The quantum Schur superalgebra

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    - Doty-Giaquinto, Du-Parshall,...

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$$\hat{ } : \{1, 2, \ldots, m, m+1, \ldots, m+n\} \to \mathbb{Z}_2 \tag{1.0.1}$$

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- Let $M(m+n, r)$ be the set of $(m+n) \times (m+n)$ matrices $A = (a_{i,j})$ with $a_{i,j} \in \mathbb{N}$ and $\sum a_{ij} = r$, and let $M(m+n) = \cup_{r \ge 0} M(m+n, r)$.

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$$\begin{aligned} M(m|n, r) &= \{(a_{ij}) \in M(m+n, r) \colon a_{ij} \in \mathbb{Z}_2 \text{ if } \hat{i} + \hat{j} = 1\}, \\ M(m|n) &= \bigcup_{r \ge 0} M(m|n, r). \end{aligned} \tag{1.0.2}$$

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$$\Lambda(m|n, r) = \bigcup_{r_1 + r_2 = r} \Lambda(m, r_1) \times \Lambda(n, r_2).$$

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- The Hecke algebra $\mathcal{H} = \mathcal{H}(r)$ associated to the symmetric group $\mathfrak{S}_r$ is an associative $\mathcal{Z}$-algebra generated by $T_i, 1 \leq i \leq r-1$ subject to the relations (where $\mathbf{q} = \boldsymbol{v}^2$)

$$\begin{cases} T_i^2 = (\mathbf{q}-1)T_i + \mathbf{q}, & \text{for } 1 \leq i \leq r-1. \\ T_iT_j = T_jT_i, & \text{for } 1 \leq i < j \leq r-1, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & \text{for } 1 \leq i \leq r-2. \end{cases} \tag{1.0.3}$$

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- For each $\lambda|\mu \in \Lambda(m|n, r)$, define in $\mathcal{H}_R$

$$x_\lambda = \sum_{w \in \mathfrak{S}_{\lambda^*}} T_w, \quad y_\mu = \sum_{w \in \mathfrak{S}_{*_\mu}} (-q)^{-l(w)} T_w$$

where $l(w)$ is the length of w.

Definition

Let

$$\mathcal{S}(m|n, r; R) := \mathrm{End}_{\mathcal{H}_R}(\oplus_{\lambda|\mu\in\Lambda(m|n,r)} x_\lambda y_\mu \mathcal{H}_R)$$

and define a $\mathbb{Z}_2$-grading by setting

$$\mathcal{S}(m|n, r)_i = \bigoplus_{\substack{\lambda|\mu,\xi|\eta \\ |\mu|+|\eta|\equiv i (\mathrm{mod} 2)}} \mathrm{Hom}(x_\lambda y_\mu \mathcal{H}_R, x_\xi y_\eta \mathcal{H}_R). \tag{1.0.4}$$

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- Let $T(n,r) = \oplus_{\lambda|\mu\in\Lambda(m|n,r)} x_\lambda y_\mu \mathcal{H}_R$.
- There is a $\mathcal{Z}_2$ grading on $T(n,r)$ with

$$T(n,r)_0 = \bigoplus_{\substack{\lambda|\mu \\ |\mu|\equiv 0(\mathrm{mod}2)}} x_\lambda y_\mu \mathcal{H}_R, \qquad T(n,r)_1 = \bigoplus_{\substack{\lambda|\mu \\ |\mu|\equiv 1(\mathrm{mod}2)}} x_\lambda y_\mu \mathcal{H}_R.$$

- Let $V(m|n)$ be a free R-module of rank $m+n$ with basis $e_1, e_2, \ldots, e_{m+n}$. There is a $\mathbb{Z}_2$-grading on $V(m|n) = V_0 \oplus V_1$ where V_0 is spanned by $e_1, e_2, \ldots, e_m$ and V_1 by $e_{m+1}, e_{m+2}, \cdots, e_{m+n}$.

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- Following (e.g.) Mitsuhashi, let $\check{\mathcal{R}} : V(m|n)^{\otimes 2} \to V(m|n)^{\otimes 2}$ be defined by

$$(e_c \otimes e_d)\check{\mathcal{R}} = \begin{cases} v e_c \otimes e_c, & \text{if } c = d \le m, \\ -v^{-1} e_c \otimes e_c & \text{if } m+1 \le c = d, \\ (-1)^{\hat{c}\hat{d}} e_d \otimes e_c + (v - v^{-1}) e_c \otimes e_d, & \text{if } c > d, \\ (-1)^{\hat{c}\hat{d}} e_d \otimes e_c, & \text{if } c < d. \end{cases}$$

- Let $V(m|n)$ be a free R-module of rank $m+n$ with basis $e_1, e_2, \ldots, e_{m+n}$. There is a $\mathbb{Z}_2$-grading on $V(m|n) = V_0 \oplus V_1$ where V_0 is spanned by $e_1, e_2, \ldots, e_m$ and V_1 by $e_{m+1}, e_{m+2}, \cdots, e_{m+n}$.
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- Via $\check{\mathcal{R}}$, $V(m|n)^{\otimes r}$ becomes a right $\mathcal{H}$-module where is action of $\tilde{T}_i = \boldsymbol{v}^{-1} T_i$ is given by $1^{\otimes i-1} \otimes \check{\mathcal{R}} \otimes 1^{r-i-1}$.

Proposition

There is a (graded) $\mathcal{H}_R$-module isomorphism $T(n,r) \cong V(m|n)^{\otimes r}$. In particular, $\mathcal{S}(m|n,r) \cong \mathrm{End}_{\mathcal{H}}(V(m|n)^{\otimes r})$.

Definition (Manin '89)

Let $A_v(m|n)$ be the associative superalgebra over $\mathcal{Z}$ generated by x_{ij}, $1 \le i,j \le m+n$ subject to the following relations:

- $x_{i,j}^2 = 0$, for $\hat{i} + \hat{j} = 1$,
- $x_{ij}x_{ik} = (-1)^{(\hat{i}+\hat{j})(\hat{i}+\hat{k})} v^{(-1)^{\hat{i}+1}} x_{ik}x_{ij}$, for $j < k$,
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- Equipped with comultiplication Δ and co-unit ε defined by

$$\Delta(x_{ik}) = \sum_{j=1}^{m+n} x_{ij} \otimes x_{jk}, \text{ and } \varepsilon(x_{ij}) = \delta_{ij}, \forall 1 \leq i, j, k \leq m+n,$$

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  $A_v(m|n)$ becomes a superbialgebra.
- There is an superalgebra isomorphism $A_v(m|n,r)^* \cong \mathcal{S}(m|n,r)$.

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- Let $\mathfrak{D}_\nu$ (resp. $\mathfrak{D}_\nu^+$) be the set of right $\mathfrak{S}_\nu$-coset representatives of minimal (resp., maximal) length. Thus, for $\rho \models r$, the set

$$\mathfrak{D}_{\nu,\rho} = \mathfrak{D}_\nu \cap \mathfrak{D}_\rho^{-1} \quad (\text{resp.}, \mathfrak{D}_{\nu,\rho}^+ = \mathfrak{D}_\nu^+ \cap (\mathfrak{D}_\rho^+)^{-1}))$$

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  consists of minimal (resp. maximal) double coset representatives of double cosets in $\mathfrak{S}_\nu \backslash \mathfrak{S}_r / \mathfrak{S}_\rho$.
- For $\lambda|\mu, \xi|\eta \in \Lambda(m|n,r)$, define

$$\mathfrak{D}_{\lambda|\mu,\xi|\eta}^{+,-} = \mathfrak{D}_{\lambda,\xi}^+ \cap \mathfrak{D}_{\mu,\eta} = \left\{ x \in \mathfrak{S}_r : \begin{array}{l} sx < x, tx > x, \ \forall s \in \lambda, t \in \mu \\ xs < x, xt > x, \ \forall s \in \xi, t \in \eta \end{array} \right\}$$

$$\mathfrak{D}_{\lambda|\mu,\xi|\eta}^{-,+} = \mathfrak{D}_{\lambda,\xi} \cap \mathfrak{D}_{\mu,\eta}^+ = \left\{ x \in \mathfrak{S}_r : \begin{array}{l} sx > x, tx < x, \ \forall s \in \lambda, t \in \mu \\ xs > x, xt < x, \ \forall s \in \xi, t \in \eta \end{array} \right\}.$$

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Let $\lambda|\mu, \xi|\eta \in \Lambda(m|n, r)$. *For any* $d \in \mathfrak{D}_{\lambda|\mu,\xi|\eta}$,

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Moreover, $|\mathfrak{D}^{+,-}_{\lambda|\mu,\xi|\eta} \cap \mathfrak{S}_{\lambda|\mu} d \mathfrak{S}_{\xi|\eta}| \leq 1$ *(resp.,* $|\mathfrak{D}^{-,+}_{\lambda|\mu,\xi|\eta} \cap \mathfrak{S}_{\lambda|\mu} d \mathfrak{S}_{\xi|\eta}| \leq 1$*).*

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Let $\mathfrak{J}(m|n,r)^{\pm,\mp} = \bigcup_{\lambda|\mu,\xi|\eta \in \Lambda(m|n,r)} \{(\lambda|\mu, w, \xi|\eta) \colon w \in \mathfrak{D}^{\pm,\mp}_{\lambda|\mu,\xi|\eta}\}.$

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Proposition

There are bijections

$$\jmath^{+,-} : \mathfrak{J}(m|n,r)^{+,-} \longrightarrow M(m|n,r)$$
$$\jmath^{-,+} : \mathfrak{J}(m|n,r)^{-,+} \longrightarrow M(m|n,r)$$

such that, if $A = \jmath^{+,-}(\lambda|\mu, w, \xi|\eta) = \jmath^{-,+}(\lambda|\mu, w', \xi|\eta)$, *then* $\mathrm{ro}(A) = \lambda|\mu$ *and* $\mathrm{co}(A) = \xi|\eta$.

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Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

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Assume $A \in M(m|n,10)$ for $m = 1$ and $n = 2$, $\mathrm{ro}(A) = \lambda|\mu = (3)|(3,4)$, $\mathrm{co}(A) = \xi|\eta = (4)|(4,2)$. Form

$$A_{+,-} = \begin{pmatrix} (3,2) & \emptyset & 1 \\ 4 & (5,6) & \emptyset \\ 7 & (8,9) & 10 \end{pmatrix}, \qquad A_{-,+} = \begin{pmatrix} (1,2) & \emptyset & 3 \\ 6 & (5,4) & \emptyset \\ 10 & (9,8) & 7 \end{pmatrix}$$

Then $w_A^{+,-} = (7,4,3,2,5,6,8,9,1,10)$, $w_A^{-,+} = (1,2,6,10,9,8,5,4,7,3)$.

Lemma

Let $\lambda|\mu \in \Lambda(m|n, r)$.

(1) *The right* $\mathcal{H}_R$*-module* $x_\lambda y_\mu \mathcal{H}_R$ *is free with basis* $\{x_\lambda y_\mu T_d\}_{d \in \mathfrak{D}_{\lambda|\mu}}$.

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where $\nu^* = \nu \vee (1^{r-|\nu|})$ and $^*\nu = (1^{r-|\nu|}) \vee \nu$.
- Define
$$T_D = \sum_{u|v \in \mathfrak{S}_{\xi|\eta} \cap \mathfrak{D}_{\alpha|\beta}} (-q)^{-l(v)} x_\lambda y_\mu T_d T_u T_v (= h x_\xi y_\mu).$$

Let
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If $\mathcal{H}^{+,-}_{\lambda|\mu,\xi|\eta}$ denotes the free R-submodule of $\mathcal{H}_R$ spanned by T_D for all $D \in M(m|n,r)_{\lambda|\mu,\xi|\eta}$, then

$$\begin{aligned}\mathcal{H}^{+,-}_{\lambda|\mu,\xi|\eta} &= x_\lambda y_\mu \mathcal{H}_R \cap \mathcal{H}_R x_\xi y_\eta \\ &= \{h \in \mathcal{H} \colon T_{s_1} h = h T_{t_1} = qh, T_{s_2} h = h T_{t_2} = -h, \\ &\qquad \forall s_1 \in \lambda, s_2 \in \mu, t_1 \in \xi, t_2 \in \eta\}.\end{aligned}$$

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- For $D = \jmath(\lambda|\mu, w, \xi|\eta) \in M(m|n,r)$, define $\phi_A = \phi^d_{\lambda|\mu,\xi|\eta} \in \mathcal{S}(m|n,r)$ by

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for all $\alpha|\beta \in \Lambda(m|n,r)$ and $h \in \mathcal{H}_R$.
- Observe
$$\mathrm{Hom}(x_\xi y_\eta \mathcal{H}_R, x_\lambda y_\mu \mathcal{H}_R) \cong x_\lambda y_\mu \mathcal{H}_R \cap \mathcal{H}_R x_\xi y_\eta.$$

Proposition

For any commutative ring R which is a $\mathcal{Z}$-module, $\mathcal{S}(m|n, r; R)$ is R-free of rank $|M(m|n, r)|$. Moreover, the set

$$\{\phi_A \colon A \in M(m|n, r)\}$$

forms a basis for $\mathcal{S}(m|n, r; R)$. Thus, $\mathcal{S}(m|n, r) \otimes R \cong \mathcal{S}(m|n, r; R)$. Moreover, there is an algebra anti-involution $\tau : \mathcal{S}(m|n, r; R) \to \mathcal{S}(m|n, r; R)$ satisfying $\tau(\phi_A) = \phi_{A^T}$.

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- If $A = \jmath(\lambda|\mu, d, \xi|\eta)$, we put $\hat{A} = |\mu| + |\eta| (\bmod 2)$.

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- If $A = \jmath(\lambda|\mu, d, \xi|\eta)$, we put $\hat{A} = |\mu| + |\eta| (\text{mod}\, 2)$.
- The supermultiplication is given by

$$\phi_A \phi_B = (-1)^{\hat{A}\hat{B}} \phi_A \circ \phi_B, \quad \text{for all } A, B \in M(m|n,r),$$

We use a canonical basis for $x_\lambda y_\mu \mathcal{H}_R \cap \mathcal{H}_R x_\xi y_\eta$ and the isomorphism $\mathrm{Hom}(x_\xi y_\eta \mathcal{H}_R, x_\lambda y_\mu \mathcal{H}_R) \cong x_\lambda y_\mu \mathcal{H}_R \cap \mathcal{H}_R x_\xi y_\eta$ to define the canonical basis for $\mathcal{S}(m|n, r)$.

- Let $^- : \mathcal{H} \to \mathcal{H}$ be the $\mathbb{Z}$-linear involution on $\mathcal{H}$ such that $\overline{\boldsymbol{v}} = \boldsymbol{v}^{-1}$ and $\overline{T_w} = T_{w^{-1}}^{-1}$.

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- For $D \in M(m|n,r)_{\lambda|\mu,\xi|\eta}$, let $d^* \in \mathfrak{D}^+_{\lambda^*,\xi^*} \cap \mathfrak{S}_{\lambda^*} d \mathfrak{S}_{\xi^*}$ and ${}^*d \in \mathfrak{D}^+_{{}^*\mu,{}^*\eta} \cap \mathfrak{S}_{{}^*\mu} d \mathfrak{S}_{{}^*\eta}$.

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- Define

$$\mathcal{T}_D = \boldsymbol{v}^{-l(d^*)} \boldsymbol{v}^{l(^*d)-l(d)} T_D.$$

Lemma

The restriction of the bar involution $^-$ on $\mathcal{H}$ induces a bar involution $^-$ on $\mathcal{H}^{+,-}_{\lambda|\mu,\xi|\eta}$. Moreover, for $D, C \in M(m|n,r)_{\lambda|\mu,\xi|\eta}$, there exist $r_{C,D} \in \mathcal{Z}$ such that $r_{D,D} = 1$ and

$$\overline{\mathcal{T}}_D = \sum_{\substack{C \in M(m|n,r)_{\lambda|\mu,\xi|\eta} \\ C \le D}} r_{C,D} \mathcal{T}_C.$$

Proposition

There exists a unique $\mathcal{Z}$-basis $\{\mathsf{C}_D\}_{D\in M(m|n,r)_{\lambda|\mu,\xi|\eta}}$ for $\mathcal{H}^{+,-}_{\lambda|\mu,\xi|\eta}$ such that $\overline{\mathsf{C}}_D = \mathsf{C}_D$ and $\mathsf{C}_D = \sum_{C\leq D} p_{C,D}\mathcal{T}_C$, where $p_{D,D} = 1$ and $p_{C,D} \in \boldsymbol{v}^{-1}\mathbb{Z}[\boldsymbol{v}^{-1}]$ if $C < D$. Moreover, if $D = \mathfrak{S}_{\lambda|\mu}$, then $\mathsf{C}_D = \mathcal{T}_{\mathfrak{S}_{\lambda|\mu}}$.

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Theorem

The bar involution $^- : \mathcal{Z} \to \mathcal{Z}$ can be extended to a ring homomorphism $^- : \mathcal{S}(m|n,r) \to \mathcal{S}(m|n,r)$ defined by linearly extending the action:

$$\overline{\varphi}_D = \sum_C r_{C,D}\varphi_C.$$

In particular, there is a unique basis $\{\Theta_D\}_{D\in M(m|n,r)}$ satisfying

$$\overline{\Theta}_D = \Theta_D,\ \ \Theta_D - \varphi_D \in \sum_{C<D} \boldsymbol{v}^{-1}\mathbb{Z}[\boldsymbol{v}^{-1}]\varphi_C.$$

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There exists a unique $\mathcal{Z}$-basis $\{\mathsf{C}_D\}_{D\in M(m|n,r)_{\lambda|\mu,\xi|\eta}}$ for $\mathcal{H}^{+,-}_{\lambda|\mu,\xi|\eta}$ such that $\overline{\mathsf{C}}_D = \mathsf{C}_D$ and $\mathsf{C}_D = \sum_{C\le D} p_{C,D}\mathcal{T}_C$, where $p_{D,D}=1$ and $p_{C,D}\in \boldsymbol{v}^{-1}\mathbb{Z}[\boldsymbol{v}^{-1}]$ if $C<D$. Moreover, if $D=\mathfrak{S}_{\lambda|\mu}$, then $\mathsf{C}_D = \mathcal{T}_{\mathfrak{S}_{\lambda|\mu}}$.

Theorem

The bar involution $^- : \mathcal{Z}\to\mathcal{Z}$ can be extended to a ring homomorphism $^- : \mathcal{S}(m|n,r)\to\mathcal{S}(m|n,r)$ defined by linearly extending the action:

$$\overline{\varphi}_D = \sum_C r_{C,D}\varphi_C.$$

In particular, there is a unique basis $\{\Theta_D\}_{D\in M(m|n,r)}$ satisfying

$$\overline{\Theta}_D = \Theta_D,\;\; \Theta_D-\varphi_D \in \sum_{C<D}\boldsymbol{v}^{-1}\mathbb{Z}[\boldsymbol{v}^{-1}]\varphi_C.$$

Note that Θ_D is the element satisfying $\Theta_D(\mathcal{T}_{\mathfrak{S}_{\mathrm{co}(D)}}) = \mathsf{C}_D$.

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$$\mathcal{T}'_D = y'_\mu \mathcal{T}_{D^*} y'_\eta,$$
where $D^* = \mathfrak{S}_{\lambda^*} w \mathfrak{S}_{\xi^*}$ and $\mathcal{T}_{D^*} = \boldsymbol{v}^{-l(w)} \sum_{x \in D^*} T_x$.

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$$\mathcal{T}'_D = \boldsymbol{v}^{l(w_{0,\beta})} P_{\mathfrak{S}_\beta}(\mathbf{q}^{-1}) \mathcal{T}_D, \tag{1.8.1}$$
where $P_{\mathfrak{S}_\beta}(\mathbf{q})$ is the Poincaré polynomial of $\mathfrak{S}_\beta$ with $\beta = \beta(D)$.

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- Let $\mathcal{L}^{+,-}_{\lambda|\mu,\xi|\eta}$ be the $\mathcal{Z}$-span of $\mathcal{T}'_D$, $D \in M(m|n, r)_{\lambda|\mu,\xi|\eta}$. This is a $\mathcal{Z}$-submodule of $\mathcal{H}^{+,-}_{\lambda|\mu,\xi|\eta}$.

Proposition

For any $C, D \in M(m|n,r)_{\lambda|\mu,\xi|\eta}$*, there exist* $r^*_{C,D} \in \mathcal{Z}$ *such that* $r^*_{D,D} = 1$ *and*

$$\overline{\mathcal{T}}'_D = \sum_{\substack{C \in M(m|n,r)_{\lambda|\mu,\xi|\eta} \\ C \leq D}} r^*_{C,D} y'_\lambda \mathcal{T}_{C^*} y'_\eta.$$

Moreover, if $\{\mathrm{C}'_D\}_{D \in M(m|n,r)_{\lambda|\mu,\xi|\eta}}$ *denotes the associated canonical basis for* $\mathcal{L}^{+,-}_{\lambda|\mu,\xi|\eta}$*, then* $\mathrm{C}'_D := y'_\lambda C_{d^*} y'_\eta$*, where* d^* *is the longest element in* D^**.*

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- Let $\boldsymbol{\mathcal{S}}(m|n,r) = \mathcal{S}(m|n,r) \otimes \mathbb{Q}(\boldsymbol{v})$.
- For every $D \in M(m|n,r)$ with $\mathrm{ro}(D) = \lambda|\mu, \mathrm{co}(D) = \xi|\eta$, define $\Theta'_D \in \mathcal{S}(m|n,r)$ by setting

$$\Theta'_D(x'_\xi y'_\eta) = y'_\mu C_{d^*} y'_\eta .$$

Corollary

The set $\{\Theta'_D\}_{D \in M(m|n,r)}$ *forms a* $\mathbb{Q}(\boldsymbol{v})$*-basis for* $\boldsymbol{\mathcal{S}}(m|n,r)$*.*

- For a partition $\lambda \in \Lambda^+(r)$, define $\mathrm{t}^\lambda, \mathrm{t}_\lambda$ as follows (for $\lambda = (4, 3, 1)$)

$$\mathrm{t}^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}, \qquad \mathrm{t}_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 8 \\ \hline 2 & 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}.$$

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Proposition

Suppose $x, y \in \mathfrak{S}_r$. Then

- $x \sim_L y$ *if and only if* $Q(x) = Q(y)$.
- $x \sim_R y$ *if and only if* $P(x) = P(y)$.
- $x \sim_{LR} y$ *if and only if* $P(x)$ *and* $P(y)$ *have the same shape.*

- For $\lambda \in \Lambda^+(n,r)$ and $\mu \in \Lambda(n,r)$, a λ-tableau S of *content* μ is the tableau obtained by inserting each box of the Young diagram with numbers $i, 1 \leq i \leq n$, such that the number i occurring in S is μ_i.

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Proposition

Suppose $\mu \in \Lambda(m|n,r)$ and $\lambda \in \Lambda(r)^+$. Let Γ_λ be the right cell of $\mathfrak{S}_r$, which contains $w_{0,\lambda}$. We have

$$\mathfrak{D}^+_{\lambda,\mu} \cap \Gamma_\lambda = \{(w_{0,\lambda})_\mathrm{T} w_{0,\mu} \mid \mathrm{T} \in \mathbf{T}^{ss}(\lambda,\mu)\}.$$

- Fix two non-negative integers m, n with $m + n > 0$, and define
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$$\lambda' = (\lambda_1, \ldots, \lambda_m), \qquad \lambda'' = (\lambda_{m+1}, \lambda_{m+2}, \ldots)^t,$$
- The map $\lambda \mapsto (\lambda', \lambda'')$ is an injective map from $\Lambda^+(r)_{m|n}$ to $\Lambda(m|n, r)$.

Definition

For $\lambda \in \Lambda^+(r)_{m|n}$, $\mu|\nu \in \Lambda(m|n, r)$, a λ-tableau S of content $\mu \vee \nu$ is called semi-standard λ-supertableau of content $\mu|\nu$ if

- the entries in S are weakly increasing in each row and each column of S;
- the subtableau obtained by removing all rows below row m is row semistandard;
- the subtableau obtained by removing all rows above row $m+1$ is column semistandard.

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Lemma

Let $\lambda \in \Lambda^+(r)$. The following are equivalent:

(1) $\mathbf{T}^{sss}(\lambda, \mu|\nu) \neq \emptyset$ *for some* $\mu|\nu \in \Lambda(m|n, r)$;

(2) *there exist* $\mu \in \Lambda(m, r_1)$ *with* $r_1 \leq |\lambda'|$, $\nu^{(1)} \in \Lambda(n, |\lambda'| - r_1)$ *and* $\nu^{(2)} \in \Lambda(n, |\lambda''|)$ *such that* $\mathbf{T}^{ss}(\lambda', \mu \vee \nu^{(1)}) \neq \emptyset$ *and* $\mathbf{T}^{ss}(\lambda'', \nu^{(2)}) \neq \emptyset$;

(3) *there exist* $\mu \in \Lambda(m, r_1)$ *with* $r_1 \leq |\lambda'|$, $\nu^{(1)} \in \Lambda(n, |\lambda'| - r_1)$ *and* $\nu^{(2)} \in \Lambda(n, |\lambda''|)$ *such that* $\lambda' \trianglerighteq \mu \vee \nu^{(1)}$ *and* $\lambda'' \trianglerighteq \nu^{(2)}$ *under the dominance order* $\trianglerighteq$;

(4) $\lambda \in \Lambda^+(r)_{m|n}$.

Proposition

Suppose $\mu|\nu \in \Lambda(m|n, r)$ and $\lambda \in \Lambda(r)^+$. Let Γ_λ be the right cell of $\mathfrak{S}_r$, which contains $w_{0,\lambda}$. We have

$$\mathfrak{D}^{+,-}_{\boldsymbol{\lambda},\mu|\nu} \cap \Gamma_\lambda = \{(w_{0,\lambda})_{\mathsf{T}} w_{0,\mu} \mid \mathsf{T} \in \mathbf{T}^{sss}(\lambda, \mu|\nu)\},$$

where $\boldsymbol{\lambda} = \lambda|0$.

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Corollary

(1) For $\mu|\nu \in \Lambda(m|n, r)$, $\mathfrak{D}^{+,-}_{\omega,\mu|\nu}$ *is a union of left cells. If* K_λ *denotes the two-sided cell containing* $w_{0,\lambda}$, *where* $\lambda \vdash r$, *then the number* $m_{\lambda,\mu|\nu}$ *of left cells in* $K_\lambda \cap \mathfrak{D}^{+,-}_{\omega,\mu|\nu}$ *is* $|\mathbf{T}^{sss}(\lambda, \mu|\nu)|$.

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(2) For any $\mu|\nu \in \Lambda(m|n,r)$, we have

$$x_\mu y_\nu \mathcal{H}_{\mathbb{Q}(\boldsymbol{v})} \cong S^{\tilde{\mu}^*}_{\mathbb{Q}(\boldsymbol{v})} \oplus \bigoplus_{\lambda \in \Lambda^+(r)_{m|n}, \lambda \rhd \tilde{\mu}^*} m_{\lambda,\mu|\nu} S^{\lambda}_{\mathbb{Q}(\boldsymbol{v})}.$$

Definition

For $A, B \in M(m|n, r)$ with $A = \jmath^{+,-}(\alpha|\beta, y, \gamma|\delta)$ and $B = \jmath^{+,-}(\lambda|\mu, w, \xi|\eta)$, define

$$A \leq_L B \iff y \leq_L w \text{ and } \xi|\eta = \gamma|\delta \text{ (or } \mathrm{co}(A) = \mathrm{co}(B)).$$

Define $A \leq_R B$ if $A^T \leq_R B^T$. Let $\leq_{LR}$ be the preorder generated by $\leq_L$ and $\leq_R$. The relations give rise to three equivalence relations $\sim_L, \sim_R$ and $\sim_{LR}$. Thus, $A \sim_X B$ if and only if $A \leq_X B \leq_X A$ for all $X \in \{L, R, LR\}$. The corresponding equivalence classes in $M(m|n, r)$ with respect to $\sim_L, \sim_R$ and $\sim_{LR}$ are called *left cells, right cells* and *two-sided cells*, respectively.

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- for $A, B \in M(m|n,r)$, if $\Theta'_A\Theta'_B = \sum_{C \in M(m|n,r)} f_{A,B,C}\Theta'_C$, then $f_{A,B,C} \neq 0$ implies $C \leq_L B$.

- Suppose $w \in \mathfrak{D}^{+,-}_{\lambda|\mu,\xi|\eta}$ with $w \xrightarrow{\mathrm{RS}} (P(w), Q(w)) = (\mathsf{s}, \mathsf{t})$. Let ν^t be the shape of s where ν^t is the partition dual to ν.

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- Define $x, y \in \mathfrak{S}_r$ such that $P(x^{-1}) = \mathsf{s}$, $Q(x^{-1}) = \mathsf{t}_{\nu^t}$, $P(y) = \mathsf{t}_{\nu^t}$ and $Q(y) = \mathsf{t}$.

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- Then $w_{0,\nu} \sim_L x^{-1} \sim_R w$ and $w_{0,\nu} \sim_R y \sim_L w$. Thus, $\mathcal{R}(x) = \mathcal{L}(w)(= \mathcal{R}(w^{-1}))$ and $\mathcal{R}(y) = \mathcal{R}(w)(:= \{s \in S \mid ws < w\})$

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- This implies that $x \in \mathfrak{D}^{+,-}_{\nu,\lambda|\mu} \cap \Gamma_\nu$ and $y \in \mathfrak{D}^{+,-}_{\nu,\xi|\eta} \cap \Gamma_\nu$, where Γ_ν is the right cell of $\mathfrak{S}_r$ which contains $w_{0,\nu}$.

- Suppose $w \in \mathfrak{D}^{+,-}_{\lambda|\mu,\xi|\eta}$ with $w \xrightarrow{\mathrm{RS}} (P(w), Q(w)) = (\mathsf{s}, \mathsf{t})$. Let ν^t be the shape of s where ν^t is the partition dual to ν.
- Define $x, y \in \mathfrak{S}_r$ such that $P(x^{-1}) = \mathsf{s}$, $Q(x^{-1}) = \mathsf{t}_{\nu^t}$, $P(y) = \mathsf{t}_{\nu^t}$ and $Q(y) = \mathsf{t}$.
- Then $w_{0,\nu} \sim_L x^{-1} \sim_R w$ and $w_{0,\nu} \sim_R y \sim_L w$. Thus, $\mathcal{R}(x) = \mathcal{L}(w)(= \mathcal{R}(w^{-1}))$ and $\mathcal{R}(y) = \mathcal{R}(w)(:= \{s \in S \mid ws < w\})$
- This implies that $x \in \mathfrak{D}^{+,-}_{\nu,\lambda|\mu} \cap \Gamma_\nu$ and $y \in \mathfrak{D}^{+,-}_{\nu,\xi|\eta} \cap \Gamma_\nu$, where Γ_ν is the right cell of $\mathfrak{S}_r$ which contains $w_{0,\nu}$.
- There is a pair of semi-standard ν-tableaux $(\mathsf{S}_w, \mathsf{T}_w) \in \mathbf{T}^{sss}(\nu, \lambda|\mu) \times \mathbf{T}^{sss}(\nu, \xi|\eta)$, which are determined uniquely by x and y, respectively.

Lemma

The maps $\partial^{+,-}_{\lambda|\mu,\xi|\eta}$, for any $\lambda|\mu, \xi|\eta \in \Lambda(m|n,r)$, are bijection which induce a bijective correspondence

$$M(m|n,r) \longrightarrow \bigcup_{\substack{\lambda|\mu,\xi|\eta\in\Lambda(m|n,r)\\ \nu\in\Lambda^+(r)_{m|n}}} \mathbf{T}^{sss}(\nu,\lambda|\mu)\times\mathbf{T}^{sss}(\nu,\xi|\eta),\ A \longmapsto (\mathsf{S}(A), \mathsf{T}(A)).$$

- For $\nu \in \Lambda^+(r)_{m|n}$, let

$$I(\nu) = \bigcup_{\lambda|\mu \in \Lambda(m|n,r)} \mathbf{T}^{sss}(\nu, \lambda|\mu)$$

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- By the Robinson–Schensted-Knuth supercorrespondence, if $A \xrightarrow{\text{RSKs}} (\mathsf{S}, \mathsf{T}) \in I(\nu)$, we relabel the basis element Θ'_A as

$$\Theta'_{\mathsf{S},\mathsf{T}} = \Theta'^{\nu}_{\mathsf{S},\mathsf{T}} := \Theta'_A.$$

Theorem

The basis $\{\Theta'^{\nu}_{\mathsf{S},\mathsf{T}} \mid \nu \in \Lambda^+(r)_{m|n}, \mathsf{S}, \mathsf{T} \in I(\nu)\} = \{\Theta'_A \mid A \in M(m|n,r)\}$ *is a cellular basis.*

- For each $\nu \in \Lambda^+(r)_{m|n}$, let

$$C(\nu) = \boldsymbol{\mathcal{S}}(m|n,r)^{\trianglerighteq\nu,\mathsf{T}_\nu} / \boldsymbol{\mathcal{S}}(m|n,r)^{\triangleright\nu,\mathsf{T}_\nu},$$

where $\boldsymbol{\mathcal{S}}(m|n,r)^{\trianglerighteq\nu,\mathsf{T}_\nu}$ is the $\mathbb{Q}(\boldsymbol{v})$-space spanned by $\boldsymbol{\mathcal{S}}(m|n,r)^{\triangleright\nu}$ and $\Theta'^{\nu}_{\mathsf{S},\mathsf{T}_\nu}$, $\mathsf{S} \in I(\nu)$, and T_ν is the unique element in $\mathbf{T}^{sss}(\nu,\nu'|\nu'')$. These are called cell modules.

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Theorem

The set $\{C(\nu) \mid \nu \in \Lambda^+(r)_{m|n}\}$ is a complete set of pair-wise non-isomorphic irreducible $\boldsymbol{\mathcal{S}}(m|n,r)$-supermodules.

- For each $\nu \in \Lambda^+(r)_{m|n}$, let

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Theorem

The set $\{\mathsf{C}(\nu) \mid \nu \in \Lambda^+(r)_{m|n}\}$ is a complete set of pair-wise non-isomorphic irreducible $\boldsymbol{\mathcal{S}}(m|n,r)$-supermodules.

- Applications:
    - Category equivalence
    - Realization problem?
    - ....