order g using complex variables and duality properties. Then we introduce the notions of strongly $\mathbb{C}/\mathbb{Z}$- and $\mathbb{C}$-graded generalized g-twisted V-module in the general case in Section 3. In Section 4, we study the exponentials of the negative-power parts of vertex operators. The construction of strongly $\mathbb{C}$-graded generalized g_u-twisted V-modules for $u \in V_{(1)}$ is given in Section 5. The explicit examples related to the triplet W-algebras are also given in this section.

Acknowledgments I would like to thank Antun Milas for pointing out that Theorem 5.5 can be used to show that certain intertwining operators constructed in [AM] are in fact the twisted vertex operator maps for certain generalized twisted modules in the sense of the definition given in the present paper.

2 Twisted modules for finite-order automorphisms of a vertex operator algebra

The notion of g-twisted module associated to a finite order automorphism g of V is formulated by collecting the properties of twisted vertex operators obtained in [Le1], [FLM2], [Le2] and [FLM3]; in particular, it uses a twisted Jacobi identity (see (2.1) below) as the main axiom. See [FLM3], [D] and [DL]. For an infinite-order automorphism g of V, we cannot write down a generalization of the twisted Jacobi identity in the case of finite-order automorphisms. The twisted Jacobi identity can also be replaced by the weak commutativity and (twisted) weak associativity in the case of finite-order automorphisms (see (2.3) below). But the side of the weak associativity containing the iterate of the twisted vertex operator map cannot be generalized to the case of an (necessarily infinite-order) automorphism which acts on the vertex operator algebra or the twisted module to be defined nonsemisimply. These difficulties force us to use the complex variable approach. To motivate our notion of g-twisted module for a general automorphism g of V, we reformulate the definition of g-twisted module in terms of complex variables in this section.

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and g an automorphism of V of order $k \in \mathbb{Z}_+$. Then $V = \coprod_{j\in\mathbb{Z}/k\mathbb{Z}} V^j$ where $V^j = \{v \in V \mid gv = \eta^j v\}$ for $j \in \mathbb{Z}/k\mathbb{Z}$, where $\eta = e^{\frac{2\pi\sqrt{-1}}{k}}$. We first recall the notion of g-twisted

V-module.

Definition 2.1. A *g-twisted V-module* W is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a linear map

$$\begin{aligned} Y^g : V \otimes W &\rightarrow W[[x^{1/k}, x^{-1/k}]] \\ v \otimes w &\mapsto Y^g(v,x)w = \sum_{n\in\frac{1}{k}\mathbb{Z}} Y^g_n(v)wx^{-n-1} \end{aligned}$$

satisfying the following conditions:

1. The *grading-restriction condition*: For each $n \in \mathbb{C}$, $\dim W_{(n)} < \infty$ and $W_{(n+l/k)} = 0$ for all sufficiently negative integers l.
2. The *formal monodromy condition*: For $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$, $Y^g(v,x) = \sum_{n\in j/k+\mathbb{Z}} Y^g_n(v)x^{-n-1}$.
3. The *lower-truncation condition*: For $v \in V$ and $w \in W$, $Y^g_n(v)w = 0$ for n sufficiently large.
4. The *identity property*: $Y^g(\mathbf{1}, x) = 1$.
5. The *twisted Jacobi identity*: For $u, v \in V$ and $w \in W$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) & Y^g(u,x_1)Y^g(v,x_2) \\ & - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y^g(v,x_2)Y^g(u,x_1) \\ = \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} & x_2^{-1}\delta\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) Y^g(Y(g^j u, x_0)v, x_2). \end{aligned} \tag{2.1}$$

6. The *$L^g(0)$-grading condition*: For $n \in \mathbb{Z}$, let $L^g(n) = Y^g_{n+1}(\omega)$, that, is, $Y^g(\omega, x) = \sum_{n\in\mathbb{Z}} L^g(n)x^{-n-2}$. Then $L^g(0)w = mw$ for $w \in W_{(m)}$.
7. The *$L^g(-1)$-derivative condition*: For $v \in V$,

$$\frac{d}{dx}Y^g(v,x) = Y^g(L(-1)v, x).$$

We denote a g-twisted V-module defined above by (W, Y^g) (or briefly, by W). Note that in the definition above, we use $Y_n^g(u)$ instead of u_n^g for $n \in \frac{1}{k}\mathbb{Z}$ to denote the components of the vertex operator $Y^g(u, x)$. In the present paper, we shall always use such notations to denote the componenets of a vertex operator. For example, for our vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, we shall use $Y_n(u)$ instead of u_n for $n \in \mathbb{Z}$ to denote the compoenets of the vertex operator $Y(u, x)$. Similarly, for a V-module (W, Y_W), we shall use $(Y_W)_n(u)$ instead of u_n for $n \in \mathbb{Z}$ to denote the compoenets of the vertex operator $Y_W(u, x)$.

This definition of g-twisted V-module is a natural generalization of the definition of V-module given in [Bo], [FLM3] and [FHL]. The main axiom is the twisted Jacobi identity. The following result is proved by Li [Li]:

Proposition 2.2. *The twisted Jacobi identity in Definition 2.1 can be replaced by the following two properties:*

1. *The weak commutativity or locality: For $u, v \in V$, there exists $N \in \mathbb{Z}_+$ such that*

$$(x_1 - x_2)^N Y^g(u, x_1) Y^g(v, x_2) = (x_2 - x_1)^N Y^g(v, x_2) Y^g(u, x_1). \quad (2.2)$$

2. *The weak associativity: For $u \in V^j$ and $w \in W$, there exists $M \in \mathbb{Z}_+$ such that*

$$(x_2+x_0)^{\tilde{j}/k+M} Y^g(Y(u, x_0 v, x_2) = (x_0+x_2)^{\tilde{j}/k+M} Y^g(u, x_0+x_2) Y^g(v, x_2) \quad (2.3)$$

for $v \in V$, where $\tilde{j} \in j$ satisfies $0 \leq \tilde{j} < k$.

One advantage of the definition, the weak commutativity and the weak associativity above is that they use only formal variables so that one can discuss the multivaluedness of the correlation functions algebraically. But as we mentioned above, this formulation of g-twisted V-module for g of finite order cannot be generalized to the case that g is of infinite order. So we first reformulate this definition using complex variables and a duality property.

We need a complex variable formulation of vertex operator maps and some notations first. For any $z \in \mathbb{C}^\times$, we shall use $\log z$ to denote $\log |z| + \sqrt{-1} \arg z$ where $0 \leq \arg z < 2\pi$. In general, we shall use $l_k(z)$ to denote $\log z + 2\sqrt{-1}\pi$ for $k \in \mathbb{Z}$. Let W_1, W_2 and W_3 be $\mathbb{C}$-graded vector spaces. The $\mathbb{C}$-gradings on W_1 and W_2 induce a $\mathbb{C}$-grading on $W_1 \otimes W_2$. For a formal variable $x^{1/k}$, we define its *degree* to be $-1/k$. Then the grading on W_3 and the degree of x together give $W_3[[x^{1/k}, x^{-1/k}]]$ a $\mathbb{C}$-graded vector space structure. Let

$$\begin{aligned} X : W_1 \otimes W_2 &\to W_3[[x^{1/k}, x^{-1/k}]] \\ w_1 \otimes w_2 &\mapsto X(w_1, x)w_2 = \sum_{n \in \frac{1}{k}\mathbb{Z}} X_n(w_1)w_2 x^{-n-1} \end{aligned}$$

is a linear map preserving the gradings. For any $w_1 \in W_1$ and $w_2 \in W_2$, we define $X^p(w_1, z)w_2$ for $p \in \mathbb{Z}/k\mathbb{Z}$ to be the elements $X(w_1, x)w_2\Big|_{x^n = e^{nl_{\tilde{p}}(z)}}$ of $\overline{W}_3 = \prod_{n \in \mathbb{C}} (W_3)_{(n)}$ where for $p \in \mathbb{Z}/k\mathbb{Z}$, $\tilde{p}$ is the integer in p satisfying $0 \leq \tilde{p} < k$. When $p = 0$, for simplicity, we shall denote $X^0(w_1, z)w_2$ simply as $X(w_1, z)w_2$. For $z \in \mathbb{C}^\times$, we have the linear maps

$$\begin{aligned} X^p(\cdot, z)\cdot : W_1 \otimes W_2 &\to \overline{W}_3, \\ w_1 \otimes w_2 &\mapsto X^p(w_1, z)w_2 \end{aligned} \tag{2.4}$$

and we shall denote $X^0(\cdot, z)\cdot$ simply as $X(\cdot, z)\cdot$. For any $w_3' \in W_3'$,

$$\langle w_3', X^p(w_1, z)w_2 \rangle$$

are branches of a multivalued analytic function defined on $\mathbb{C}^\times$. We can view this multivalued analytic function on $\mathbb{C}^\times$ as a single-valued analytic function defined on a k-fold covering space of $\mathbb{C}^\times$. For $p \in \mathbb{Z}/k\mathbb{Z}$, we have a map

$$\begin{aligned} X^p : \mathbb{C}^\times &\to \mathrm{Hom}(W_1 \otimes W_2, \overline{W}_3) \\ z &\mapsto X^p(\cdot, z) \cdot . \end{aligned}$$

We shall call X^p the *p-th analytic branch* of the map X.

In particular, for the twisted vertex operator map Y^g, we have its p-th analytic branch $Y^{g;p}$. In terms of these analytic branches, we have:

Proposition 2.3. *The formal monodromy condition in Definition 2.1 can be replaced by the following condition: For $p \in \mathbb{Z}/k\mathbb{Z}$, $z \in \mathbb{C}^\times$ and $v \in V$,*

$$Y^{g;p+1}(gv, z) = Y^{g;p}(v, z). \tag{2.5}$$

Proof. Let W be a g-twisted V-module satisfying Definition 2.1. Then for $v \in V^j$, we have $gv = \eta^j v \in V^j$. Then by the formal monodromy condition, for $p \in \mathbb{Z}/k\mathbb{Z}$, $z \in \mathbb{C}^\times$ and $v \in V^j$,

$$\begin{aligned}
Y^{g;p+1}(gv, z) &= \sum_{n\in\frac{1}{k}\mathbb{Z}} Y_n^g(gv)e^{(-n-1)l_{p+1}(z)} \\
&= \sum_{n\in\frac{j}{k}+\mathbb{Z}} Y_n^g(v)\eta^j e^{(-n-1)l_{p+1}(z)} \\
&= \sum_{n\in\frac{j}{k}+\mathbb{Z}} Y_n^g(v)e^{\frac{2\pi j\sqrt{-1}}{k}} e^{(-n-1)l_{p+1}(z)} \\
&= \sum_{n\in\frac{j}{k}+\mathbb{Z}} Y_n^g(v)e^{(-n-1)(l_{p+1}(z)-2\pi\sqrt{-1})} \\
&= \sum_{n\in\frac{j}{k}+\mathbb{Z}} Y_n^g(v)e^{(-n-1)l_p(z)} \\
&= Y^{g;p}(v, z).
\end{aligned}$$

Conversely, assume that W satisfies the modification of Definition 2.1 by replacing the formal monodromy condition with the equivariance condition. Then for $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$,

$$\begin{aligned}
\sum_{n\in\frac{1}{k}\mathbb{Z}} Y_n^g(v)\eta^j e^{(-n-1)l_{p+1}(z)} &= \sum_{n\in\frac{1}{k}\mathbb{Z}} Y_n^g(gv)e^{(-n-1)l_{p+1}(z)} \\
&= Y^{g;p+1}(gv, z) \\
&= Y^{g;p}(v, z) \\
&= \sum_{n\in\frac{1}{k}\mathbb{Z}} Y_n^g(v)e^{(-n-1)l_p(z)} \\
&= \sum_{n\in\frac{1}{k}\mathbb{Z}} Y_n^g(v)e^{\frac{2\pi(n+1)\sqrt{-1}}{k}} e^{(-n-1)l_{p+1}(z)}.
\end{aligned}$$

So we have

$$\sum_{n\in\frac{1}{k}\mathbb{Z}} Y_n^g(v)(\eta^j - e^{\frac{2\pi(-n-1)\sqrt{-1}}{k}})e^{(n+1)l_p(z)} = 0,$$

which implies that

$$Y_n^g(v)(\eta^j - e^{\frac{2\pi(n+1)\sqrt{-1}}{k}}) = 0$$

for $n \in \frac{1}{k}\mathbb{Z}$. So we have either $Y_n^g(v) = 0$ or $\eta^j = e^{\frac{2\pi(n+1)\sqrt{-1}}{k}}$, that is, either $Y_n^g(v) = 0$ or $n \in j/k + \mathbb{Z}$. So we obtain $Y^g(v, x) = \sum_{n\in j/k+\mathbb{Z}} Y_n^g(v)x^{-n-1}$. ∎

We shall call (2.5) in Proposition 2.3 the *equivariance property*.

For $n \in \mathbb{C}$, let $\pi_n : W \to W_{(n)}$ be the projection. We have:

Theorem 2.4. *The twisted Jacobi identity in Definition 2.1 can be replaced by the following property: For any $u, v \in V$, $w \in W$ and $w' \in W'$, there exists a multivalued analytic function of the form*

$$f(z_1, z_2) = \sum_{r,s=N_1}^{N_2} a_{rs} z_1^{r/k} z_2^{s/k} (z_1 - z_2)^{-N}$$

for $N_1, N_2, N \in \mathbb{Z}_+$, such that the series

$$\langle w', Y^{g;p}(u, z_1) Y^{g;p}(v, z_2) w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g;p}(u, z_1) \pi_n Y^{g;p}(v, z_2) w \rangle, \quad (2.6)$$

$$\langle w', Y^{g;p}(v, z_2) Y^{g;p}(u, z_1) w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g;p}(v, z_2) \pi_n Y^{g;p}(u, z_1) w \rangle, \quad (2.7)$$

$$\langle w', Y^{g;p}(Y(u, z_1 - z_2) v, z_2) w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g;p}(\pi_n Y(u, z_1 - z_2) v, z_2) w \rangle \quad (2.8)$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to the branch

$$\sum_{r,s=N_1}^{N_2} a_{rs} e^{(r/k) l_p(z_1)} e^{(s/k) l_p(z_2)} (z_1 - z_2)^{-N} \quad (2.9)$$

of $f(z_1, z_2)$.

Proof. Assume that W is a g-twisted V-module satisfying Definition 2.1. By Proposition 2.2, there exists $N \in \mathbb{Z}_+$ such that (2.2) holds.

For $w \in W$ and $w' \in W'$, we know that

$$\langle w', Y^g(u, x_1) Y^g(v, x_2) w \rangle \quad (2.10)$$

involves finitely many positive powers of $x_1^{1/k}$ and fnitely many negative powers of $x_2^{1/k}$ and

$$\langle w', Y^g(v, x_2) Y^g(u, x_1) w \rangle \quad (2.11)$$

involves finitely many negative powers of $x_1^{1/k}$ and fnitely many positive powers of $x_2^{1/k}$. Since $(x_1 - x_2)^N$ is a polynomial in $x_1^{1/k}$ and $x_2^{1/k}$, the same statements are also true for

$$(x_1 - x_2)^N \langle w', Y^g(u, x_1)Y^g(v, x_2)w\rangle$$

and

$$(x_1 - x_2)^N \langle w', Y^g(v, x_2)Y^g(u, x_1)w\rangle,$$

respectively. By (2.2), they are acrually equal. Thus they are both equal to

$$\sum_{r,s=N_1}^{N_2} a_{rs} x_1^{r/k} x_2^{s/k} \in \mathbb{C}[, x_1^{1/k}, x_1^{-1/k}, x_2^{1/k}, x_2^{-1/k}].$$

Since (2.10) involves finitely many positive powers of $x_1^{1/k}$ and fnitely many negative powers of $x_2^{1/k}$, it must be equal to the expansion of

$$\sum_{r,s=N_1}^{N_2} a_{rs} x_1^{r/k} x_2^{s/k} (x_1 - x_2)^{-N}$$

as a series of the same form. Thus the expansion of (2.9) in the region $|z_1| > |z_2| > 0$ is equal to (2.6). The same argument shows that the expansion of (2.9) in the region $|z_2| > |z_1| > 0$ is equal to (2.7).

For $j \in \mathbb{Z}/k\mathbb{Z}$, let $\tilde{j} \in j$ satisfying $0 \leq \tilde{j} < k$. For $u \in V^j$, by Proposition 2.2 there exists $M \in \mathbb{Z}_+$ such that (2.3) holds.

For $w \in W$ and $w' \in W'$, we know that (2.10) involves finitely many positive powers of $x_1^{1/k}$ and fnitely many negative powers of $x_2^{1/k}$. So

$$\langle w', Y^g(u, x_0 + x_2)Y^g(v, x_2)w\rangle \tag{2.12}$$

involves finitely many positive powers of $x_0^{1/k}$ and fnitely many negative powers of $x_2^{1/k}$. We also know that

$$\langle w', Y^g(Y(u, x_0)v, x_2)w\rangle \tag{2.13}$$

involves finitely many negative powers of $x_0^{1/k}$ and fnitely many positive powers of $x_2^{1/k}$. Since $(x_0 + x_2)^{\tilde{j}/k+M}$ involves finitely many positive powers of $x_0^{1/k}$ and no negative powers of $x_2^{1/k}$ and $(x_2 + x_0)^{\tilde{j}/k+M}$ involves finitely many positive powers of $x_2^{1/k}$ and no negative powers of $x_2^{1/k}$, the same statements above for (2.12) and (2.13) are also true for

$$(x_0+x_2)^{\tilde{j}/k+M}\langle w', Y^g(u, x_0+x_2)Y^g(v, x_2)w\rangle$$

and

$$(x_2+x_0)^{\tilde{j}/k+M}\langle w', Y^g(Y(u, x_0)v, x_2)w\rangle,$$

respectively. Since we have proved that (2.10) is equal to

$$\sum_{r,s=N_1}^{N_2} a_{rs}x_1^{r/k}x_2^{s/k}(x_1-x_2)^{-N},$$

we obtain

$$\begin{aligned}(x_0+x_2)^{\tilde{j}/k+M}&\langle w', Y^g(u, x_0+x_2)Y^g(v, x_2)w\rangle\\ &= \sum_{r,s=N_1}^{N_2} a_{rs}(x_0+x_2)^{r/k+\tilde{j}/k+M}x_2^{s/k}x_0^{-N}.\end{aligned}$$

By (2.3), we also have

$$\begin{aligned}(x_2+x_0)^{\tilde{j}/k+M}&\langle w', Y^g(Y(u, x_0)v, x_2)w\rangle\\ &= \sum_{r,s=N_1}^{N_2} a_{rs}(x_2+x_0)^{r/k+\tilde{j}/k+M}x_2^{s/k}x_0^{-N}.\end{aligned}$$

Thus the expansion of (2.9) in the region $|z_2| > |z_1 - z_2| > 0$ is equal to (2.8).

Conversely, assume that the property in the theorem holds. Let

$$f_j(x_1, x_2, x_0) = \sum_{r,s=N_1}^{N_2} a_{rs}\eta^{-jr}x_1^{r/k}x_2^{s/k}x_0^{-N}$$

for $j \in \mathbb{Z}/kZ$. Then this property gives

$$\begin{aligned}\langle w', Y^g(u, x_1)Y^g(v, x_2)w\rangle &= f_0(x_1, x_2, x_1-x_2), &(2.14)\\ \langle w', Y^g(v, x_2)Y^g(u, x_1)w\rangle &= f_0(x_1, x_2, -x_2+x_1), &(2.15)\\ \langle w', Y^g(Y(u, z_1-z_2)v, z_2)w\rangle &= f_0(x_2+x_0, x_2, x_0) &(2.16)\end{aligned}$$

for $u, v \in V$, $w \in W$ and $w' \in W'$. From (2.14), (2.16) and the equivariance property, we have

$$\langle w', Y^g(Y(g^j u, z_1 - z_2)v, z_2)w\rangle = f_j(x_2 + x_0, x_2, x_0) \tag{2.17}$$

for $u, v \in V$, $w \in W$ and $w' \in W'$.

We have the formal variable identity

$$x_0^{-1}\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\left(\frac{x_2 - x_1}{-x_0}\right) = \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right).$$

Using this identity and the property of the delta-function, we have

$$\begin{aligned}
x_0^{-1}&\left(\frac{x_1 - x_2}{x_0}\right) f_0(x_1, x_2, x_1 - x_2) - x_0^{-1}\left(\frac{x_2 - x_1}{-x_0}\right) f_0(x_1, x_2, -x_2 + x_1)\\
&= x_0^{-1}\left(\frac{x_1 - x_2}{x_0}\right) f_0(x_1, x_2, x_0) - x_0^{-1}\left(\frac{x_2 - x_1}{-x_0}\right) f_0(x_1, x_2, x_0)\\
&= \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) f_0(x_1, x_2, x_0)\\
&= \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) \sum_{r,s=N_1}^{N_2} a_{rs} x_1^{r/k} x_2^{s/k} x_0^{-N}\\
&= \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) \sum_{r,s=N_1}^{N_2} a_{rs}((x_1 - x_0) + x_0)^{r/k} x_2^{s/k} x_0^{-N}\\
&= \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right)\cdot\\
&\qquad\qquad \cdot\sum_{r,s=N_1}^{N_2} a_{rs}(x_1 - x_0)^{r/k}\left(1 + \frac{x_0}{x_1 - x_0}\right)^{r/k} x_2^{s/k} x_0^{-N}\\
&= \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) \sum_{r,s=N_1}^{N_2} a_{rs}\eta^{-jr} x_2^{r/k}\left(1 + \frac{x_0}{x_2}\right)^{r/k} x_2^{s/k} x_0^{-N}\\
&= \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) \sum_{r,s=N_1}^{N_2} a_{rs}\eta^{-jr}(x_2 + x_0)^{r/k} x_2^{s/k} x_0^{-N}
\end{aligned}$$

$$= \frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} x_2^{-1} \left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) f_j(x_2 + x_0, x_2, x_0). \tag{2.18}$$

From (2.14)–(2.16) and (2.18), we obtain

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) \langle w', Y^g(u, x_1)Y^g(v, x_2)w\rangle \\ -x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \langle w', Y^g(v, x_2)Y^g(u, x_1)w\rangle \\ = \frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} x_2^{-1}\delta\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) \langle w', Y^g(Y(g^j u, x_0)v, x_2)w\rangle \end{aligned}$$

for $u, v \in V$, $w \in W$ and $w' \in W'$. Since $w \in W$ and $w' \in W'$ are arbitrary, we obtain the twisted Jacobi identity (2.1) for $u, v \in V$. ∎

We shall call the property in Theorem 2.4 the *duality property.*

Remark 2.5. For simplicity, we give only the definition of twisted module for a vertex operator algebra, not for a vertex operator superalgebra. But it is trivial to generalize it to that of twisted module for a vertex operator superalgebra.

3 Definitions of strongly $\mathbb{C}/\mathbb{Z}$- and $\mathbb{C}$-graded generalized g-twisted module

We give a definition of strongly $\mathbb{C}/\mathbb{Z}$-graded generalized g-twisted V-module for an automorphism g of V of possibly infinite order in this section. In the case that V admits an additional $\mathbb{C}$-grading compatible with g, we also give a definition of strongly $\mathbb{C}$-graded generalized g-twisted V-module. In the case that g is of finite order, g always acts on V semisimply so that we have $V = \coprod_{j \in \mathbb{Z}/k\mathbb{Z}} V^j$. When the order of g is infinite, it does not have to act on V semisimply. In particular, we have to allow not only nonintegral powers of x in the twisted vertex operators, but also integral powers of $\log x$ in these operators. As we have mentioned in the introduction, the left-hand side of the twisted Jacobi identity cannot be generalized to the case that the order of g is infinite. Also we cannot straightforwardly generalize the formal monodromy condition to the general case. On the other hand, the equivariance property and the duality property in Proposition 2.3 and Theorem 2.4, respectively, can be generalized easily. Our definition will use the generalizations of these axioms.

First we need to further generalize our consideration of analytic branches of the map X in the preceding section. For formal variables x and $\log x$, we define their *degrees* to be -1 and 0. Let W_1, W_2 and W_3 be $\mathbb{C}$-graded vector spaces. Then the grading on W_3 and the degrees of x and $\log x$ together give $W_3\{x\}[\log x]$ a $\mathbb{C}$-grading. Let

$$\begin{aligned} X : W_1 \otimes W_2 &\to W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto X(w_1, x)w_2 = \sum_{n\in\mathbb{C}} \sum_{k=1}^{K} X_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k \end{aligned}$$

is a linear map preserving the gradings. For any $w_1 \in W_1$ and $w_2 \in W_2$, we define $X^p(w_1, z)w_2$ for $p \in \mathbb{Z}$ to be the elements $X(w_1, x)w_2\Big|_{x^n=e^{nl_p(z)},\ \log x=l_p(z)}$ of $\overline{W}_3$. When $p = 0$, we shall denote $X^0(w_1, z)w_2$ simply as $X(w_1, z)w_2$. For $z \in \mathbb{C}^\times$, we have the linear maps

$$\begin{aligned} X^p(\cdot, z)\cdot : W_1 \otimes W_2 &\to \overline{W}_3, \\ w_1 \otimes w_2 &\mapsto X^p(w_1, z)w_2 \end{aligned} \tag{3.1}$$

and we shall denote $X^0(\cdot, z)\cdot$ simply as $X(\cdot, z)\cdot$. For any $w_3' \in W_3'$,

$$\langle w_3', X^p(w_1, z)w_2\rangle$$

are branches of a multivalued analytic function defined on $\mathbb{C}^\times$. We can view this multivalued analytic function on $\mathbb{C}^\times$ as a single-valued analytic function defined on a covering space of $\mathbb{C}^\times$. For $p \in \mathbb{Z}$, we have a map

$$\begin{aligned} X^p : \mathbb{C}^\times &\to \mathrm{Hom}(W_1 \otimes W_2, \overline{W}_3) \\ z &\mapsto X^p(\cdot, z)\cdot . \end{aligned}$$

We shall call X^p the *p-th analytic branch* of the map X.

We also need a $\mathbb{C}^\times$-, or equivalently, $\mathbb{C}/\mathbb{Z}$-grading structure on V given by an automorphism g of V. Since g preserve $V_{(n)}$ for $n \in \mathbb{Z}$ and $V_{(n)}$ for $n \in \mathbb{Z}$ are all finite dimensional, we have

$$V_{(n)} = \coprod_{a\in\mathbb{C}^\times} V_{(n)}^{[a]}$$

for $n \in \mathbb{Z}$, where for $n \in \mathbb{Z}$ and $a \in \mathbb{C}^\times$, if a is an eigenvalue for the operator g on $V_{(n)}$, $V_{(n)}^{[a]}$ is the generalized eigenspace of $V_{(n)}$ for g with the eigenvalue a, and if a is not an eigenvalue for g on $V_{(n)}$, $V_{(n)}^{[a]} = 0$. Since the multiplicative abelian group $\mathbb{C}^\times$ is isomorphic to the additive abelian group $\mathbb{C}/\mathbb{Z}$ through the isomorphism $a \mapsto \frac{1}{2\pi\sqrt{-1}} \log a + \mathbb{Z}$, we can use $\mathbb{C}/\mathbb{Z}$ instead of $\mathbb{C}^\times$ so that

$$V_{(n)} = \coprod_{\alpha \in \mathbb{C}/\mathbb{Z}} V_{(n)}^{[\alpha]}$$

and V is doubly graded by $\mathbb{C}$ and $\mathbb{C}/\mathbb{Z}$ as

$$V = \coprod_{n \in \mathbb{Z}, \alpha \in \mathbb{C}/\mathbb{Z}} V_{(n)}^{[\alpha]}.$$

Let $V^{[\alpha]} = \coprod_{n \in \mathbb{Z}} V_{(n)}^{[\alpha]}$ for $\alpha \in \mathbb{C}/\mathbb{Z}$. Then $V = \coprod_{\alpha \in \mathbb{C}/\mathbb{Z}} V^{[\alpha]}$ and for $v \in V^{[\alpha]}$, $gv = e^{2\pi\sqrt{-1}\,\alpha} v$. It is easy to see that the vertex operator algebra V together with this $\mathbb{C}/\mathbb{Z}$-grading is a strongly $\mathbb{C}/\mathbb{Z}$-graded vertex operator algebra in the sense of [HLZ2].

If, in addition to the $\mathbb{C}$-grading given by the eigenvalues of $L(0)$, V has another $\mathbb{C}$-grading $V = \coprod_{\alpha \in \mathbb{C}} V^{[\alpha]}$ which is compatible with g (that is, for $v \in V^{[\alpha]}$, $gv = e^{2\pi\sqrt{-1}\,\alpha} v$), then $V = \coprod_{n, \alpha \in \mathbb{C}} V_{(n)}^{[\alpha]}$ equipped with this $\mathbb{C}$-grading has a strongly $\mathbb{C}$-graded vertex operator algebra structure.

For a strongly $\mathbb{C}/\mathbb{Z}$-graded (or $\mathbb{C}$-graded) vertex operator algebra, a natural module category is the category of strongly $\mathbb{C}/\mathbb{Z}$-graded (or $\mathbb{C}$-graded, respectively) generalized V-modules. Moreover, if the $\mathbb{C}/\mathbb{Z}$-grading (or $\mathbb{C}$-graded) is given by (or compatible with) g, it is natural to consider strongly $\mathbb{C}/\mathbb{Z}$-graded (or $\mathbb{C}$-graded) generalized V-modules with actions of g such that the $\mathbb{C}^\times$-gradings (or $\mathbb{C}$-gradings) are given by (or compatible with) generalized eigenspaces for g in the obvious sense. The category of the generalizations of g-twisted modules should be a generalization of the category of such strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized V-modules. In particular, there should also be an action of g and also a $\mathbb{C}/\mathbb{Z}$-grading (or $\mathbb{C}$-grading) given by (or compatible with) generalized eigenspaces for the action of g. Here is our generalization of g-twisted V-module for finite-order g to the general case:

Definition 3.1. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and let g be an automorphism of V. A *strongly $\mathbb{C}/\mathbb{Z}$-graded generalized g-twisted V-module* is a $\mathbb{C}\times\mathbb{C}/\mathbb{Z}$-graded vector space $W=\coprod_{n\in\mathbb{C},\alpha\in\mathbb{C}/\mathbb{Z}}W_{[n]}^{[\alpha]}$ equipped with a linear map

$$\begin{aligned} Y^g: V\otimes W &\to W\{x\}[\log x]\\ v\otimes w &\mapsto Y^g(v,x)w \end{aligned}$$

and an action of g satisfying the following conditions:

1. The *grading-restriction condition*: For each $n\in\mathbb{C}$, $\alpha\in\mathbb{C}/\mathbb{Z}$, $\dim W_{[n]}^{[\alpha]}<\infty$ and $W_{[n+l]}^{[\alpha]}=0$ for all sufficiently negative real number l.

2. The *equivariance property*: For $p\in\mathbb{Z}$, $z\in\mathbb{C}^\times$, $v\in V$ and $w\in W$, $Y^{g;p+1}(gv,z)w=Y^{g;p}(v,z)w$, where for $p\in\mathbb{Z}$, $Y^{g;p}$ is the p-th analytic branch of Y^g.

3. The *identity property*: For $w\in W$, $Y^g(\mathbf{1},x)w=w$.

4. The *duality property*: Let $W'=\coprod_{n\in\mathbb{C},\alpha\in\mathbb{C}/\mathbb{Z}}(W_{[n]}^{[\alpha]})^*$ and, for $n\in\mathbb{C}$, $\pi_n: W\to W_{[n]}$ be the projection. For any $u,v\in V$, $w\in W$ and $w'\in W'$, there exists a multivalued analytic function of the form

$$f(z_1,z_2)=\sum_{i,j,k,l=0}^{N}a_{ijkl}z_1^{m_i}z_2^{n_i}(\log z_1)^k(\log z_2)^l(z_1-z_2)^{-t}$$

for $N\in\mathbb{N}$, $m_1,\dots,m_N,n_1,\dots,n_N\in\mathbb{C}$ and $t\in\mathbb{Z}_+$, such that the series

$$\langle w',Y^{g;p}(u,z_1)Y^{g;p}(v,z_2)w\rangle=\sum_{n\in\mathbb{C}}\langle w',Y^{g;p}(u,z_1)\pi_nY^{g;p}(v,z_2)w\rangle, \quad (3.2)$$

$$\langle w',Y^{g;p}(v,z_2)Y^{g;p}(u,z_1)w\rangle=\sum_{n\in\mathbb{C}}\langle w',Y^{g;p}(v,z_2)\pi_nY^{g;p}(u,z_1)w\rangle, \quad (3.3)$$

$$\langle w',Y^{g;p}(Y(u,z_1-z_2)v,z_2)w\rangle=\sum_{n\in\mathbb{C}}\langle w',Y^{g;p}(\pi_nY(u,z_1-z_2)v,z_2)w\rangle \quad (3.4)$$

are absolutely convergent in the regions $|z_1|>|z_2|>0$, $|z_2|>|z_1|>0$, $|z_2|>|z_1-z_2|>0$, respectively, to the branch

$$\sum_{i,j,k,l=0}^{N}a_{ijkl}e^{m_il_p(z_1)}e^{n_jl_{p(z_2)}}l_p(z_1)^kl_p(z_2)^l(z_1-z_2)^{-t}$$

of $f(z_1,z_2)$.

5. The $L^g(0)$- *and g-grading conditions*: Let $L^g(n)$ for $n \in \mathbb{Z}$ be the operator on W given by $Y^g(\omega, x) = \sum_{n\in\mathbb{Z}} L^g(n)^{-n-2}$. Then for $n \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$, $w \in W_{[n]}^{[\alpha]}$, there exists $K, \Lambda \in \mathbb{Z}_+$ such that $(L^g(0) - n)^K w = (g - e^{2\pi\sqrt{-1}\,\alpha})^\Lambda w = 0$.

6. The $L(-1)$-*derivative condition*: For $v \in V$,

$$\frac{d}{dx}Y^g(v, x) = Y^g(L(-1)v, x).$$

If V has a strongly $\mathbb{C}$-graded vertex operator algebra structure compatible with g, then a *strongly $\mathbb{C}$-graded generalized g-twisted V-module* is a $\mathbb{C} \times \mathbb{C}$-graded vector space $W = \coprod_{n,\alpha\in\mathbb{C}} W_{[n]}^{[\alpha]}$ equipped with a linear map

$$\begin{array}{rcl} Y^g : V \otimes W & \rightarrow & W\{x\}[\log x] \\ v \otimes w & \mapsto & Y^g(v, x)w \end{array}$$

and an action of g, satisfying the same axioms above except that $\mathbb{C}/\mathbb{Z}$ is replaced by $\mathbb{C}$ and the g-grading condition is replaced by the following *grading-compatibility condition*: For $\alpha, \beta \in \mathbb{C}$, $v \in V^{[\alpha]}$ and $w \in W^{[\beta]}$, $Y^g(v, x)w \in W^{[\alpha+\beta]}\{x\}[\log x]$.

Remark 3.2. As in the preceding section, for simplicity, we give only the definition of such generalized twisted module for a vertex operator algebra, not for a vertex operator superalgebra. It is also trivial in this general case to generalize the definition above to that of twisted module for a vertex operator superalgebra.

4 Exponentiating integrals of negative parts of vertex operators

In this section we shall exponentiate integrals of negative parts of vertex operators and prove formulas needed in the next section. We need first several commutator formulas.

The first is the commutator formula for vertex operators which is the special case of the Jacobi identity when we take the coefficients of x_0^{-1} and expand the formal delta-function on the right-hand side explicitly:

Proposition 4.1. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and (W, Y_W) a weak V-module. The for $u, v \in V$, we have the commutator formula*

$$[Y_W(u, x_1), Y_W(v, x_2)] = \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \binom{m}{k} x_1^{-m-1} x_2^{m-k} Y_W(Y_k(u)v, x_2). \qquad (4.1)$$

Let $u \in V$. In this section, we do not require that the conformal weight of u be 1. Let $Y_W^{\leq -2}(u, x)$ be the part of $Y_W(u, x)$ consisting all the monomials in x with powers of x less than or equal to -2, that is,

$$Y_W^{\leq -2}(u, x) = \sum_{m\in\mathbb{Z}_+} Y_m(u) x^{-m-1}.$$

For any vector space W and

$$f(x) = \sum_{n\in\mathbb{Z}\setminus\{-1\}} a_n x^n \in W[[x]] + x^{-2} W[[x^{-1}]],$$

let

$$\int_0^x f(y) = \sum_{n\in\mathbb{Z}\setminus\{-1\}} \frac{a_n}{n+1} x^{n+1}.$$

Then we obtain a linear map (*integration from* 0 *to* x)

$$\int_0^x : W[[x]] + x^{-2} W[[x^{-1}]] \to xW[[x]] + x^{-1} W[[x^{-1}]].$$

We have:

Proposition 4.2. *For $u, v \in V$,*

$$\left[\left(\int_0^{x_1} Y_W^{\leq -2}(u, y)\right), Y_W(v, x_2)\right] = Y_W\left(\int_0^{-x_1-x_2} Y^{\leq -2}(u, y)v, x_2\right) + \log\left(1 + \frac{x_2}{x_1}\right) Y_W(Y_0(u)v, x_2), \qquad (4.2)$$

where $\int_0^{-x_1-x_2} Y^{\leq -2}(u, y)v$ is defined by expanding the series in powers of $-x_1 - x_2$ as a series in positive powers of x_2 and in powers of x_1.

Proof. From (4.1), we obtain

$$
\begin{aligned}
[Y_{\overline{W}}^{\leq -2}(u, x_1), Y_W(v, x_2)] &= \sum_{m\in\mathbb{Z}_+}\sum_{k\in\mathbb{N}}\binom{m}{k} x_1^{-m-1}x_2^{m-k}Y_W(Y_k(u)v, x_2)\\
&= \sum_{k,l\in\mathbb{N}, k+l\neq 0}\binom{k+l}{k} x_1^{-k-l-1}x_2^{l}Y_W(Y_k(u)v, x_2).
\end{aligned}
\tag{4.3}
$$

Then

$$
\begin{aligned}
&\left[\int_0^{-x_1} Y_{\overline{W}}^{\leq -2}(u, y), Y_W(v, x_2)\right]\\
&\quad= \sum_{k,l\in\mathbb{N}, k+l\neq 0}\binom{k+l}{k}\frac{(-1)^{k+l+1}}{k+l} x_1^{-k-l}x_2^{l}Y_W(Y_k(u)v, x_2)\\
&\quad= \sum_{k,l\in\mathbb{N}, k+l\neq 0}\frac{(k+l)\cdots(k+l-l-1)}{l!}\frac{(-1)^{k+l+1}}{k+l} x_1^{-k-l}x_2^{l}Y_W(Y_k(u)v, x_2)\\
&\quad= \sum_{k,l\in\mathbb{N}, k+l\neq 0}\frac{(-k)\cdots(-k-l+1)}{l!}\frac{(-1)^{k}}{-k} x_1^{-k-l}x_2^{l}Y_W(Y_k(u)v, x_2)\\
&\quad= \sum_{k\in\mathbb{Z}_+}\frac{(-1)^{k}}{-k}\sum_{l\in\mathbb{N}}\binom{-k}{l} x_1^{-k-l}x_2^{l}Y_W(Y_k(u)v, x_2)\\
&\qquad+\sum_{l\in\mathbb{Z}_+}\frac{(-1)^{l+1}}{l} x_1^{-l}x_2^{l}Y_W(Y_0(u)v, x_2)\\
&\quad= \sum_{k\in\mathbb{Z}_+}\frac{(-1)^{k}}{-k}(x_1+x_2)^{-k}Y_W(Y_k(u)v, x_2) + \log\left(1+\frac{x_2}{x_1}\right)Y_W(Y_0(u)v, x_2)\\
&\quad= Y_W\left(\int_0^{-x_1-x_2} Y^{\leq -2}(u, y)v, x_2\right) + \log\left(1+\frac{x_2}{x_1}\right)Y_W(Y_0(u)v, x_2).
\end{aligned}
$$

∎

We also have:

Proposition 4.3. *For $u \in V$,*

$$
\left[L(-1), \int_0^{-x} Y_{\overline{W}}^{\leq -2}(u, y)\right] = -\frac{d}{dx}\int_0^{x} Y_{\overline{W}}^{\leq -2}(u, y) - Y_0(u)x^{-1}. \tag{4.4}
$$

Proof. By the $L(-1)$-derivative property, we have

$$[L(-1), Y_W(u, y)] = \frac{d}{dy} Y_W(u, x),$$

which gives

$$[L(-1), Y_{\overline{W}}^{\leq -2}(u, y)] = \frac{d}{dy} Y_{\overline{W}}^{\leq -2}(u, y) - Y_0(u) y^{-2}. \tag{4.5}$$

Note that the integration $\int_0^x : W[[x]] + x^{-2}W[[x^{-1}]] \to xW[[x]] + x^{-1}W[[x^{-1}]]$ defined above is invertible and its inverse is in fact the restriction of $\frac{d}{dx}$ to the space $xW[[x]] + x^{-1}W[[x^{-1}]]$. Integrating both sides of (4.5) with respect to y and then substituting $-x$ for y, we obtain

$$\begin{aligned}\left[L(-1), \int_0^{-x} Y_{\overline{W}}^{\leq -2}(u, y)\right] &= Y_{\overline{W}}^{\leq -2}(u, -x) - Y_0(u) x^{-1} \\ &= -\frac{d}{dx} \int_0^{-x} Y_{\overline{W}}^{\leq -2}(u, y) - Y_0(u) x^{-1}.\end{aligned}$$

∎

Let

$$\widetilde{\Delta}_W^{(u)}(x) = \exp\left(\left(\int_0^{-x} Y_{\overline{W}}^{\leq -2}(u, y)\right) + Y_0(u) \log x\right).$$

Then we see that $\Delta_W^{(u)}(x) \in (\mathrm{End})[[x^{-1}]][[\log x]]$.

Proposition 4.4. *For $u, v \in V$,*

$$\widetilde{\Delta}_W^{(u)}(x) Y(v, x_2) = Y(v, x_2)(\widetilde{\Delta}_W^{(u)}(x + x_2) v, x_2) \widetilde{\Delta}_W^{(u)}(x), \tag{4.6}$$

$$[L(-1), \widetilde{\Delta}_W^{(u)}(x)] = -\frac{d}{dx} \widetilde{\Delta}_W^{(u)}(x). \tag{4.7}$$

Proof. By (4.2) and (4.4), we obtain (4.6) and (4.7), respectively. ∎

5 Generalized twisted V-modules associated to weight 1 elements of V

In this section, we construct strongly $\mathbb{C}$-graded generalized twisted V-modules associated to $u \in V_{(1)}$ under the assumption that $V_{(0)} = \mathbb{C}\mathbf{1}$ and $V_{(n)} = 0$ for $n < 0$ and that $L(1)u = 0$.

For such an element u, we have an operator $Y_0(u) = \mathrm{Res}_x Y(u, x)$ on V. Since the conformal weight of $Y_0(u)$ is 0, it preserves the finite-dimensional homogeneous subspaces $V_{(n)}$ for $n \in \mathbb{C}$. In particular, for any $n \in \mathbb{C}$, $V_{(n)} = \coprod_{\alpha\in\mathbb{C}} V_{(n)}^{[\alpha]}$ where $V_{(n)}^{[\alpha]}$ is the generalized eigenspcaes for $Y_0(u)$ with eigenvalue α if α is an eigenvalue of $Y_0(u)$ and is 0 for other $\alpha \in \mathbb{C}$. Thus V has an additional $\mathbb{C}$-grading and, equipped with this $\mathbb{C}$-grading, is a strongly $\mathbb{C}$-graded vertex operator algebra.

Given a strongly $\mathbb{C}$-graded generalized V-module, we would like to construct a strongly $\mathbb{C}$-graded generalized twisted V-module by modifying the vertex operator map for the module using the operators $\widetilde{\Delta}_V^{(u)}(x)$ constructed in the preceding section. But in general $\widetilde{\Delta}_V^{(u)}(x)v \in V[[x^{-1}]][[\log x]]$ for $v \in V$. On the other hand, in our definition of strongly $\mathbb{C}^\times$-graded generalized twisted module, the image of the vertex operator map must be in $V\{x\}[\log x]$. So we cannot use $\widetilde{\Delta}_V^{(u)}(x)$ directly. Instead, we first need to construct a series $\Delta_V^{(u)}(x)$ by modifying $\widetilde{\Delta}_V^{(u)}(x)$ and discuss under what assumptions, $\Delta_V^{(u)}(x)v$ is in $V\{x\}[\log x]$ for $v \in V$.

Let $(Y_0(u))_{ss}$ be the semisimple part of $Y_0(u)$, that is, for any generalized eigenvector $v \in V$ for $Y_0(u)$ with eigenvalue λ, $(Y_0(u))_{ss}v = \lambda v$. Then $Y_0(u) = (Y_0(u))_{ss}+(Y_0(u)-(Y_0(u))_{ss})$ and $(Y_0(u)-(Y_0(u))_{ss})^{N_u} = 0$. The commutator formula (4.1) for vertex operators gives

$$[Y_0(u), Y(v, x_2)] = Y(Y_0(u)v, x_2) \tag{5.1}$$

for any $v \in V$. The commutator formula (5.1) gives

$$\begin{aligned} [(Y_0(u))_{ss}, Y(v, x_2)] &= Y((Y_0(u))_{ss}v, x_2), & (5.2)\\ [(Y_0(u) - (Y_0(u))_{ss}), Y(v, x_2)] &= Y((Y_0(u) - (Y_0(u))_{ss})v, x_2. & (5.3) \end{aligned}$$

For any generalized eigenvector $v \in V$ for $Y_0(u)$ with eigenvalue λ, we have $e^{tY_0(u)}v = e^{t(Y_0(u))_{ss}}e^{tY_0(u)-t(Y_0(u))_{ss}}v = e^{t\lambda}e^{tY_0(u)-t(Y_0(u))_{ss}}v$. Since V is spanned by such generalized eigenvector $v \in V$ for $Y_0(u)$, we have a well-defined operator $e^{tY_0(u)}$ on V for every $t \in \mathbb{C}$.

Proposition 5.1. *For $t \in \mathbb{C}$, the operator $e^{tY_0(u)}$ is an automorphism of V.*

Proof. Since $Y_0(u)$ preserve the grading of V, $e^{tY_0(u)}$ also preserve the grading of V. From (5.2) and (5.3), we obtain

$$e^{tY_0(u)}Y(v, x) = Y(e^{tY_0(u)}v, x)e^{tY_0(u)}.$$

The first half of the creation property says that $Y(u,x)\mathbf{1} \in V[[x]]$. In particular, $Y_0(u)\mathbf{1} = 0$. So $e^{tY_0(u)}\mathbf{1} = \mathbf{1}$. Since $u \in V_{(1)}$ and $V_{(n)} = 0$ for $n < 0$, $L(n)u = 0$ for $n \geq 2$. We also have $L(1)u = 0$. So u is a lowest weight vector of weight 1 and $Y(u,x)$ is a primary field. Thus we have

$$[L(-2), Y(u,x)] = x^{-1}\frac{d}{dx}Y(u,x) - z^{-2}Y(u,x).$$

Taking the coefficients of x^{-1} of both sides of this formula, we obtain

$$[L(-2), Y_0(u)] = 0.$$

Thus $Y_0(u)\omega = L(-2)Y_0(u)\mathbf{1} = 0$ which implies $e^{tY_0(u)}\omega = \omega$. ∎

By the skew-symmetry for V, we have

$$Y(u,x)u = e^{xL(-1)}Y(u,-x)u.$$

Taking Res_x of both sides, we have

$$Y_0(u)u = \sum_{m\in\mathbb{N}} \frac{(-1)^{-m-1}}{m!}(L(-1))^m Y_m(u)u.$$

Since $\mathrm{wt}\; Y_m(u)u = 1 - m - 1 + 1 = 1 - m < 0$ and $V_{(n)} = 0$ when $n < 0$, $Y_m(u)u = 0$ for $m > 1$. Since $V_{(0)} = \mathbb{C}\mathbf{1}$, $Y_1(u)u$ is proportional to $\mathbf{1}$ since $\mathrm{wt}\; Y_1(u)u = 1 - 1 - 1 + 1 = 0$. So $(L(-1))^m Y_m(u)u = 0$ for $m \geq 0$. Thus we obtain $Y_0(u)u = -Y_0(u)u$ which implies $Y_0(u)u = 0$. By the component form of the commutator formula (5.1), we obtain

$$\begin{aligned} [Y_0(u), Y_n(u)] &= Y_n((Y_0(u)u)) \\ &= 0 \end{aligned}$$

for $n \in \mathbb{Z}$. Then we can define

$$\begin{aligned} \Delta_V^{(u)}(x) &= x^{Y_0(u)} e^{\int_0^{-x} Y^{\leq -2}(u,y)} \\ &= x^{(Y_0(u))_{ss}} e^{(Y_0(u) - (Y_0(u))_{ss}) \log x} e^{\int_0^{-x} Y^{\leq -2}(u,y)}, \end{aligned}$$

where $(Y_0(u))_{ss}$ is the semisimple part of $Y_0(u)$ and, on an eigenvector of $(Y_0(u))_{ss}$ (that is, a genralized eigenvector of $Y_0(u)$), $x^{(Y_0(u))_{ss}}$ is defined to be x^a if the eigenvalue of the eigenvector is a.

Theorem 5.2. *For any $v \in V$, there exist $m_1, \dots, m_l \in \mathbb{R}$ such that*

$$\Delta_V^{(u)}(x)v \in x^{m_1}V[x^{-1}][\log x] + \cdots + x^{m_l}V[x^{-1}][\log x]$$

and the series $\Delta_V^{(u)}(x)$ satisfies

$$\Delta_V^{(u)}(x)Y(v, x_2) = Y(v, x_2)(\Delta_V^{(u)}(x + x_2)v, x_2)\Delta_V^{(u)}(x) \tag{5.4}$$

and

$$[L(-1), \Delta_V^{(u)}(x)] = -\frac{d}{dx}\Delta_V^{(u)}(x). \tag{5.5}$$

Proof. For any $v \in V$,

$$\begin{aligned}
&e^{\int_0^{-x} Y^{\leq -2}(u,y)v}v \\
&\quad = \sum_{k\in\mathbb{N}} \frac{1}{k!}\left(\int_0^{-x} Y^{\leq -2}(u, y)\right)^k v \\
&\quad = \sum_{k\in\mathbb{N}} \frac{1}{k!}\left(\sum_{n\in\mathbb{Z}_+} \frac{(-1)^n}{-n} Y_n(u)x^{-n}\right)^k v \\
&\quad = \sum_{k\in\mathbb{N}} \frac{1}{k!} \sum_{n_1,\dots,n_k\in\mathbb{Z}_+,\ n_1+\cdots+n_k=l} \frac{(-1)^l}{(-n_1)\cdots(-n_k)} Y_{n_1}(u)\cdots Y_{n_k}(u)vx^{-l}.
\end{aligned} \tag{5.6}$$

Since

$$\begin{aligned}
\text{wt } Y_{n_1}(u)\cdots Y_{n_k}(u) &= (1 - n_1 - 1) + \cdots + (1 - n_k - 1)) \\
&= -n_1 - \cdots - n_k \\
&= -l,
\end{aligned}$$

we see that $Y_{n_1}(u)\cdots Y_{n_k}(u)w = 0$ when l is sufficiently large. Thus the right-hand side of (5.6) is an element of $V[x^{-1}]$ and our first conclusion

$$\Delta_V^{(u)}(x)v \in x^{m_1}V[x^{-1}][\log x] + \cdots + x^{m_l}V[x^{-1}][\log x]$$

follows immediately.

From (4.2), we obtain

$$
\begin{aligned}
&e^{\int_0^{-x} Y^{\leq -2}(u,y)v} Y(v, x_2) e^{-\int_0^{-x} Y^{\leq -2}(u,y)v}\\
&\quad = Y\left(e^{\int_0^{-x-x_2} Y^{\leq -2}(u,y)v + Y_0(u)\log\left(1+\frac{x_2}{x_1}\right)} v, x_2\right)\\
&\quad = Y\left(e^{Y_0(u)\log\left(1+\frac{x_2}{x_1}\right)} e^{\int_0^{-x-x_2} Y^{\leq -2}(u,y)} v, x_2\right). \qquad (5.7)
\end{aligned}
$$

On the other hand, by (5.2) and (5.3), we have

$$
\begin{aligned}
&x^{Y_0(u)} Y(v, x_2)\\
&\quad = x^{(Y_0(u))_{ss}} e^{(Y_0(u)-(Y_0(u))_{ss})\log x} Y(v, x_2)\\
&\quad = Y(x^{(Y_0(u))_{ss}} e^{(Y_0(u)-(Y_0(u))_{ss})\log x} v, x_2) x^{(Y_0(u))_{ss}} e^{(Y_0(u)-(Y_0(u))_{ss})\log x}\\
&\quad = Y(x^{Y_0(u)} v, x_2) x^{Y_0(u)}. \qquad (5.8)
\end{aligned}
$$

Using (5.7) and (5.8), we obtain (5.4).

Finally, notice that both $[L(-1), \cdot]$ and $-\frac{d}{dx}$ are derivatives on the space $\mathbb{C}[[\int_0^{-x} Y^{\leq -2}(u,y)]]$ of power series in $\int_0^{-x} Y^{\leq -2}(u,y)$. So from (4.4), we obtain

$$
\begin{aligned}
&[L(-1), e^{\int_0^{-x} Y^{\leq -2}(u,y)}] + \frac{d}{dx} e^{\int_0^{-x} Y^{\leq -2}(u,y)}\\
&\quad = \left[L(-1), \int_0^{-x} Y^{\leq -2}(u,y)\right] e^{\int_0^{-x} Y^{\leq -2}(u,y)}\\
&\qquad + \left(\frac{d}{dx}\int_0^{-x} Y^{\leq -2}(u,y)\right) e^{\int_0^{-x} Y^{\leq -2}(u,y)}\\
&\quad = -Y_0(u) x^{-1} e^{\int_0^{-x} Y^{\leq -2}(u,y)}. \qquad (5.9)
\end{aligned}
$$

The equality (5.9) is equivalent to (5.5). ∎

Let $W = \coprod_{n,\alpha\in\mathbb{C}} W_{\langle n\rangle}^{[\alpha]}$ be a strongly $\mathbb{C}$-graded generalized V-module (see [HLZ1] and [HLZ2]) with the action $e^{2\pi\sqrt{-1}\,(Y_W)_0(u)}$ of the automorphism $e^{2\pi\sqrt{-1}\,Y_0(u)}$ of V such that the $\mathbb{C}$-grading is given by the generalized eigenspaces of $(Y_W)_0(u)$ and thus is compatible with the generalized eigenspaces of the action $e^{2\pi\sqrt{-1}\,(Y_W)_0(u)}$ of $e^{2\pi\sqrt{-1}\,Y_0(u)}$. For example, when W is a V-module, W has a natural $\mathbb{C}$-grading given by the generalized eigenspaces for $(Y_W)_0(u)$ and together with this grading, W is such a generalized V-module. We define a linear map

$$\begin{array}{rcl} Y_W^{(u)} : V \otimes W & \to & W\{x\}[\log x] \\ v \otimes w & \mapsto & Y_W^{(u)}(v, x)w \end{array}$$

by

$$Y_W^{(u)}(v, x)w = Y_W(\Delta_V^{(u)}(x)v, x).$$

First, we have:

Lemma 5.3. *The vertex operator* $Y_W^{(u)}(\omega, x)$ *is in* $W[[x, x^{-1}]]$ *and if we write*

$$Y_W^{(u)}(\omega, x) = \sum_{n \in \mathbb{Z}} L_W^{(u)}(n) x^{-n-2},$$

then

$$L_W^{(u)}(0) = L_W(0) + (Y_W)_0(u) - \frac{1}{2}\mu,$$

where $\mu \in \mathbb{C}$ *is given by*

$$(Y_W)_1(u)u = \mu \mathbf{1}$$

(note that since $(Y_W)_1(u)u \in V_{(0)} = \mathbb{C}\mathbf{1}$*, it must be proportional to* $\mathbf{1}$*). In particular, for* $n \in \mathbb{C}$ *and* α *an eigenvalue of* $(Y_W)_0(u)$*,* $W^{[\alpha]}_{\langle n - \alpha + \frac{1}{2}\mu \rangle}$ *is the generalized eigenspace for both* $L_W^{(u)}(0)$ *and* $(Y_W)_0(u)$ *with the eigenvalues* n *and* α*, respectively.*

Proof. By (4.1), we have

$$[Y(\omega, x_1), Y(u, x_2)] = \mathrm{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(\omega, x_0)u, x_2).$$

Taking the coefficients of x_1^0 in both sides and noticing that from our assumption, $L(n)u = 0$ for $n > 0$, we obtain

$$\begin{array}{rcl} [L(-2), Y(u, x_2)] & = & -x_2^{-1} Y(L(-1)u, x_2) + x_2^{-2} Y(L(0)u, x_2) \\ & = & -x_2^{-1} \frac{d}{dx_2} Y(u, x_2) + x_2^{-2} Y(u, x_2). \end{array}$$

Taking the coefficients of x_2^{-m-1} in both sides, we obtain

$$[L(-2), Y_m(u)] = mY_{m-2}(u)$$

for $m \in \mathbb{Z}$. Then we have

$$\begin{aligned} Y_m(u)\omega &= Y_m(u)L(-2)\mathbf{1} \\ &= L(-2)Y_m(u)\mathbf{1} - mY_{m-2}(u)\mathbf{1} \end{aligned}$$

for $m \in \mathbb{Z}$. Since $Y_m(u)\mathbf{1} = 0$ for $m \geq 0$ and $Y_{-1}(u)\mathbf{1} = u$, we obtain

$$Y_m(u)\omega = 0$$

for $m \in \mathbb{N}$ but $m \neq 1$ and

$$Y_1(u)\omega = -u.$$

In particular, $Y_0(u)\omega = (Y_0(u))_{ss}\omega$. We also know that $Y_m(u)u = 0$ for $m > 1$ and $Y_1(u)u = \mu\mathbf{1}$. So

$$e^{\int_0^{-x} Y^{\leq -2}(u,y)}\omega = \omega + ux^{-1} - \frac{1}{2}\mu\mathbf{1}x^{-2}$$

and

$$\begin{aligned} \Delta_V^{(u)}(x)\omega &= x^{(Y_0(u))_{ss}}(\omega + ux^{-1} - \frac{1}{2}\mu\mathbf{1}x^{-2}) \\ &= \omega + ux^{-1} - \frac{1}{2}\mu\mathbf{1}x^{-2}. \end{aligned}$$

Thus

$$Y_W^{(u)}(\omega, x) = Y_W(\omega, x) + Y(u, x)x^{-1} - \frac{1}{2}\mu x^{-2} \in W[[x, x^{-1}]]$$

and $L_W^{(u)}(0) = L_W(0) + (Y_W)_0(u) - \frac{1}{2}\mu$.

The second conclusion follows immediately. ∎

We have the following consequence:

Corollary 5.4. *The space W is also doubly graded by the generalized eigenspaces for $L_W^u(0)$ and $(Y_W)_0(u)$, that is, W has a $\mathbb{C} \times \mathbb{C}$-grading $W = \coprod_{n,\alpha\in\mathbb{C}} W_{[n]}^{[\alpha]}$ where for $n, \alpha \in \mathbb{C}$,*

$$W_{[n]}^{[\alpha]} = W^{[\alpha]}_{\langle n-\alpha+\frac{1}{2}\mu\rangle},$$

that is, $w \in W_{[n]}^{[\alpha]}$ for $n, \alpha \in \mathbb{C}$ if and only if w is a generalized eigenvector for $L_W^u(0)$ with the eigenvalue n and a generalized eigenvector for $(Y_W)_0(u)$ with the eigenvalue α.

Proof. By Lemma 5.3, every element of W is a finite sum of vectors which are generalized eigenvecotrs for both $L_W^{(u)}(0)$ and $(Y_W)_0(u)$. ∎

The following theorem is the main result of the present paper:

Theorem 5.5. *The pair* $(W, Y_W^{(u)})$*, where* W *is equipped with the* $\mathbb{C}\times\mathbb{C}$*-grading given in Corollary 5.4, is a strongly* $\mathbb{C}$*-graded generalized* $e^{2\pi\sqrt{-1}\,Y_0(u)}$*-twisted* V*-module.*

Proof. We first prove that W satisfies the grading-restriction condition. By Lemma 5.3, $L_W(0) = L_W^{(u)}(0) - (Y_W)_0(u) + \frac{1}{2}\mu$. Since the generalized V-module W satisfies the grading-restriction condition, $\dim W_{\langle n\rangle}^{[\alpha]} < \infty$ for $n, \alpha \in \mathbb{C}$ and for any fixed $n, \alpha \in \mathbb{C}$, $W_{\langle n+l\rangle}^{[\alpha]} = 0$ when l is a sufficiently negative integer. So for $n, \alpha \in \mathbb{C}$, $W_{[n]}^{[\alpha]} = W_{\langle n-\alpha+\frac{1}{2}\mu\rangle}^{[\alpha]}$ are finite dimensional and for any fixed $n, \alpha \in \mathbb{C}$, $W_{[n+l]}^{[\alpha]} = W_{\langle n+l-\alpha+\frac{1}{2}\mu\rangle}^{[\alpha]} = 0$ when l is a sufficiently negative integer.

From the first part of the creation property, $Y(u,x)\mathbf{1} \in V[[x]]$ for vertex operator algebra, we have $Y_n(u)\mathbf{1} = 0$ for $n \geq 0$. In particular, $(Y_0(u))_{ss}\mathbf{1} = (Y_0(u) - (Y_0(u))_{ss})\mathbf{1} = 0$. We obtain

$$\begin{aligned} \Delta_V^{(u)}(x)\mathbf{1} &= x^{(Y_0(u))_{ss}} e^{(Y_0(u)-(Y_0(u))_{ss})\log x} e^{\int_0^{-x} Y^{\leq -2}(u,y)}\mathbf{1} \\ &= \mathbf{1}. \end{aligned}$$

For $w \in W$,

$$\begin{aligned} Y_W^{(u)}(\mathbf{1}, x)w &= Y_W(\Delta_V^{(u)}(x)\mathbf{1}, x)w \\ &= Y_W(\mathbf{1}, x)w \\ &= w, \end{aligned}$$

proving the identity property.

Let $v \in V$ be a generalized eigenvector for $Y_0(u)$ with eigenvalue λ. Then for $v \in V$ and $z \in \mathbb{C}^\times$,

$$\begin{aligned} &\Delta_V^{(u)}(x) e^{2\pi i Y_0(u)} v \Big|_{x^n = e^{n l_p(z)},\ \log x = l_p(z)} \\ &= x^{\lambda} e^{(Y_0(u)-(Y_0(u))_{ss})\log x} e^{2\pi i\lambda} e^{2\pi i (Y_0(u)-(Y_0(u))_{ss})} v \Big|_{x^n = e^{n l_p(z)},\ \log x = l_p(z)} \\ &= x^{\lambda} e^{(Y_0(u)-(Y_0(u))_{ss})\log x} v \Big|_{x^n = e^{n l_{p+1}(z)},\ \log x = l_{p+1}(z)} \\ &= \Delta_V^{(u)}(x) v \Big|_{x^n = e^{n l_{p+1}(z)},\ \log x = l_{p+1}(z)}. \end{aligned}$$

Thus

$$\begin{aligned}
&(Y_W^{(u)})^{e^{2\pi i Y_0(u)};p}(e^{2\pi i Y_0(u)}v, z)w\\
&\quad = Y_W(\Delta_V^{(u)}(x)e^{2\pi i Y_0(u)}v, x)w\Big|_{x^n=e^{nl_p(z)},\ \log x=l_p(z)}\\
&\quad = Y_W(\Delta_V^{(u)}(x)v, x)w\Big|_{x^n=e^{nl_{p+1}(z)},\ \log x=l_{p+1}(z)}\\
&\quad = (Y_W^{(u)})^{e^{2\pi i Y_0(u)};p+1}(v, z)w
\end{aligned}$$

for $v \in V$ and $w \in W$, that is, the equivariance property holds.

By Corollary 5.4, the $L^{(u)}(0)$-grading condition and the grading-compatibility condition are satisfied.

By the $L(-1)$-derivative property for the vertex operator map Y_W and (5.5), we have

$$\begin{aligned}
\frac{d}{dx}Y_W^{(u)}(v, x) &= \frac{d}{dx}Y_W(\Delta_V^{(u)}(x)v, x)\\
&= Y_W(L(-1)\Delta_V^{(u)}(x)v, x) + Y_W\left(\left(\frac{d}{dx}\Delta_V^{(u)}(x)v\right), x\right)\\
&= Y_W(L(-1)\Delta_V^{(u)}(x)v, x) + Y_W(-[L(-1), \Delta_V^{(u)}(x)]v), x)\\
&= Y_W(\Delta_V^{(u)}(x)L(-1)v, x)\\
&= Y_W^{(u)}(L(-1)v, x),
\end{aligned}$$

for $v \in V$, proving the $L(-1)$-derivative property.

Finally we prove the duality property. By Corollary 5.4, $W_{[n]}^{[\alpha]} = W^{[\alpha]}_{\langle n-\alpha+\frac{1}{2}\mu\rangle}$. So the graded dual of W with respect to the grading $W = \coprod_{n,\alpha\in\mathbb{C}} W^{[\alpha]}_{\langle n\rangle}$ and the graded dual with respect the new $\mathbb{C}\times\mathbb{C}$-grading $W = \coprod_{n,\alpha\in\mathbb{C}} W^{[\alpha]}_{[n]}$ in Corollary 5.4 are equal as vector spaces, though their gradings are different. We shall use W' to denote the underlying vecotr space of these two graded duals. It will be clear from the context which grading we will be using.

For $v_1, v_2 \in V$, $w \in W$ and $w' \in W'$, we know that there exist $m_1, \dots, m_r$, $n_1, \dots, n_s \in \mathbb{R}$ such that

$$\Delta_V^{(u)}(x_1)e^{2\pi i Y_0(u)}v_1 \in x_1^{m_1}V[x_1^{-1}][\log x_1] + \cdots + x_1^{m_r}V[x_1^{-1}][\log x_1]$$

and

$$\Delta_V^{(u)}(x_2)e^{2\pi i Y_0(u)}v_2 \in x_2^{n_1}V[x_2^{-1}][\log x_2] + \cdots + x_2^{n_s}V[x_2^{-1}][\log x_2].$$

Thus, using the rationality, commutativity and associativity properties for the V-module W and the fact that in the region $|z_2| > |z_1 - z_2| > 0$

$$\Delta_V^{(u)}(x_2 + x_0)v_1\Big|_{x_0^n=(z_1-z_2)^n,\ x_2^n=e^{nl_p(z_2)},\ \log x_2=l_p(z_2)}$$
$$= \Delta_V^{(u)}(x_1)v_1\Big|_{x_1^n=e^{nl_p(z_1)},\ \log x_1=l_p(z_1)},$$

we see that there exists $a_{ijkl} \in \mathbb{C}$ for $i, j, k, l = 0, \ldots, N$, $\alpha_i, \beta_j \in \mathbb{C}$ for $i, j = 0, \ldots, N$ and $t \in \mathbb{N}$ such that the series

$$\langle w', (Y_W^{(u)})^{e^{2\pi i Y_0(u)};p}(v_1, z_1)(Y_W^{(u)})^{e^{2\pi i Y_0(u)};p}(v_2, z_2)w\rangle$$
$$= \langle w', Y_W(\Delta_V^{(u)}(x_1)v_1, x_1) \cdot$$
$$\cdot Y_W(\Delta_V^{(u)}(x_2)v_2, x_2)w\rangle\Big|_{x_1^n=e^{nl_p(z_1)},\ \log x_1=l_p(z_1),\ x_2^n=e^{nl_p(z_2)},\ \log x_2=l_p(z_2)},$$
$$\langle w', (Y_W^{(u)})^{e^{2\pi i Y_0(u)};p}(v_2, z_2)(Y_W^{(u)})^{e^{2\pi i Y_0(u)};p}(v_1, z_1)w\rangle$$
$$= \langle w', Y_W(\Delta_V^{(u)}(x_2)v_2, x_2) \cdot$$
$$\cdot Y_W(\Delta_V^{(u)}(x_1)v_1, x_1)w\rangle\Big|_{x_1^n=e^{nl_p(z_1)},\ \log x_1=l_p(z_1),\ x_2^n=e^{nl_p(z_2)},\ \log x_2=l_p(z_2)},$$
$$\langle w', Y_W(Y(\Delta_V^{(u)}(x_2 + x_0)v_1, x_0) \cdot$$
$$\cdot \Delta_V^{(u)}(x_2)v_2, x_2)w\rangle\Big|_{x_0^n=(z_1-z_2)^n,\ x_2^n=e^{nl_p(z_2)},\ \log x_2=l_p(z_2)}$$

are absoluely convergent in the regions $|z_1| > |z_2| > 0$, $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to the branch

$$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_{p(z_2)}} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$

of the multivalued analytic function

$$\sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}.$$

By (5.4),

$$\langle w', Y_W(Y(\Delta_V^{(u)}(x_2 + x_0)v_1, x_0)\cdot$$

$$\cdot\Delta_V^{(u)}(x_2)v_2, x_2)w\rangle\Big|_{x_0^n=(z_1-z_2)^n,\ x_2^n=e^{nl_p(z_2)},\ \log x_2=l_p(z_2)}$$
$$= \langle w', Y_W(\Delta_V^{(u)}(x_2)Y(v_1,x_0)v_2, x_2)w\rangle\Big|_{x_0^n=(z_1-z_2)^n,\ x_2^n=e^{nl_p(z_2)},\ \log x_2=l_p(z_2)}$$
$$= \langle w', (Y_W^{(u)})^{e^{2\pi i Y_0(u)};p}(Y(v_1, z_1-z_2)v_2, z_2)w\rangle.$$

Thus the suality property is proved. ∎

Remark 5.6. In the case that $\mathrm{Res}_x Y(u,x)$ acts on V semisimply and has eigenvalues belonging to $\frac{1}{k}\mathbb{Z}$, the theorem above reduces to the construction by Li in [Li].

We have the following immediate consequence:

Corollary 5.7. *The correspondence given by* $(W, Y_W) \to (W, Y_W^{(u)})$ *is a functor from the category of strongly* $\mathbb{C}$*-graded generalized* V*-modules to the category of strongly* $\mathbb{C}$*-graded generalized* $e^{2\pi\sqrt{-1}\,Y_0(u)}$*-twisted* V*-modules.* ∎

We now apply the theorem above to give some explicit examples. Let p and q be a pair of coprime positive integers, L the one-dimensional positive-definite even lattice generated by γ with the bilinear form $\langle\cdot,\cdot\rangle$ given by $\langle\gamma,\gamma\rangle = 2pq$, and $\widetilde{L}$ its dual lattice. Let V_L be the vertex algebra associated to L and $V_{\widetilde{L}}$ the generalized vertex algebra associated to $\widetilde{L}$ (see [DL]). The element
$$\omega = \frac{1}{pq}\gamma(-1)^2\mathbf{1} + \frac{p-q}{2pq}\gamma(-2)\mathbf{1} \in V_L \subset V_{\widetilde{L}}$$
is a conformal element. If instead of the usual conformal element $\frac{1}{pq}\gamma(-1)^2\mathbf{1}$, we take ω to be the conformal element for V_L and $V_{\widetilde{L}}$, we obtain a vertex operator algebra (still denoted V_L) and a generalized vertex operator algebra (still denoted $V_{\widetilde{L}}$), respectively. Note that in the grading given by the conformal element ω, the weights of $e^{\gamma/q}$ and $e^{-\gamma/p}$ are 1. Consider the V_L-modules $V_{L-\frac{\gamma}{p}}$ and $V_{L+\frac{\gamma}{q}}$ which are both graded by $\mathbb{Z}$. Then we have vertex-operator-algebraic extensions $\mathcal{V}_0 = V_L \oplus V_{L-\frac{\gamma}{p}}$, $\mathcal{V}_0^o = V_L \oplus V_{L+\frac{\gamma}{q}}$ and $\mathcal{V}(p,q) = V_L \oplus V_{L-\frac{\gamma}{p}} \oplus V_{L+\frac{\gamma}{q}}$ of V_L for which the vertex operators are given by

$$\begin{aligned} Y_{\mathcal{V}_0}(u_1+v_1,x)(u_2+v_2) &= Y_{V_{\widetilde{L}}}(u_1+v_1,x)u_2 + Y_{V_{\widetilde{L}}}(u_1,x)v_2,\\ Y_{\mathcal{V}_0^o}(u_1+w_1,x)(u_2+w_2) &= Y_{V_{\widetilde{L}}}(u_1+w_1,x)u_2 + Y_{V_{\widetilde{L}}}(u_1,x)w_2, \end{aligned}$$

$$Y_{\mathcal{V}(p,q)}(u_1+v_1+w_1,x)(u_2+v_2+w_2) = Y_{V_{\tilde{L}}}(u_1+v_1+w_1,x)u_2 + Y_{V_{\tilde{L}}}(u_1,x)(v_2+w_2)$$

for $u_1, u_2 \in V_L$ and $v_1, v_2 \in V_{L-\frac{\gamma}{p}}$ and $w_1, w_2 \in V_{L+\frac{\gamma}{q}}$. Note that $\mathcal{V}_0$ and $\mathcal{V}_0^o$ are vertex operator subalgebras of $\mathcal{V}(p,q)$ and that $e^{-\gamma/p} \in \mathcal{V}_0$ and $e^{\gamma/q} \in \mathcal{V}_0^o$. Also note that $(\mathcal{V}_0)_{(0)} = (\mathcal{V}_0^o)_{(0)} = (\mathcal{V}(p,q))_{(0)} = \mathbb{C}\mathbf{1}$ and $(\mathcal{V}_0)_{(n)} = (\mathcal{V}_0^o)_{(n)} = (\mathcal{V}(p,q))_{(n)} = 0$ for $n < 0$.

Let $Q = \mathrm{Res}_x Y_{\mathcal{V}(p,q)}(e^{\gamma/q}, x)$ and $\tilde{Q} = \mathrm{Res}_x Y_{\mathcal{V}(p,q)}(e^{-\gamma/p}, x)$. These operators are called *sceening operators*. Then $e^{2\pi\sqrt{-1}\,\tilde{Q}}$ and $e^{2\pi\sqrt{-1}\,Q}$ are automorphisms of $\mathcal{V}_0$ and $\mathcal{V}_0^o$, respectively, and are both automorphisms of $\mathcal{V}(p,q)$. In fact, the triplet vertex operator algebra $\mathcal{W}(p,q)$ (see [FGST] and [AM]) is a vertex operator subalegbra of the fixed-point subalgebra of $\mathcal{V}(p,q)$ under the group generated by $e^{2\pi\sqrt{-1}\,Q}$ and $e^{2\pi\sqrt{-1}\,\tilde{Q}}$.

Let $W = \coprod_{n,\alpha\in\mathbb{C}} W_{\langle n\rangle}^{[\alpha]}$ be a strongly $\mathbb{C}$-graded generalized $\mathcal{V}_0$-, $\mathcal{V}_0^o$- or $\mathcal{V}(p,q)$-module with an action $e^{2\pi\sqrt{-1}\,(Y_W)_0(e^{\gamma/q})}$ of $e^{2\pi\sqrt{-1}\,\tilde{Q}}$, $e^{2\pi\sqrt{-1}\,(Y_W)_0(e^{-\gamma/p})}$ of $e^{2\pi\sqrt{-1}\,Q}$ or either of them, respectively, such that the $\mathbb{C}$-grading is given by the generalized eigenspaces of $(Y_W)_0(e^{\gamma/q})$ or $(Y_W)_0(e^{-\gamma/p})$. For example, we can take W to be $\mathcal{V}_0$, $\mathcal{V}_0^o$ or $\mathcal{V}(p,q)$ themselves. In [AM] (Theorem 9.1), Adamović and Milas in [AM] proved that $Y_W^{(e^{-\gamma/p})}$ and $Y_W^{(e^{\gamma/q})}$ are intertwining operators of suitable types. Applying Theorem 5.5, we obtain immediately:

Theorem 5.8. *For a strongly $\mathbb{C}$-graded generalized $\mathcal{V}_0$-module (or $\mathcal{V}_0^o$- or $\mathcal{V}(p,q)$-module) (W, Y_W), the pair $(W, Y_W^{(e^{\gamma/q})})$ (or the pair $(W, Y_W^{(e^{-\gamma/p})})$ or the pairs $(W, Y_W^{(e^{\gamma/q})})$ and $(W, Y_W^{(e^{-\gamma/p})})$) is a strongly $\mathbb{C}$-graded generalized $e^{2\pi\sqrt{-1}\,\tilde{Q}}$-twisted $\mathcal{V}_0$-modules (or is a strongly $\mathbb{C}$-graded generalized $e^{2\pi\sqrt{-1}\,Q}$-twisted $\mathcal{V}_0^o$-module or are strongly $\mathbb{C}$-graded generalized $e^{2\pi\sqrt{-1}\,\tilde{Q}}$- and $e^{2\pi\sqrt{-1}\,Q}$-twisted $\mathcal{V}(p,q)$-modules, respectively).* ∎

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