Irreducible characters of general linear superalgebra and super duality

Shun-Jen Cheng

Institute of Mathematics

Academia Sinica, Taiwan

1. Review of irreducible characters of $\mathfrak{gl}(m+n)$

1. Review of irreducible characters of $\mathfrak{gl}(m+n)$

- Let V be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$. We denote the Lie algebra of linear transformations by $\mathfrak{g} = \mathfrak{gl}(m+n)$.

1. Review of irreducible characters of $\mathfrak{gl}(m+n)$

- Let V be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$. We denote the Lie algebra of linear transformations by $\mathfrak{g} = \mathfrak{gl}(m+n)$.

  With respect to the above ordered basis, the standard Borel subalgebra $\mathfrak{b}$ is the subalgebra of upper triangular matrices and the standard Cartan subalgebra $\mathfrak{h}$ is the space of diagonal matrices E_{ii}, with dual basis ϵ_i, $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$.

1. Review of irreducible characters of $\mathfrak{gl}(m+n)$

- Let V be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$. We denote the Lie algebra of linear transformations by $\mathfrak{g} = \mathfrak{gl}(m+n)$.

  With respect to the above ordered basis, the standard Borel subalgebra $\mathfrak{b}$ is the subalgebra of upper triangular matrices and the standard Cartan subalgebra $\mathfrak{h}$ is the space of diagonal matrices E_{ii}, with dual basis ϵ_i, $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$.

- Recall the Bernstein-Gelfand-Gelfand category $\mathcal{O}$, which consists of finitely generated modules that are semisimple over $\mathfrak{h}$ and locally finite over $\mathfrak{b}$. (We assume all our modules have integral weights.)

1. Review of irreducible characters of $\mathfrak{gl}(m+n)$

- Let V be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$. We denote the Lie algebra of linear transformations by $\mathfrak{g} = \mathfrak{gl}(m+n)$.

  With respect to the above ordered basis, the standard Borel subalgebra $\mathfrak{b}$ is the subalgebra of upper triangular matrices and the standard Cartan subalgebra $\mathfrak{h}$ is the space of diagonal matrices E_{ii}, with dual basis ϵ_i, $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$.

- Recall the Bernstein-Gelfand-Gelfand category $\mathcal{O}$, which consists of finitely generated modules that are semisimple over $\mathfrak{h}$ and locally finite over $\mathfrak{b}$. (We assume all our modules have integral weights.)

  The simple objects in $\mathcal{O}$ are indexed by the highest weights, which are elements in $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$ denote by $L(\lambda)$ this irreducible object.

1. Review of irreducible characters of $\mathfrak{gl}(m+n)$

- Let V be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$. We denote the Lie algebra of linear transformations by $\mathfrak{g} = \mathfrak{gl}(m+n)$.

  With respect to the above ordered basis, the standard Borel subalgebra $\mathfrak{b}$ is the subalgebra of upper triangular matrices and the standard Cartan subalgebra $\mathfrak{h}$ is the space of diagonal matrices E_{ii}, with dual basis ϵ_i, $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$.

- Recall the Bernstein-Gelfand-Gelfand category $\mathcal{O}$, which consists of finitely generated modules that are semisimple over $\mathfrak{h}$ and locally finite over $\mathfrak{b}$. (We assume all our modules have integral weights.)

  The simple objects in $\mathcal{O}$ are indexed by the highest weights, which are elements in $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$ denote by $L(\lambda)$ this irreducible object.

  For an $\mathfrak{h}$-semisimple module M we set for $\mu \in \mathfrak{h}^*$

$$M_\mu = \{m \in M | hm = \mu(h)m, \forall h \in \mathfrak{h}\}.$$

1. Review of irreducible characters of $\mathfrak{gl}(m+n)$

- Let V be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$. We denote the Lie algebra of linear transformations by $\mathfrak{g} = \mathfrak{gl}(m+n)$.

  With respect to the above ordered basis, the standard Borel subalgebra $\mathfrak{b}$ is the subalgebra of upper triangular matrices and the standard Cartan subalgebra $\mathfrak{h}$ is the space of diagonal matrices E_{ii}, with dual basis ϵ_i, $i = -m, -m+1, \cdots, -1, 1, 2, \cdots, n$.

- Recall the Bernstein-Gelfand-Gelfand category $\mathcal{O}$, which consists of finitely generated modules that are semisimple over $\mathfrak{h}$ and locally finite over $\mathfrak{b}$. (We assume all our modules have integral weights.)

  The simple objects in $\mathcal{O}$ are indexed by the highest weights, which are elements in $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$ denote by $L(\lambda)$ this irreducible object.

  For an $\mathfrak{h}$-semisimple module M we set for $\mu \in \mathfrak{h}^*$

$$M_\mu = \{m \in M | hm = \mu(h)m, \forall h \in \mathfrak{h}\}.$$

  and for an indeterminate e we denote the character of M by

$$\mathrm{ch}M = \sum_\mu \dim M_\mu e^\mu.$$

- For $L(\lambda) \in \mathcal{O}$ ch$L(\lambda)$ is expressed in terms of the Kazhdan-Lusztig polynomials $Q_{\lambda\mu}(q)$ as follows:

- For $L(\lambda) \in \mathcal{O}$ ch$L(\lambda)$ is expressed in terms of the Kazhdan-Lusztig polynomials $Q_{\lambda\mu}(q)$ as follows:

  We set $V(\lambda)$ be the Verma module corresponding to $\lambda \in \mathfrak{h}^*$.

- For $L(\lambda) \in \mathcal{O}$ ch$L(\lambda)$ is expressed in terms of the Kazhdan-Lusztig polynomials $Q_{\lambda\mu}(q)$ as follows:

  We set $V(\lambda)$ be the Verma module corresponding to $\lambda \in \mathfrak{h}^*$.

  Now $\{[V(\lambda)]\}$ and $\{[L(\lambda)]\}$ are two sets of basis for the Grothendieck group of $\mathcal{O}$, and hence we can write

$$[L(\lambda)] = \sum_{\mu} a_{\mu\lambda}[V(\lambda)], \quad a_{\mu\lambda} \in \mathbb{Z}.$$

  Thus $\mathrm{ch}L(\lambda) = \sum_{\mu} a_{\mu\lambda}\mathrm{ch}V(\lambda), \quad a_{\mu\lambda} \in \mathbb{Z}.$

- For $L(\lambda) \in \mathcal{O}$ ch$L(\lambda)$ is expressed in terms of the Kazhdan-Lusztig polynomials $Q_{\lambda\mu}(q)$ as follows:

  We set $V(\lambda)$ be the Verma module corresponding to $\lambda \in \mathfrak{h}^*$.

  Now $\{[V(\lambda)]\}$ and $\{[L(\lambda)]\}$ are two sets of basis for the Grothendieck group of $\mathcal{O}$, and hence we can write

$$[L(\lambda)] = \sum_{\mu} a_{\mu\lambda}[V(\lambda)], \quad a_{\mu\lambda} \in \mathbb{Z}.$$

  Thus ch$L(\lambda) = \sum_{\mu} a_{\mu\lambda}chV(\lambda), \quad a_{\mu\lambda} \in \mathbb{Z}.$

  Then one has

$$Q_{\mu\lambda}(1) = a_{\mu\lambda}.$$

- For $L(\lambda) \in \mathcal{O}$ ch$L(\lambda)$ is expressed in terms of the Kazhdan-Lusztig polynomials $Q_{\lambda\mu}(q)$ as follows:

  We set $V(\lambda)$ be the Verma module corresponding to $\lambda \in \mathfrak{h}^*$.

  Now $\{[V(\lambda)]\}$ and $\{[L(\lambda)]\}$ are two sets of basis for the Grothendieck group of $\mathcal{O}$, and hence we can write

$$[L(\lambda)] = \sum_{\mu} a_{\mu\lambda}[V(\lambda)], \quad a_{\mu\lambda} \in \mathbb{Z}.$$

  Thus $\mathrm{ch}L(\lambda) = \sum_{\mu} a_{\mu\lambda}\mathrm{ch}V(\lambda), \quad a_{\mu\lambda} \in \mathbb{Z}.$

  Then one has

$$Q_{\mu\lambda}(1) = a_{\mu\lambda}.$$

  Since ch$V(\lambda)$ is easy, once one knows $Q_{\mu\lambda}(q)$, one has solved the character problem.

- In this talk we will restrict ourselves to a smaller parabolic subcategory $\mathcal{O}^{\mathfrak{l}}$ of the category $\mathcal{O}$ defined as follows.

- In this talk we will restrict ourselves to a smaller parabolic subcategory $\mathcal{O}^{\mathfrak{l}}$ of the category $\mathcal{O}$ defined as follows.

  Namely, consider the Levi subalgebra $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \subseteq \mathfrak{gl}(m+n)$.

- In this talk we will restrict ourselves to a smaller parabolic subcategory $\mathcal{O}^{\mathfrak{l}}$ of the category $\mathcal{O}$ defined as follows.

  Namely, consider the Levi subalgebra $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \subseteq \mathfrak{gl}(m+n)$.

  Let $\mathfrak{l}^{<0}$ be a Levi subalgebra of $\mathfrak{gl}(m)$ so that $\mathfrak{l} := \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$ is a Levi of $\mathfrak{gl}(m+n)$. Corresponding to $\mathfrak{l}$ we have

  $\mathfrak{p}$ parabolic subalgebra, $\mathfrak{u}$ nilradical, $\mathfrak{u}_-$ opposite nilradical.

- In this talk we will restrict ourselves to a smaller parabolic subcategory $\mathcal{O}^{\mathfrak{l}}$ of the category $\mathcal{O}$ defined as follows.

  Namely, consider the Levi subalgebra $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \subseteq \mathfrak{gl}(m+n)$.

  Let $\mathfrak{l}^{<0}$ be a Levi subalgebra of $\mathfrak{gl}(m)$ so that $\mathfrak{l} := \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$ is a Levi of $\mathfrak{gl}(m+n)$. Corresponding to $\mathfrak{l}$ we have

  $$\mathfrak{p} \text{ parabolic subalgebra}, \quad \mathfrak{u} \text{ nilradical}, \quad \mathfrak{u}_- \text{ opposite nilradical}.$$

  $\mathcal{O}^{\mathfrak{l}}$ is the subcategory of $\mathcal{O}$ consisting of finitely generated modules that are semisimple over $\mathfrak{l}$ and locally finite over $\mathfrak{p}$.

- In this talk we will restrict ourselves to a smaller parabolic subcategory $\mathcal{O}^{\mathfrak{l}}$ of the category $\mathcal{O}$ defined as follows.

  Namely, consider the Levi subalgebra $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \subseteq \mathfrak{gl}(m+n)$.

  Let $\mathfrak{l}^{<0}$ be a Levi subalgebra of $\mathfrak{gl}(m)$ so that $\mathfrak{l} := \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$ is a Levi of $\mathfrak{gl}(m+n)$. Corresponding to $\mathfrak{l}$ we have

  $$\mathfrak{p} \text{ parabolic subalgebra}, \quad \mathfrak{u} \text{ nilradical}, \quad \mathfrak{u}_- \text{ opposite nilradical}.$$

  $\mathcal{O}^{\mathfrak{l}}$ is the subcategory of $\mathcal{O}$ consisting of finitely generated modules that are semisimple over $\mathfrak{l}$ and locally finite over $\mathfrak{p}$.

  We have parabolic Verma modules $K(\lambda)$ and irreducible $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ subject to the restriction that λ is a finite-dimensional highest weight of $\mathfrak{l}$.

- In this talk we will restrict ourselves to a smaller parabolic subcategory $\mathcal{O}^{\mathfrak{l}}$ of the category $\mathcal{O}$ defined as follows.

  Namely, consider the Levi subalgebra $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \subseteq \mathfrak{gl}(m+n)$.

  Let $\mathfrak{l}^{<0}$ be a Levi subalgebra of $\mathfrak{gl}(m)$ so that $\mathfrak{l} := \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$ is a Levi of $\mathfrak{gl}(m+n)$. Corresponding to $\mathfrak{l}$ we have

  $$\mathfrak{p} \text{ parabolic subalgebra}, \quad \mathfrak{u} \text{ nilradical}, \quad \mathfrak{u}_- \text{ opposite nilradical}.$$

  $\mathcal{O}^{\mathfrak{l}}$ is the subcategory of $\mathcal{O}$ consisting of finitely generated modules that are semisimple over $\mathfrak{l}$ and locally finite over $\mathfrak{p}$.

  We have parabolic Verma modules $K(\lambda)$ and irreducible $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ subject to the restriction that λ is a finite-dimensional highest weight of $\mathfrak{l}$.

  Similarly one has

  $$[L(\lambda)] = \sum_{\mu} P_{\mu\lambda}(1)[K(\lambda)], \quad a_{\mu\lambda} \in \mathbb{Z},$$

- In this talk we will restrict ourselves to a smaller parabolic subcategory $\mathcal{O}^{\mathfrak{l}}$ of the category $\mathcal{O}$ defined as follows.

  Namely, consider the Levi subalgebra $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \subseteq \mathfrak{gl}(m+n)$.

  Let $\mathfrak{l}^{<0}$ be a Levi subalgebra of $\mathfrak{gl}(m)$ so that $\mathfrak{l} := \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$ is a Levi of $\mathfrak{gl}(m+n)$. Corresponding to $\mathfrak{l}$ we have

  $$\mathfrak{p} \text{ parabolic subalgebra}, \quad \mathfrak{u} \text{ nilradical}, \quad \mathfrak{u}_- \text{ opposite nilradical}.$$

  $\mathcal{O}^{\mathfrak{l}}$ is the subcategory of $\mathcal{O}$ consisting of finitely generated modules that are semisimple over $\mathfrak{l}$ and locally finite over $\mathfrak{p}$.

  We have parabolic Verma modules $K(\lambda)$ and irreducible $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ subject to the restriction that λ is a finite-dimensional highest weight of $\mathfrak{l}$.

  Similarly one has

  $$[L(\lambda)] = \sum_{\mu} P_{\mu\lambda}(1)[K(\lambda)], \quad a_{\mu\lambda} \in \mathbb{Z},$$

  where $P_{\mu\lambda}(q)$ are parabolic Kazhdan-Lusztig polynomials, which can be expressed in terms of the usual KL-polynomials.

2. Representation Theory of $\mathfrak{gl}(m|n)$

2. Representation Theory of $\mathfrak{gl}(m|n)$

- Let $\overline{V}$ be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1/2, 3/2, \cdots, n-1/2$. We declare a $\mathbb{Z}_2$-gradation by setting $\deg(v_i) \equiv 2i \pmod 2$. The linear transformations are naturally $\mathbb{Z}_2$-graded, which, by defining

$$[X, Y] := XY - (-1)^{|X||Y|} YX$$

becomes a Lie superalgebra.

2. Representation Theory of $\mathfrak{gl}(m|n)$

- Let $\overline{V}$ be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1/2, 3/2, \cdots, n-1/2$. We declare a $\mathbb{Z}_2$-gradation by setting $\deg(v_i) \equiv 2i \pmod 2$. The linear transformations are naturally $\mathbb{Z}_2$-graded, which, by defining

$$[X, Y] := XY - (-1)^{|X||Y|} YX$$

becomes a Lie superalgebra.

Here for a homogeneous element X, $|X| = \deg X$.

2. Representation Theory of $\mathfrak{gl}(m|n)$

- Let $\overline{V}$ be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1/2, 3/2, \cdots, n-1/2$. We declare a $\mathbb{Z}_2$-gradation by setting $\deg(v_i) \equiv 2i \pmod 2$. The linear transformations are naturally $\mathbb{Z}_2$-graded, which, by defining

$$[X, Y] := XY - (-1)^{|X||Y|} YX$$

becomes a Lie superalgebra.

Here for a homogeneous element X, $|X| = \deg X$.

With respect to the ordered basis above the standard Borel and standard Cartan subalgebras are again the spaces of upper triangular and diagonal matrices, which we denote by $\overline{\mathfrak{b}}$ and $\overline{\mathfrak{h}}$, respectively.

2. Representation Theory of $\mathfrak{gl}(m|n)$

- Let $\overline{V}$ be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1/2, 3/2, \cdots, n-1/2$. We declare a $\mathbb{Z}_2$-gradation by setting $\deg(v_i) \equiv 2i \pmod 2$. The linear transformations are naturally $\mathbb{Z}_2$-graded, which, by defining

$$[X, Y] := XY - (-1)^{|X||Y|} YX$$

becomes a Lie superalgebra.

Here for a homogeneous element X, $|X| = \deg X$.

With respect to the ordered basis above the standard Borel and standard Cartan subalgebras are again the spaces of upper triangular and diagonal matrices, which we denote by $\overline{\mathfrak{b}}$ and $\overline{\mathfrak{h}}$, respectively.

- Again we have the analogue of the BGG category $\mathcal{O}$ with simple objects $\overline{L}(\lambda)$, $\lambda \in \overline{\mathfrak{h}}^*$. We denote this category by $\overline{\mathcal{O}}$. (Again we consider only integral weight modules.)

2. Representation Theory of $\mathfrak{gl}(m|n)$

- Let $\overline{V}$ be a complex vector space with ordered basis v_i with $i = -m, -m+1, \cdots, -1, 1/2, 3/2, \cdots, n-1/2$. We declare a $\mathbb{Z}_2$-gradation by setting $\deg(v_i) \equiv 2i \pmod 2$. The linear transformations are naturally $\mathbb{Z}_2$-graded, which, by defining

$$[X, Y] := XY - (-1)^{|X||Y|} YX$$

  becomes a Lie superalgebra.

  Here for a homogeneous element X, $|X| = \deg X$.

  With respect to the ordered basis above the standard Borel and standard Cartan subalgebras are again the spaces of upper triangular and diagonal matrices, which we denote by $\overline{\mathfrak{b}}$ and $\overline{\mathfrak{h}}$, respectively.

- Again we have the analogue of the BGG category $\mathcal{O}$ with simple objects $\overline{L}(\lambda)$, $\lambda \in \overline{\mathfrak{h}}^*$. We denote this category by $\overline{\mathcal{O}}$. (Again we consider only integral weight modules.)

- One of the main problems is the irreducible character problem of $\mathfrak{gl}(m|n)$ in $\overline{\mathcal{O}}$.

2.1. Finite-dimensional irreducible characters

2.1. Finite-dimensional irreducible characters

- Even for finite-dimensional modules the irreducible character problem turns out to be challenging, and a solution of it is only known 20 years after Kac's 1977 classification of finite-dimensional complex Lie superalgebras.

2.1. Finite-dimensional irreducible characters

- Even for finite-dimensional modules the irreducible character problem turns out to be challenging, and a solution of it is only known 20 years after Kac's 1977 classification of finite-dimensional complex Lie superalgebras.

- 1996: Serganova used Vogan's cohomological interpretation of the classical KL polynomials to define KL polynomials for Lie superalgebras $\overline{P}_{\mu\lambda}(q)$.

2.1. Finite-dimensional irreducible characters

- Even for finite-dimensional modules the irreducible character problem turns out to be challenging, and a solution of it is only known 20 years after Kac's 1977 classification of finite-dimensional complex Lie superalgebras.

- 1996: Serganova used Vogan's cohomological interpretation of the classical KL polynomials to define KL polynomials for Lie superalgebras $\overline{P}_{\mu\lambda}(q)$.

  Serganova then gave an algorithm to compute $\overline{P}_{\mu\lambda}(q)$ in the case that when the weight λ and μ are finite-dimensional highest weights. Thus she gets an algorithm to compute the characters of finite-dimensional irreducible modules over $\mathfrak{gl}(m|n)$.

2.1. Finite-dimensional irreducible characters

- Even for finite-dimensional modules the irreducible character problem turns out to be challenging, and a solution of it is only known 20 years after Kac's 1977 classification of finite-dimensional complex Lie superalgebras.

- 1996: Serganova used Vogan's cohomological interpretation of the classical KL polynomials to define KL polynomials for Lie superalgebras $\overline{P}_{\mu\lambda}(q)$.

  Serganova then gave an algorithm to compute $\overline{P}_{\mu\lambda}(q)$ in the case that when the weight λ and μ are finite-dimensional highest weights. Thus she gets an algorithm to compute the characters of finite-dimensional irreducible modules over $\mathfrak{gl}(m|n)$.

  We remark again that the finite-dimensional irreducible character problem for $\mathfrak{gl}(m+n)$ is comparably trivial. (Weyl character formula)

2.1. Finite-dimensional irreducible characters

- Even for finite-dimensional modules the irreducible character problem turns out to be challenging, and a solution of it is only known 20 years after Kac's 1977 classification of finite-dimensional complex Lie superalgebras.

- 1996: Serganova used Vogan's cohomological interpretation of the classical KL polynomials to define KL polynomials for Lie superalgebras $\overline{P}_{\mu\lambda}(q)$.

  Serganova then gave an algorithm to compute $\overline{P}_{\mu\lambda}(q)$ in the case that when the weight λ and μ are finite-dimensional highest weights. Thus she gets an algorithm to compute the characters of finite-dimensional irreducible modules over $\mathfrak{gl}(m|n)$.

  We remark again that the finite-dimensional irreducible character problem for $\mathfrak{gl}(m+n)$ is comparably trivial. (Weyl character formula)

- 2003: Brundan gave a new and more conceptual solution of the finite-dimensional irreducible character problem by relating these $\overline{P}_{\mu\lambda}(q)$ in the finite-dimensional case to Lusztig's canonical and dual canonical basis on a certain Fock space.

- 2004: Using Brundan's results in a joint work with R. Zhang and W. Wang we showed in [CWZ] that Brundan Kazhdan-Lusztig polynomials coincide with the classical type A parabolic Kazhdan-Lusztig polynomials for Lie algebras.

- 2004: Using Brundan's results in a joint work with R. Zhang and W. Wang we showed in [CWZ] that Brundan Kazhdan-Lusztig polynomials coincide with the classical type A parabolic Kazhdan-Lusztig polynomials for Lie algebras.

  Parabolic here means that they correspond to the maximal parabolic subalgebra with Levi $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

- 2004: Using Brundan's results in a joint work with R. Zhang and W. Wang we showed in [CWZ] that Brundan Kazhdan-Lusztig polynomials coincide with the classical type A parabolic Kazhdan-Lusztig polynomials for Lie algebras.

  Parabolic here means that they correspond to the maximal parabolic subalgebra with Levi $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

  To be more precise one needs to take appropriate limit $n \to \infty$.

- 2004: Using Brundan's results in a joint work with R. Zhang and W. Wang we showed in [CWZ] that Brundan Kazhdan-Lusztig polynomials coincide with the classical type A parabolic Kazhdan-Lusztig polynomials for Lie algebras.

  Parabolic here means that they correspond to the maximal parabolic subalgebra with Levi $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

  To be more precise one needs to take appropriate limit $n \to \infty$.

  Nevertheless the moral of the story is that the classical parabolic Kazhdan-Lusztig polynomials also solves the finite-dimensional irreducible character problem for $\mathfrak{gl}(m|n)$.

- 2004: Using Brundan's results in a joint work with R. Zhang and W. Wang we showed in [CWZ] that Brundan Kazhdan-Lusztig polynomials coincide with the classical type A parabolic Kazhdan-Lusztig polynomials for Lie algebras.

  Parabolic here means that they correspond to the maximal parabolic subalgebra with Levi $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

  To be more precise one needs to take appropriate limit $n \to \infty$.

  Nevertheless the moral of the story is that the classical parabolic Kazhdan-Lusztig polynomials also solves the finite-dimensional irreducible character problem for $\mathfrak{gl}(m|n)$.

  This is to say that

$$\overline{P}_{\mu\lambda}(q) = P_{\mu^\natural \lambda^\natural}(q),$$

  for some bijection $\natural$ from the sets of finite-dimensional integral highest weights of $\mathfrak{gl}(m|n)$ to the set of weights in the maximal parabolic subcategory of $\mathfrak{gl}(m+n)$-modules (in a suitable limit $n \to \infty$.)

Let $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ and let $\overline{\mathcal{O}}^{\mathfrak{l}}$ denote the category of finite-dimensional $\mathfrak{gl}(m|n)$-modules.

Let $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ and let $\overline{\mathcal{O}}^{\mathfrak{l}}$ denote the category of finite-dimensional $\mathfrak{gl}(m|n)$-modules.

The following was conjectured in [CWZ]:

Let $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ and let $\overline{\mathcal{O}}^{\mathfrak{l}}$ denote the category of finite-dimensional $\mathfrak{gl}(m|n)$-modules.

The following was conjectured in [CWZ]:

Conjecture 1.(Super Duality Conjecture) [CWZ] Let $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. Then in the limit $n \longrightarrow \infty$ the categories $\mathcal{O}^{\mathfrak{l}}$ and $\overline{\mathcal{O}}^{\mathfrak{l}}$ are equivalent.

2.2. Irreducible Characters in $\overline{\mathcal{O}}^{\iota}$

2.2. Irreducible Characters in $\overline{\mathcal{O}}^{\mathfrak{l}}$

- In the same 2003 paper Brundan also gave a conjecture for the irreducible character for $\mathfrak{gl}(m|n)$-modules in the category $\overline{\mathcal{O}}$.

2.2. Irreducible Characters in $\overline{\mathcal{O}}^{\mathfrak{l}}$

- In the same 2003 paper Brundan also gave a conjecture for the irreducible character for $\mathfrak{gl}(m|n)$-modules in the category $\overline{\mathcal{O}}$.
- 2008: Motivated by [CWZ] in a joint work with W. Wang in [CW] we compare the category $\mathcal{O}^{\mathfrak{l}}$ of $\mathfrak{gl}(m+n)$-modules with corresponding subcategories $\overline{\mathcal{O}}^{\mathfrak{l}}$ in $\overline{\mathcal{O}}$, where $\mathfrak{l} = \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$.

2.2. Irreducible Characters in $\overline{\mathcal{O}}^{\mathfrak{l}}$

- In the same 2003 paper Brundan also gave a conjecture for the irreducible character for $\mathfrak{gl}(m|n)$-modules in the category $\overline{\mathcal{O}}$.

- 2008: Motivated by [CWZ] in a joint work with W. Wang in [CW] we compare the category $\mathcal{O}^{\mathfrak{l}}$ of $\mathfrak{gl}(m+n)$-modules with corresponding subcategories $\overline{\mathcal{O}}^{\mathfrak{l}}$ in $\overline{\mathcal{O}}$, where $\mathfrak{l} = \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$.

  We note that $\mathfrak{l} = \bar{\mathfrak{l}}$ is also a Levi subalgebra of the Lie superalgebra $\mathfrak{gl}(m|n)$ and hence gives rise to a corresponding parabolic subcategory of $\overline{\mathcal{O}}^{\mathfrak{l}}$ with corresponding parabolic subalgebra $\overline{\mathfrak{p}}$, nilradical $\overline{\mathfrak{u}}$, and opposite nilradical $\overline{\mathfrak{u}}_-$.

2.2. Irreducible Characters in $\overline{\mathcal{O}}^{\mathfrak{l}}$

- In the same 2003 paper Brundan also gave a conjecture for the irreducible character for $\mathfrak{gl}(m|n)$-modules in the category $\overline{\mathcal{O}}$.

- 2008: Motivated by [CWZ] in a joint work with W. Wang in [CW] we compare the category $\mathcal{O}^{\mathfrak{l}}$ of $\mathfrak{gl}(m+n)$-modules with corresponding subcategories $\overline{\mathcal{O}}^{\mathfrak{l}}$ in $\overline{\mathcal{O}}$, where $\mathfrak{l} = \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$.

  We note that $\mathfrak{l} = \bar{\mathfrak{l}}$ is also a Levi subalgebra of the Lie superalgebra $\mathfrak{gl}(m|n)$ and hence gives rise to a corresponding parabolic subcategory of $\overline{\mathcal{O}}^{\mathfrak{l}}$ with corresponding parabolic subalgebra $\overline{\mathfrak{p}}$, nilradical $\overline{\mathfrak{u}}$, and opposite nilradical $\overline{\mathfrak{u}}_-$.

- For example when $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{gl}(n)$ let us denote this category of $\mathfrak{gl}(m|n)$-modules by $\overline{\mathcal{O}}^{\mathfrak{h} \oplus \mathfrak{gl}(n)}$.

2.2. Irreducible Characters in $\overline{\mathcal{O}}^{\mathfrak{l}}$

- In the same 2003 paper Brundan also gave a conjecture for the irreducible character for $\mathfrak{gl}(m|n)$-modules in the category $\overline{\mathcal{O}}$.

- 2008: Motivated by [CWZ] in a joint work with W. Wang in [CW] we compare the category $\mathcal{O}^{\mathfrak{l}}$ of $\mathfrak{gl}(m+n)$-modules with corresponding subcategories $\overline{\mathcal{O}}^{\mathfrak{l}}$ in $\overline{\mathcal{O}}$, where $\mathfrak{l} = \mathfrak{l}^{<0} \oplus \mathfrak{gl}(n)$.

  We note that $\mathfrak{l} = \bar{\mathfrak{l}}$ is also a Levi subalgebra of the Lie superalgebra $\mathfrak{gl}(m|n)$ and hence gives rise to a corresponding parabolic subcategory of $\overline{\mathcal{O}}^{\mathfrak{l}}$ with corresponding parabolic subalgebra $\overline{\mathfrak{p}}$, nilradical $\overline{\mathfrak{u}}$, and opposite nilradical $\overline{\mathfrak{u}}_-$.

- For example when $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{gl}(n)$ let us denote this category of $\mathfrak{gl}(m|n)$-modules by $\overline{\mathcal{O}}^{\mathfrak{h} \oplus \mathfrak{gl}(n)}$.

  Then $\overline{\mathcal{O}}^{\mathfrak{h} \oplus \mathfrak{gl}(n)}$ consists of $\mathfrak{gl}(m|n)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m|n)$-modules with highest weights of the form

$$\lambda := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda_j \epsilon_{j-\frac{1}{2}},$$

  where $\lambda_i, \lambda_j \in \mathbb{Z}$ and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

The corresponding subcategory of $\mathcal{O}$ is $\mathcal{O}^{\mathfrak{h}\oplus\mathfrak{gl}(n)}$, which consists of $\mathfrak{gl}(m+n)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m+n)$-modules with highest weights of the form

$$\lambda^\natural := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda'_j \epsilon_j,$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$ and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

The corresponding subcategory of $\mathcal{O}$ is $\mathcal{O}^{\mathfrak{h}\oplus\mathfrak{gl}(n)}$, which consists of $\mathfrak{gl}(m+n)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m+n)$-modules with highest weights of the form

$$\lambda^\natural := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda'_j \epsilon_j,$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$ and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

- As another example when $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ the category $\overline{\mathcal{O}}^{\mathfrak{l}}$ is the category of "finite-dimensional" $\mathfrak{gl}(m|n)$-modules.

The corresponding subcategory of $\mathcal{O}$ is $\mathcal{O}^{\mathfrak{h}\oplus\mathfrak{gl}(n)}$, which consists of $\mathfrak{gl}(m+n)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m+n)$-modules with highest weights of the form

$$\lambda^\natural := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda'_j \epsilon_j,$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$ and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

- As another example when $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ the category $\overline{\mathcal{O}}^{\mathfrak{l}}$ is the category of "finite-dimensional" $\mathfrak{gl}(m|n)$-modules.

  That is, $\overline{\mathcal{O}}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$ consists of modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m|n)$-modules with highest weights of the form

$$\lambda := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda_j \epsilon_{j-\frac{1}{2}},$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$, $\lambda_{-m} \geq \cdots \geq \lambda_{-1}$ and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

The corresponding subcategory of $\mathcal{O}$ is $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$, which consists of $\mathfrak{gl}(m+\infty)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m+n)$-modules with highest weights of the form

$$\lambda^\natural := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda'_j \epsilon_j,$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$, $\lambda_{-m} \geq \cdots \geq \lambda_{-1}$ and and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

The corresponding subcategory of $\mathcal{O}$ is $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$, which consists of $\mathfrak{gl}(m+\infty)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m+n)$-modules with highest weights of the form

$$\lambda^\natural := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda'_j \epsilon_j,$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$, $\lambda_{-m} \geq \cdots \geq \lambda_{-1}$ and and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

- The following was stated as in [CW]:

The corresponding subcategory of $\mathcal{O}$ is $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$, which consists of $\mathfrak{gl}(m+\infty)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m+n)$-modules with highest weights of the form

$$\lambda^\natural := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda'_j \epsilon_j,$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$, $\lambda_{-m} \geq \cdots \geq \lambda_{-1}$ and and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

- The following was stated as in [CW]:

  Conjecture 2. [CW] For all $\mathfrak{l}$ the Kazhdan-Lusztig polynomials of $\overline{\mathcal{O}}^{\mathfrak{l}}$ and of $\mathcal{O}^{\mathfrak{l}}$ coincide in the sense that $\overline{P}_{\lambda\mu}(q) = P_{\lambda^\natural\mu^\natural}(q)$ in the limit $n \to \infty$.

The corresponding subcategory of $\mathcal{O}$ is $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$, which consists of $\mathfrak{gl}(m+\infty)$-modules that have (finite) composition series with composition factors being highest weight irreducible $\mathfrak{gl}(m+n)$-modules with highest weights of the form

$$\lambda^\natural := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \lambda'_j \epsilon_j,$$

where $\lambda_i, \lambda_j \in \mathbb{Z}$, $\lambda_{-m} \geq \cdots \geq \lambda_{-1}$ and and $(\lambda_1, \lambda_2, \cdots)$ is a partition.

- The following was stated as in [CW]:

  Conjecture 2. [CW] For all $\mathfrak{l}$ the Kazhdan-Lusztig polynomials of $\overline{\mathcal{O}}^{\mathfrak{l}}$ and of $\mathcal{O}^{\mathfrak{l}}$ coincide in the sense that $\overline{P}_{\lambda\mu}(q) = P_{\lambda^\natural\mu^\natural}(q)$ in the limit $n \to \infty$.

  Note: For the case $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ Conjecture 2 is true and is the main result in [CWZ]. Also Conjecture 2 can be shown to be equivalent to a parabolic version of Brundan's conjecture on $\overline{\mathcal{O}}$.

Note: Since the polynomials $P_{\lambda^\natural \mu^\natural}$ are in principle known, the validity of Conjecture 2 would imply irreducible characters for a wide class of $\mathfrak{gl}(m|n)$-modules that includes many infinite-dimensional representations as well.

Note: Since the polynomials $P_{\lambda^\natural \mu^\natural}$ are in principle known, the validity of Conjecture 2 would imply irreducible characters for a wide class of $\mathfrak{gl}(m|n)$-modules that includes many infinite-dimensional representations as well.

- Theorem 1. (Lam - C. 2009) Conjecture 2 is true.

Note: Since the polynomials $P_{\lambda^\natural \mu^\natural}$ are in principle known, the validity of Conjecture 2 would imply irreducible characters for a wide class of $\mathfrak{gl}(m|n)$-modules that includes many infinite-dimensional representations as well.

- Theorem 1. (Lam - C. 2009) Conjecture 2 is true.

- The following was also stated as in [CW]:

Note: Since the polynomials $P_{\lambda^\natural \mu^\natural}$ are in principle known, the validity of Conjecture 2 would imply irreducible characters for a wide class of $\mathfrak{gl}(m|n)$-modules that includes many infinite-dimensional representations as well.

- Theorem 1. (Lam - C. 2009) Conjecture 2 is true.

- The following was also stated as in [CW]:

  Conjecture 3.(General Super Duality Conjecture) [CW] For all $\mathfrak{l}$ the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$ and $\mathcal{O}^{\mathfrak{l}}$ are equivalent in the limit $n \to \infty$.

Note: Since the polynomials $P_{\lambda^\natural \mu^\natural}$ are in principle known, the validity of Conjecture 2 would imply irreducible characters for a wide class of $\mathfrak{gl}(m|n)$-modules that includes many infinite-dimensional representations as well.

- Theorem 1. (Lam - C. 2009) Conjecture 2 is true.

- The following was also stated as in [CW]:

  Conjecture 3.(General Super Duality Conjecture) [CW] For all $\mathfrak{l}$ the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$ and $\mathcal{O}^{\mathfrak{l}}$ are equivalent in the limit $n \to \infty$.

  Note: In the special case $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ Conjecture 3 gives Conjecture 1.

Note: Since the polynomials $P_{\lambda^\natural \mu^\natural}$ are in principle known, the validity of Conjecture 2 would imply irreducible characters for a wide class of $\mathfrak{gl}(m|n)$-modules that includes many infinite-dimensional representations as well.

- Theorem 1. (Lam - C. 2009) Conjecture 2 is true.

- The following was also stated as in [CW]:

  Conjecture 3.(General Super Duality Conjecture) [CW] For all $\mathfrak{l}$ the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$ and $\mathcal{O}^{\mathfrak{l}}$ are equivalent in the limit $n \to \infty$.

  Note: In the special case $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ Conjecture 3 gives Conjecture 1.

- Theorem 2. (Lam - C. 2009) Conjecture 2 is true.

Note: Since the polynomials $P_{\lambda^\natural \mu^\natural}$ are in principle known, the validity of Conjecture 2 would imply irreducible characters for a wide class of $\mathfrak{gl}(m|n)$-modules that includes many infinite-dimensional representations as well.

- Theorem 1. (Lam - C. 2009) Conjecture 2 is true.

- The following was also stated as in [CW]:

  Conjecture 3.(General Super Duality Conjecture) [CW] For all $\mathfrak{l}$ the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$ and $\mathcal{O}^{\mathfrak{l}}$ are equivalent in the limit $n \to \infty$.

  Note: In the special case $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ Conjecture 3 gives Conjecture 1.

- Theorem 2. (Lam - C. 2009) Conjecture 2 is true.

- In the remainder of the talk we will give the main ideas of a proof of Theorems 1 and 2. Also in the sequel we will always assume that $n = \infty$.

3. Super Duality

3. Super Duality

- One of the main point is to introduce a new bigger Lie superalgebra that includes both $\mathfrak{gl}(m|\infty)$ and $\mathfrak{gl}(m+\infty)$, and a respective category of modules that connects their representation theories.

3. Super Duality

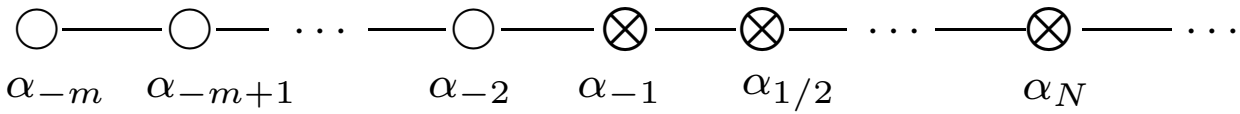
- One of the main point is to introduce a new bigger Lie superalgebra that includes both $\mathfrak{gl}(m|\infty)$ and $\mathfrak{gl}(m+\infty)$, and a respective category of modules that connects their representation theories.

- This Lie superalgebra is the general linear superalgebra acting on the superspace with ordered basis indexed by $-m, \cdots, -1, 1/2, 1, 3/2, 2, \cdots$ with Dynkin diagram

3. Super Duality

- One of the main point is to introduce a new bigger Lie superalgebra that includes both $\mathfrak{gl}(m|\infty)$ and $\mathfrak{gl}(m+\infty)$, and a respective category of modules that connects their representation theories.

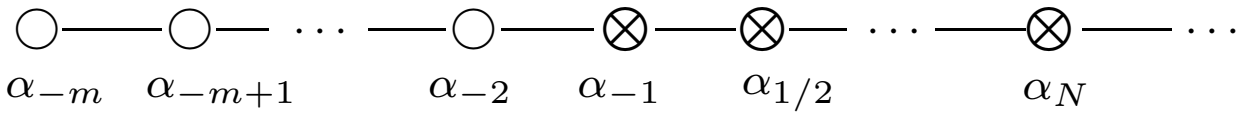
- This Lie superalgebra is the general linear superalgebra acting on the superspace with ordered basis indexed by $-m, \cdots, -1, 1/2, 1, 3/2, 2, \cdots$ with Dynkin diagram



3. Super Duality

- One of the main point is to introduce a new bigger Lie superalgebra that includes both $\mathfrak{gl}(m|\infty)$ and $\mathfrak{gl}(m+\infty)$, and a respective category of modules that connects their representation theories.

- This Lie superalgebra is the general linear superalgebra acting on the superspace with ordered basis indexed by $-m, \cdots, -1, 1/2, 1, 3/2, 2, \cdots$ with Dynkin diagram

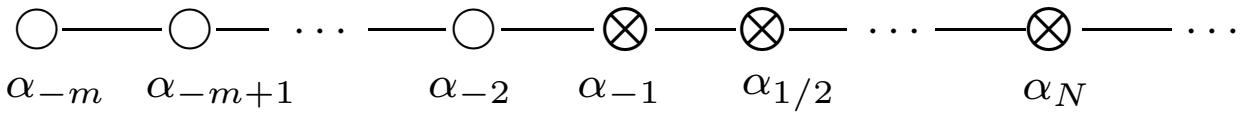


Note: $\alpha_{-m} = \epsilon_{-m} - \epsilon_{-m+1}$, $\epsilon_{-m+1} = \epsilon_{-m+1} - \epsilon_{-m+2}$, $\alpha_{-2} = \epsilon_{-2} - \epsilon_{-1}$, $\epsilon_{-1} = \epsilon_{-1} - \epsilon_{1/2}$, $\epsilon_{1/2} = \epsilon_{1/2} - \epsilon_1$ etc.

3. Super Duality

- One of the main point is to introduce a new bigger Lie superalgebra that includes both $\mathfrak{gl}(m|\infty)$ and $\mathfrak{gl}(m+\infty)$, and a respective category of modules that connects their representation theories.

- This Lie superalgebra is the general linear superalgebra acting on the superspace with ordered basis indexed by $-m, \cdots, -1, 1/2, 1, 3/2, 2, \cdots$ with Dynkin diagram



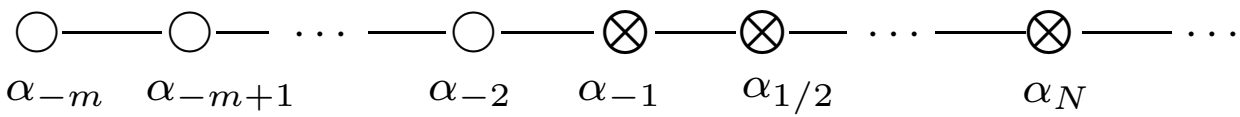
Note: $\alpha_{-m} = \epsilon_{-m} - \epsilon_{-m+1}$, $\epsilon_{-m+1} = \epsilon_{-m+1} - \epsilon_{-m+2}$, $\alpha_{-2} = \epsilon_{-2} - \epsilon_{-1}$, $\epsilon_{-1} = \epsilon_{-1} - \epsilon_{1/2}$, $\epsilon_{1/2} = \epsilon_{1/2} - \epsilon_1$ etc.

We denote this Lie superalgebra by $\widetilde{\mathfrak{g}}$.

3. Super Duality

- One of the main point is to introduce a new bigger Lie superalgebra that includes both $\mathfrak{gl}(m|\infty)$ and $\mathfrak{gl}(m+\infty)$, and a respective category of modules that connects their representation theories.

- This Lie superalgebra is the general linear superalgebra acting on the superspace with ordered basis indexed by $-m, \cdots, -1, 1/2, 1, 3/2, 2, \cdots$ with Dynkin diagram



Note: $\alpha_{-m} = \epsilon_{-m} - \epsilon_{-m+1}$, $\epsilon_{-m+1} = \epsilon_{-m+1} - \epsilon_{-m+2}$, $\alpha_{-2} = \epsilon_{-2} - \epsilon_{-1}$, $\epsilon_{-1} = \epsilon_{-1} - \epsilon_{1/2}$, $\epsilon_{1/2} = \epsilon_{1/2} - \epsilon_1$ etc.

We denote this Lie superalgebra by $\widetilde{\mathfrak{g}}$.

The corresponding Levi subalgebra of $\widetilde{\mathfrak{g}}$ is the direct sum of $\mathfrak{l}^{<0}$ and the general linear Lie superalgebra generated by α_r, $r > 0$, denoted by $\widetilde{\mathfrak{g}}^{>0}$. We denote this Levi subalgebra by $\widetilde{\mathfrak{l}}$ with parabolic subalgebra $\widetilde{\mathfrak{p}}$, nilradical $\widetilde{\mathfrak{u}}$, and opposite radical $\widetilde{\mathfrak{u}}_-$.

- The corresponding parabolic subcategory $\widetilde{\mathfrak{g}}$-modules is denoted by $\widetilde{\mathcal{O}}^{\mathfrak{l}}$. On modules in $\widetilde{\mathcal{O}}^{\mathfrak{l}}$ the Lie superalgebra $\widetilde{\mathfrak{g}}^{>0}$ acts as polynomial representations.

- The corresponding parabolic subcategory $\widetilde{\mathfrak{g}}$-modules is denoted by $\widetilde{\mathcal{O}}^{\mathfrak{l}}$. On modules in $\widetilde{\mathcal{O}}^{\mathfrak{l}}$ the Lie superalgebra $\widetilde{\mathfrak{g}}^{>0}$ acts as polynomial representations.

  For example, when $\widetilde{\mathfrak{l}} = \mathfrak{h} \oplus \widetilde{\mathfrak{g}}^{>0}$, then the corresponding category of modules is $\widetilde{\mathcal{O}}^{\mathfrak{h} \oplus \widetilde{\mathfrak{g}}^{>0}}$ whose objects are modules with finite composition series with composition factors isomorphic to irreducible highest weight modules of highest weight

$$\lambda^{\theta} := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \theta(\lambda'_+)_j \epsilon_j,$$

  where $\lambda_i, \lambda_j \in \mathbb{Z}$ and $\lambda_+ = (\lambda_1, \lambda_2, \cdots)$ is a partition.

- The corresponding parabolic subcategory $\widetilde{\mathfrak{g}}$-modules is denoted by $\widetilde{\mathcal{O}}^{\mathfrak{l}}$. On modules in $\widetilde{\mathcal{O}}^{\mathfrak{l}}$ the Lie superalgebra $\widetilde{\mathfrak{g}}^{>0}$ acts as polynomial representations.

  For example, when $\widetilde{\mathfrak{l}} = \mathfrak{h} \oplus \widetilde{\mathfrak{g}}^{>0}$, then the corresponding category of modules is $\widetilde{\mathcal{O}}^{\mathfrak{h} \oplus \widetilde{\mathfrak{g}}^{>0}}$ whose objects are modules with finite composition series with composition factors isomorphic to irreducible highest weight modules of highest weight

$$\lambda^{\theta} := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \theta(\lambda'_+)_j \epsilon_j,$$

  where $\lambda_i, \lambda_j \in \mathbb{Z}$ and $\lambda_+ = (\lambda_1, \lambda_2, \cdots)$ is a partition.

  Notation: Given a partition $\mu = (\mu_1, \mu_2, \cdots)$ we set $\theta(\mu)$ to be the sequence of integers

$$\theta(\mu) := (\theta(\mu)_{1/2}, \theta(\mu)_1, \theta(\mu)_{3/2}, \theta(\mu)_2, \cdots),$$

  where $\theta(\mu)_{i-1/2} := \langle \mu'_i - (i-1) \rangle$ and $\theta(\mu)_i := \langle \mu_i - i \rangle$, for $i \in \mathbb{N}$.

- The corresponding parabolic subcategory $\widetilde{\mathfrak{g}}$-modules is denoted by $\widetilde{\mathcal{O}}^{\mathfrak{l}}$. On modules in $\widetilde{\mathcal{O}}^{\mathfrak{l}}$ the Lie superalgebra $\widetilde{\mathfrak{g}}^{>0}$ acts as polynomial representations.

  For example, when $\widetilde{\mathfrak{l}} = \mathfrak{h} \oplus \widetilde{\mathfrak{g}}^{>0}$, then the corresponding category of modules is $\widetilde{\mathcal{O}}^{\mathfrak{h} \oplus \widetilde{\mathfrak{g}}^{>0}}$ whose objects are modules with finite composition series with composition factors isomorphic to irreducible highest weight modules of highest weight

$$\lambda^{\theta} := \sum_{i=-m}^{-1} \lambda_i \epsilon_i + \sum_{j>0} \theta(\lambda'_+)_j \epsilon_j,$$

  where $\lambda_i, \lambda_j \in \mathbb{Z}$ and $\lambda_+ = (\lambda_1, \lambda_2, \cdots)$ is a partition.

  Notation: Given a partition $\mu = (\mu_1, \mu_2, \cdots)$ we set $\theta(\mu)$ to be the sequence of integers

$$\theta(\mu) := (\theta(\mu)_{1/2}, \theta(\mu)_1, \theta(\mu)_{3/2}, \theta(\mu)_2, \cdots),$$

  where $\theta(\mu)_{i-1/2} := \langle \mu'_i - (i-1) \rangle$ and $\theta(\mu)_i := \langle \mu_i - i \rangle$, for $i \in \mathbb{N}$.

  Here $\langle k \rangle := k$, of $k \in \mathbb{N}$, and it is defined to be 0 otherwise.

- Let $T : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \mathcal{O}^{\mathfrak{l}}$ and $\overline{T} : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \overline{\mathcal{O}}^{\mathfrak{l}}$ be the functors defined by projecting onto the weight spaces corresponding to the weights $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{j\in\mathbb{N}} \mathbb{Z}\epsilon_j$ and $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{r\in\frac{1}{2}+\mathbb{Z}} \mathbb{Z}\epsilon_r$, respectively.

- Let $T : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \mathcal{O}^{\mathfrak{l}}$ and $\overline{T} : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \overline{\mathcal{O}}^{\mathfrak{l}}$ be the functors defined by projecting onto the weight spaces corresponding to the weights $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{j\in\mathbb{N}} \mathbb{Z}\epsilon_j$ and $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{r\in\frac{1}{2}+\mathbb{Z}} \mathbb{Z}\epsilon_r$, respectively.

  Let $\overline{L}(\lambda)$, $L(\lambda^{\natural})$, and $\widetilde{L}(\lambda^{\theta})$ denote the respective highest weight irreducible modules of highest weights λ, $\lambda^{\natural}$, and λ^{θ}, in the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$, $\mathcal{O}^{\mathfrak{l}}$, and $\widetilde{\mathcal{O}}^{\mathfrak{l}}$.

- Let $T : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \mathcal{O}^{\mathfrak{l}}$ and $\overline{T} : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \overline{\mathcal{O}}^{\mathfrak{l}}$ be the functors defined by projecting onto the weight spaces corresponding to the weights $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{j\in\mathbb{N}} \mathbb{Z}\epsilon_j$ and $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{r\in\frac{1}{2}+\mathbb{Z}} \mathbb{Z}\epsilon_r$, respectively.

  Let $\overline{L}(\lambda)$, $L(\lambda^{\natural})$, and $\widetilde{L}(\lambda^{\theta})$ denote the respective highest weight irreducible modules of highest weights λ, $\lambda^{\natural}$, and λ^{θ}, in the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$, $\mathcal{O}^{\mathfrak{l}}$, and $\widetilde{\mathcal{O}}^{\mathfrak{l}}$.

  Let $\overline{K}(\lambda)$, $K(\lambda^{\natural})$, and $\widetilde{K}(\lambda^{\theta})$ denote the respective parabolic Verma modules of highest weights λ, $\lambda^{\natural}$, and λ^{θ}, in the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$, $\mathcal{O}^{\mathfrak{l}}$, and $\widetilde{\mathcal{O}}^{\mathfrak{l}}$.

- Let $T : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \mathcal{O}^{\mathfrak{l}}$ and $\overline{T} : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \overline{\mathcal{O}}^{\mathfrak{l}}$ be the functors defined by projecting onto the weight spaces corresponding to the weights $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{j\in\mathbb{N}} \mathbb{Z}\epsilon_j$ and $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{r\in\frac{1}{2}+\mathbb{Z}} \mathbb{Z}\epsilon_r$, respectively.

  Let $\overline{L}(\lambda)$, $L(\lambda^\natural)$, and $\widetilde{L}(\lambda^\theta)$ denote the respective highest weight irreducible modules of highest weights λ, $\lambda^\natural$, and λ^θ, in the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$, $\mathcal{O}^{\mathfrak{l}}$, and $\widetilde{\mathcal{O}}^{\mathfrak{l}}$.

  Let $\overline{K}(\lambda)$, $K(\lambda^\natural)$, and $\widetilde{K}(\lambda^\theta)$ denote the respective parabolic Verma modules of highest weights λ, $\lambda^\natural$, and λ^θ, in the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$, $\mathcal{O}^{\mathfrak{l}}$, and $\widetilde{\mathcal{O}}^{\mathfrak{l}}$.

  Theorem 3 (Lam-C.) We have

  (i) $T(\widetilde{K}(\lambda^\theta)) = K(\lambda^\natural)$ and $T(\widetilde{L}(\lambda^\theta)) = L(\lambda^\natural)$.

  (ii) $\overline{T}(\widetilde{K}(\lambda^\theta)) = \overline{K}(\lambda)$ and $\overline{T}(\widetilde{L}(\lambda^\theta)) = \overline{L}(\lambda)$.

- Let $T : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \mathcal{O}^{\mathfrak{l}}$ and $\overline{T} : \widetilde{\mathcal{O}}^{\mathfrak{l}} \to \overline{\mathcal{O}}^{\mathfrak{l}}$ be the functors defined by projecting onto the weight spaces corresponding to the weights $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{j\in\mathbb{N}} \mathbb{Z}\epsilon_j$ and $\sum_{i=-m}^{-1} \mathbb{Z}\epsilon_i + \sum_{r\in\frac{1}{2}+\mathbb{Z}} \mathbb{Z}\epsilon_r$, respectively.

  Let $\overline{L}(\lambda)$, $L(\lambda^\natural)$, and $\widetilde{L}(\lambda^\theta)$ denote the respective highest weight irreducible modules of highest weights λ, $\lambda^\natural$, and λ^θ, in the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$, $\mathcal{O}^{\mathfrak{l}}$, and $\widetilde{\mathcal{O}}^{\mathfrak{l}}$.

  Let $\overline{K}(\lambda)$, $K(\lambda^\natural)$, and $\widetilde{K}(\lambda^\theta)$ denote the respective parabolic Verma modules of highest weights λ, $\lambda^\natural$, and λ^θ, in the categories $\overline{\mathcal{O}}^{\mathfrak{l}}$, $\mathcal{O}^{\mathfrak{l}}$, and $\widetilde{\mathcal{O}}^{\mathfrak{l}}$.

  Theorem 3 (Lam-C.) We have

  (i) $T(\widetilde{K}(\lambda^\theta)) = K(\lambda^\natural)$ and $T(\widetilde{L}(\lambda^\theta)) = L(\lambda^\natural)$.

  (ii) $\overline{T}(\widetilde{K}(\lambda^\theta)) = \overline{K}(\lambda)$ and $\overline{T}(\widetilde{L}(\lambda^\theta)) = \overline{L}(\lambda)$.

  From the above theorem it follows that the character of an irreducible $\mathfrak{gl}(m|n)$-module $\overline{L}(\lambda)$ in $\overline{\mathcal{O}}^{\mathfrak{l}}$ is determined by the character of the $\mathfrak{gl}(m+n)$-module $L(\lambda^\natural)$.

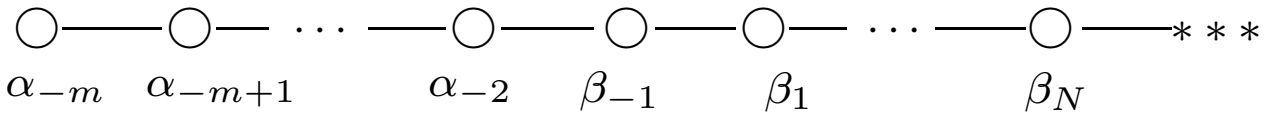
- The proof of the above theorem is remarkably elementary and relies on the combinatorics of odd reflections.

- The proof of the above theorem is remarkably elementary and relies on the combinatorics of odd reflections.

  One can apply a sequence of odd reflections so that the Dynkin diagram will be changed to the following:

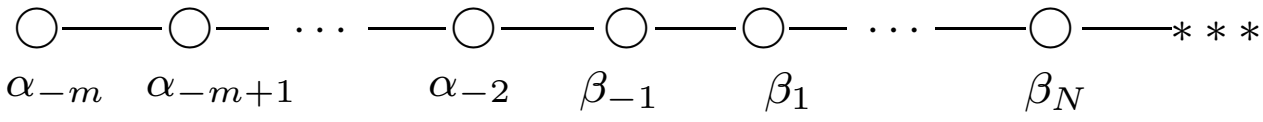
- The proof of the above theorem is remarkably elementary and relies on the combinatorics of odd reflections.

  One can apply a sequence of odd reflections so that the Dynkin diagram will be changed to the following:



- The proof of the above theorem is remarkably elementary and relies on the combinatorics of odd reflections.

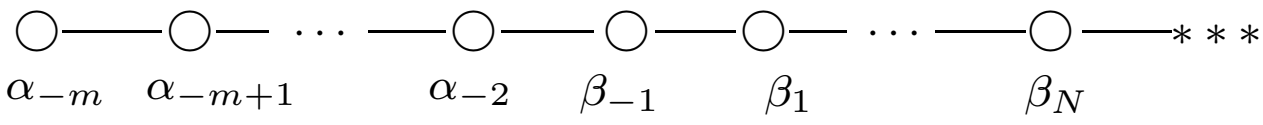
  One can apply a sequence of odd reflections so that the Dynkin diagram will be changed to the following:



  Note: $\beta_{-1} = \epsilon_{-1} - \epsilon_1$, $\beta_1 = \epsilon_1 - \epsilon_2$ etc. and $***$ is something more complicated.

- The proof of the above theorem is remarkably elementary and relies on the combinatorics of odd reflections.

  One can apply a sequence of odd reflections so that the Dynkin diagram will be changed to the following:

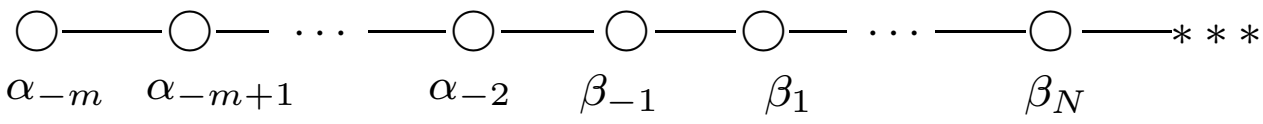


  Note: $\beta_{-1} = \epsilon_{-1} - \epsilon_1$, $\beta_1 = \epsilon_1 - \epsilon_2$ etc. and $***$ is something more complicated.

  Ignoring $***$ this is just the Dynkin diagram of $\mathfrak{gl}(m+N)$.

- The proof of the above theorem is remarkably elementary and relies on the combinatorics of odd reflections.

  One can apply a sequence of odd reflections so that the Dynkin diagram will be changed to the following:



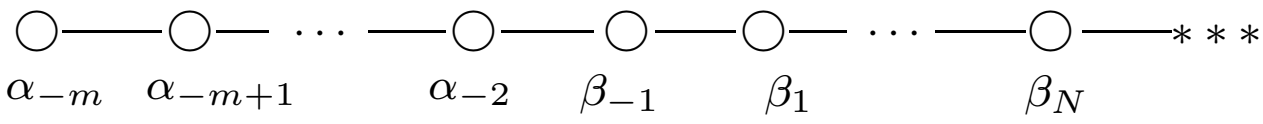
  Note: $\beta_{-1} = \epsilon_{-1} - \epsilon_1$, $\beta_1 = \epsilon_1 - \epsilon_2$ etc. and $***$ is something more complicated.

  Ignoring $***$ this is just the Dynkin diagram of $\mathfrak{gl}(m+N)$.

  The highest weight λ^θ with respect to this Borel becomes then λ.

- The proof of the above theorem is remarkably elementary and relies on the combinatorics of odd reflections.

  One can apply a sequence of odd reflections so that the Dynkin diagram will be changed to the following:



  Note: $\beta_{-1} = \epsilon_{-1} - \epsilon_1$, $\beta_1 = \epsilon_1 - \epsilon_2$ etc. and $***$ is something more complicated.

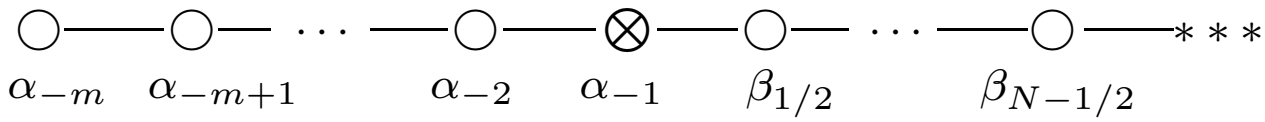
  Ignoring $***$ this is just the Dynkin diagram of $\mathfrak{gl}(m+N)$.

  The highest weight λ^θ with respect to this Borel becomes then λ.

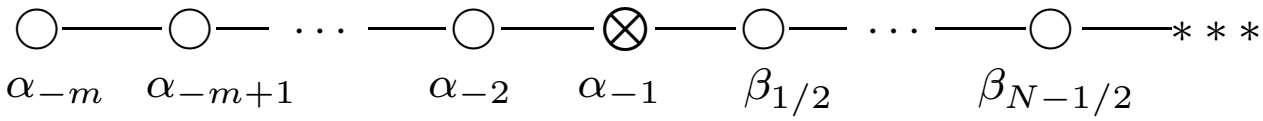
  Using this Borel it then can be shown that T sends irreducibles to irreducibles and parabolic Verma to parabolic Verma.

To prove Theorem 3 for $\overline{T}$ we use a different sequence of odd reflections to change the Borel subalgebra to the following:

To prove Theorem 3 for $\overline{T}$ we use a different sequence of odd reflections to change the Borel subalgebra to the following:

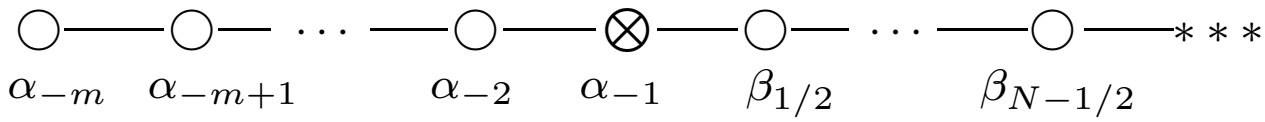


To prove Theorem 3 for $\overline{T}$ we use a different sequence of odd reflections to change the Borel subalgebra to the following:



Note: $\beta_{1/2} = \epsilon_{1/2} - \epsilon_{3/2}$, $\beta_{3/2} = \epsilon_{3/2} - \epsilon_{5/2}$ etc. and $***$ is something more complicated.

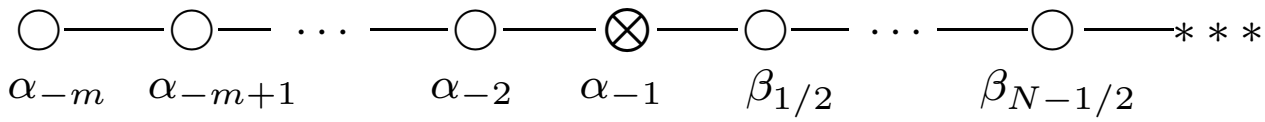
To prove Theorem 3 for $\overline{T}$ we use a different sequence of odd reflections to change the Borel subalgebra to the following:



Note: $\beta_{1/2} = \epsilon_{1/2} - \epsilon_{3/2}$, $\beta_{3/2} = \epsilon_{3/2} - \epsilon_{5/2}$ etc. and $***$ is something more complicated.

Ignoring $***$ this is just the Dynkin diagram of $\mathfrak{gl}(m|N)$.

To prove Theorem 3 for $\overline{T}$ we use a different sequence of odd reflections to change the Borel subalgebra to the following:

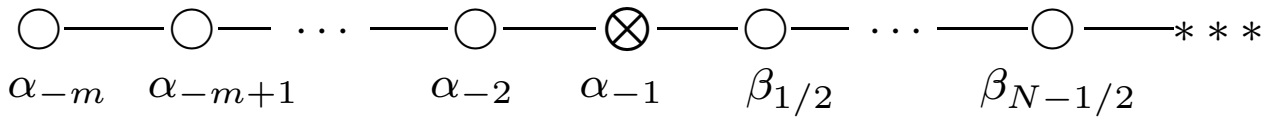


Note: $\beta_{1/2} = \epsilon_{1/2} - \epsilon_{3/2}$, $\beta_{3/2} = \epsilon_{3/2} - \epsilon_{5/2}$ etc. and $***$ is something more complicated.

Ignoring $***$ this is just the Dynkin diagram of $\mathfrak{gl}(m|N)$.

- We want to point out that the arguments are very elementary and do NOT use Serganova's or Brundan's results at all.

To prove Theorem 3 for $\overline{T}$ we use a different sequence of odd reflections to change the Borel subalgebra to the following:



Note: $\beta_{1/2} = \epsilon_{1/2} - \epsilon_{3/2}$, $\beta_{3/2} = \epsilon_{3/2} - \epsilon_{5/2}$ etc. and $***$ is something more complicated.

Ignoring $***$ this is just the Dynkin diagram of $\mathfrak{gl}(m|N)$.

- We want to point out that the arguments are very elementary and do NOT use Serganova's or Brundan's results at all.

The upshot is that if we specialize $\mathfrak{l}$ to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ we obtain in a very elementary fashion a new and independent solution of the finite-dimensional irreducible character problem for $\mathfrak{gl}(m|n)$ as well.

- Let $\widetilde{M} \in \widetilde{\mathcal{O}}$ and let $M = T(\widetilde{M})$ and $\overline{M} = \overline{T}(\widetilde{M})$.

- Let $\widetilde{M} \in \widetilde{\mathcal{O}}$ and let $M = T(\widetilde{M})$ and $\overline{M} = \overline{T}(\widetilde{M})$.

  Recall the opposite nilradical of $\widetilde{\mathfrak{g}}$, $\mathfrak{g}$, and $\overline{\mathfrak{g}}$, denoted by $\widetilde{\mathfrak{u}}_-$, $\mathfrak{u}_-$, and $\overline{\mathfrak{u}}_-$, respectively.

- Let $\widetilde{M} \in \widetilde{\mathcal{O}}$ and let $M = T(\widetilde{M})$ and $\overline{M} = \overline{T}(\widetilde{M})$.

  Recall the opposite nilradical of $\widetilde{\mathfrak{g}}$, $\mathfrak{g}$, and $\overline{\mathfrak{g}}$, denoted by $\widetilde{\mathfrak{u}}_-$, $\mathfrak{u}_-$, and $\overline{\mathfrak{u}}_-$, respectively.

  Let $\Lambda^i(\widetilde{\mathfrak{u}}_-) \otimes \widetilde{M}$, $\Lambda^i(\mathfrak{u}_-) \otimes M$, and $\Lambda^i(\overline{\mathfrak{u}}_-) \otimes \overline{M}$ denote the respective ith chains, and let $\widetilde{d}$, d and $\overline{d}$ be the respective boundary operators.

- Let $\widetilde{M} \in \widetilde{\mathcal{O}}$ and let $M = T(\widetilde{M})$ and $\overline{M} = \overline{T}(\widetilde{M})$.

  Recall the opposite nilradical of $\widetilde{\mathfrak{g}}$, $\mathfrak{g}$, and $\overline{\mathfrak{g}}$, denoted by $\widetilde{\mathfrak{u}}_-$, $\mathfrak{u}_-$, and $\overline{\mathfrak{u}}_-$, respectively.

  Let $\Lambda^i(\widetilde{\mathfrak{u}}_-) \otimes \widetilde{M}$, $\Lambda^i(\mathfrak{u}_-) \otimes M$, and $\Lambda^i(\overline{\mathfrak{u}}_-) \otimes \overline{M}$ denote the respective ith chains, and let $\widetilde{d}$, d and $\overline{d}$ be the respective boundary operators.

  Proposition 4. We have $T\widetilde{d} = dT$ and $\overline{T}\widetilde{d} = \overline{dT}$, and hence T and $\overline{T}$ induce $\mathfrak{l}$-module homomorphisms on the respective homology groups.

- Let $\widetilde{M} \in \widetilde{\mathcal{O}}$ and let $M = T(\widetilde{M})$ and $\overline{M} = \overline{T}(\widetilde{M})$.

  Recall the opposite nilradical of $\widetilde{\mathfrak{g}}$, $\mathfrak{g}$, and $\overline{\mathfrak{g}}$, denoted by $\widetilde{\mathfrak{u}}_-$, $\mathfrak{u}_-$, and $\overline{\mathfrak{u}}_-$, respectively.

  Let $\Lambda^i(\widetilde{\mathfrak{u}}_-) \otimes \widetilde{M}$, $\Lambda^i(\mathfrak{u}_-) \otimes M$, and $\Lambda^i(\overline{\mathfrak{u}}_-) \otimes \overline{M}$ denote the respective ith chains, and let $\widetilde{d}$, d and $\overline{d}$ be the respective boundary operators.

  Proposition 4. We have $T\widetilde{d} = dT$ and $\overline{T}\widetilde{d} = \overline{dT}$, and hence T and $\overline{T}$ induce $\mathfrak{l}$-module homomorphisms on the respective homology groups.

  Theorem 5. (Lam-C.) For μ an appropriate weight let $L(\mathfrak{l}, \mu)$ denote the irreducible highest weight $\mathfrak{l}$-module. For $\widetilde{M} \in \widetilde{\mathcal{O}}^{\mathfrak{l}}$ we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathfrak{l}}}\big(L(\widetilde{\mathfrak{l}}, \mu^\theta), H_n(\widetilde{\mathfrak{u}}_-; \widetilde{M})\big)\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{\overline{\mathfrak{l}}}\big(L(\overline{\mathfrak{l}}, \mu^\natural), H_n(\overline{\mathfrak{u}}_-; \overline{M})\big),$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathfrak{l}}}\big(L(\widetilde{\mathfrak{l}}, \mu^\theta), H_n(\widetilde{\mathfrak{u}}_-; \widetilde{M})\big)\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{l}}\big(L(\mathfrak{l}, \mu), H_n(\mathfrak{u}_-; M)\big).$$

- Let $\widetilde{M} \in \widetilde{\mathcal{O}}$ and let $M = T(\widetilde{M})$ and $\overline{M} = \overline{T}(\widetilde{M})$.

  Recall the opposite nilradical of $\widetilde{\mathfrak{g}}$, $\mathfrak{g}$, and $\overline{\mathfrak{g}}$, denoted by $\widetilde{\mathfrak{u}}_-$, $\mathfrak{u}_-$, and $\overline{\mathfrak{u}}_-$, respectively.

  Let $\Lambda^i(\widetilde{\mathfrak{u}}_-) \otimes \widetilde{M}$, $\Lambda^i(\mathfrak{u}_-) \otimes M$, and $\Lambda^i(\overline{\mathfrak{u}}_-) \otimes \overline{M}$ denote the respective ith chains, and let $\widetilde{d}$, d and $\overline{d}$ be the respective boundary operators.

  Proposition 4. We have $T\widetilde{d} = dT$ and $\overline{T}\widetilde{d} = \overline{dT}$, and hence T and $\overline{T}$ induce $\mathfrak{l}$-module homomorphisms on the respective homology groups.

  Theorem 5. (Lam-C.) For μ an appropriate weight let $L(\mathfrak{l}, \mu)$ denote the irreducible highest weight $\mathfrak{l}$-module. For $\widetilde{M} \in \widetilde{\mathcal{O}}^{\mathfrak{l}}$ we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathfrak{l}}}\big(L(\widetilde{\mathfrak{l}}, \mu^\theta), H_n(\widetilde{\mathfrak{u}}_-; \widetilde{M})\big)\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{\overline{\mathfrak{l}}}\big(L(\overline{\mathfrak{l}}, \mu^\natural), H_n(\overline{\mathfrak{u}}_-; \overline{M})\big),$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathfrak{l}}}\big(L(\widetilde{\mathfrak{l}}, \mu^\theta), H_n(\widetilde{\mathfrak{u}}_-; \widetilde{M})\big)\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{l}}\big(L(\mathfrak{l}, \mu), H_n(\mathfrak{u}_-; M)\big).$$

  Using analogues of Vogan's cohomological interpretation of Kazhdan-Lusztig polynomials and choosing $\widetilde{M} = \widetilde{L}(\lambda^\theta)$ Theorem 5 implies that the respective KL-polynomials $P_{\mu\lambda}(q)$ and $\overline{P}_{\mu^\natural \lambda^\natural}(q)$ coincide, thus proving Conjecture 2.

- Theorem 6 (Lam-C.) The functors T and $\overline{T}$ induce equivalence of categories. Thus $\mathcal{O}^{\mathfrak{l}} \cong \overline{\mathcal{O}}^{\mathfrak{l}}$.

- Theorem 6 (Lam-C.) The functors T and $\overline{T}$ induce equivalence of categories. Thus $\mathcal{O}^{\mathfrak{l}} \cong \overline{\mathcal{O}}^{\mathfrak{l}}$.

  Remark: Recall that the special case of $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ is Conjecture 1, i.e. the equivalence of the categories $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$ and $\overline{\mathcal{O}}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$. A proof in this special case was first announced by Brundan and Stroppel in 2008. A proof of this was posted on the web by Brundan and Stroppel very recently.

- Theorem 6 (Lam-C.) The functors T and $\overline{T}$ induce equivalence of categories. Thus $\mathcal{O}^{\mathfrak{l}} \cong \overline{\mathcal{O}}^{\mathfrak{l}}$.

  Remark: Recall that the special case of $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ is Conjecture 1, i.e. the equivalence of the categories $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$ and $\overline{\mathcal{O}}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$. A proof in this special case was first announced by Brundan and Stroppel in 2008. A proof of this was posted on the web by Brundan and Stroppel very recently.

  Our method is quite different from Brundan and Stroppel's, as it gives explicitly the functor and is also valid for any $\mathfrak{l}$. So in contrast to their approach it handles many infinite-dimensional representations of $\mathfrak{gl}(m|n)$ as well.

- Theorem 6 (Lam-C.) The functors T and $\overline{T}$ induce equivalence of categories. Thus $\mathcal{O}^{\mathfrak{l}} \cong \overline{\mathcal{O}}^{\mathfrak{l}}$.

  Remark: Recall that the special case of $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ is Conjecture 1, i.e. the equivalence of the categories $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$ and $\overline{\mathcal{O}}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$. A proof in this special case was first announced by Brundan and Stroppel in 2008. A proof of this was posted on the web by Brundan and Stroppel very recently.

  Our method is quite different from Brundan and Stroppel's, as it gives explicitly the functor and is also valid for any $\mathfrak{l}$. So in contrast to their approach it handles many infinite-dimensional representations of $\mathfrak{gl}(m|n)$ as well.

- Our method is independent of the works of Serganova and Brundan as well, and so it gives a new and independent solution to the finite-dimensional irreducible character problem for $\mathfrak{gl}(m|n)$.

- Theorem 6 (Lam-C.) The functors T and $\overline{T}$ induce equivalence of categories. Thus $\mathcal{O}^{\mathfrak{l}} \cong \overline{\mathcal{O}}^{\mathfrak{l}}$.

  Remark: Recall that the special case of $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ is Conjecture 1, i.e. the equivalence of the categories $\mathcal{O}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$ and $\overline{\mathcal{O}}^{\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)}$. A proof in this special case was first announced by Brundan and Stroppel in 2008. A proof of this was posted on the web by Brundan and Stroppel very recently.

  Our method is quite different from Brundan and Stroppel's, as it gives explicitly the functor and is also valid for any $\mathfrak{l}$. So in contrast to their approach it handles many infinite-dimensional representations of $\mathfrak{gl}(m|n)$ as well.

- Our method is independent of the works of Serganova and Brundan as well, and so it gives a new and independent solution to the finite-dimensional irreducible character problem for $\mathfrak{gl}(m|n)$.

- Furthermore it quite general and works also for other types of Lie algebras/Lie superalgebras.

- For example, it makes connections in a similar way between the representation theories of the orth-symplectic Lie superalgebras with Lie algebras of type B, C, D. This connection is more subtle and is a joint work with N. Lam and W. Wang.

- For example, it makes connections in a similar way between the representation theories of the orth-symplectic Lie superalgebras with Lie algebras of type B, C, D. This connection is more subtle and is a joint work with N. Lam and W. Wang.

- As another example, the super duality principle makes connections between the representation theories of affine Lie superalgebras with Kac-Moody Lie algebras of indefinite type. (Also joint work with N. Lam and W. Wang.)

Thank you for your attention!