

The title of  
This book\*\*\*\*\*  
ALM ?, pp. 1-?

---

© Higher Education Press  
and International Press  
Beijing-Boston

---

# Dualities for Lie Superalgebras

Shun-Jen Cheng\* and Weiqiang Wang†

## Abstract

We explain how Lie superalgebras of types  $\mathfrak{gl}$  and  $\mathfrak{osp}$  provide a natural framework generalizing the classical Schur and Howe dualities. This exposition includes a discussion of super duality, which connects the parabolic categories  $\mathcal{O}$  between classical Lie superalgebras and Lie algebras. Super duality provides a conceptual solution to the irreducible character problem for these Lie superalgebras in terms of the classical Kazhdan-Lusztig polynomials.

**2000 Mathematics Subject Classification:** 17B10

**Keywords and Phrases:** Lie superalgebras, Schur duality, Howe duality, super duality, irreducible characters

## Contents

1	Lie superalgebra ABC	5
2	Finite-dimensional modules of Lie superalgebras	11
3	Schur-Sergeev duality	18
4	Howe duality for Lie superalgebras of type $\mathfrak{gl}$	25
5	Howe duality for Lie superalgebras of type $\mathfrak{osp}$	29
6	Super duality	35

---

\*Institute of Mathematics, Academia Sinica, Taipei, Taiwan 10617

†Department of Mathematics, University of Virginia, Charlottesville, VA 22904

## Introduction

### 0.1

The study of Lie superalgebras, supergroups, and their representations was largely motivated by supersymmetry in mathematical physics, which puts bosons and fermions on the same footing. An earlier achievement is the Cartan-Killing type classification of finite-dimensional simple complex Lie superalgebras by Kac [K1] (also cf. [SNR] for an independent classification of the so-called classical Lie superalgebras). The most important basic classical Lie superalgebras consist of infinite series of types  $\mathfrak{sl}$ ,  $\mathfrak{osp}$ . The basic classical Lie superalgebras afford root systems, Dynkin diagrams, Cartan subalgebras, triangular decomposition, Verma modules, category  $\mathcal{O}$ , and so on. There has been much work on representation theory of Lie superalgebras (in particular, basic classical) in the last three decades, but conceptual approaches have been lacking until recently.

### 0.2

The aim of these lecture notes is to explain three different kinds of dualities for Lie superalgebras:

Schur duality,   Howe duality,   and Super duality.

In the superalgebra setting, the first (i.e. Schur) duality was formulated by Sergeev, and the latter two dualities have been largely developed by the authors and their collaborators. These lecture notes are also intended to serve as a road map for a forthcoming book by the authors.

The Schur-Sergeev duality is an interplay between Lie superalgebras and the symmetric groups which incorporates the trivial and sign modules in a unified framework. On the algebraic combinatorial level, there is a natural super generalization of the notion of semistandard tableaux which is a hybrid of the traditional version and its conjugate counterpart.

It has been observed that much of the study of the classical invariant theory on polynomial algebras has parallels for exterior algebras, and both admit reformulation and extension in the theory of Howe's reductive dual pairs. Lie superalgebras allow a uniform treatment of Howe duality on the polynomial and exterior algebras.

Super duality has a different flavor. It views representation theories of Lie superalgebras and Lie algebras as two sides of the same coin, and it is an unexpected yet powerful approach developed in the past few years which allow us to overcome various superalgebra difficulties. We give an exposition on the new development on super duality which is an equivalence between parabolic categories  $\mathcal{O}$  of Lie superalgebras and Lie algebras. Super duality provides a conceptual solution to the long-standing irreducible character problem for a wide class of modules over (a wide class of) Lie superalgebras in terms of Kazhdan-Lusztig polynomials. This is achieved despite the fact that there are no obvious Weyl groups controlling the linkage for super representation theory.

### 0.3

In Section 1, we give some basic constructions and structures of the general linear and the ortho-symplectic Lie superalgebras. We emphasize the super phenomena that are not observed in the ordinary Lie algebra setting, such as odd roots, non-conjugate Borel subalgebras, and so on. In Section 2, we present Kac's classification of finite-dimensional simple  $\mathfrak{g}$ -modules [K2]. The classification is very easy for type  $A$ , but nontrivial for  $\mathfrak{osp}$ . In the latter case we explain a new odd reflection approach by Shu and the second author [SW], using a more natural labeling of these modules by hook partitions. We note that odd reflection is also one of the main technical tools in super duality. In addition, we present the *typical* finite-dimensional irreducible character formula, following [K2].

### 0.4

The classical Schur duality relates the representation theory of the general linear Lie algebras and that of the symmetric groups. In Section 3, we explain Sergeev's generalization [Sv1] of Schur duality for the general linear Lie superalgebras  $\mathfrak{gl}(m|n)$  (also see Berele and Regev [BeR] for additional insight and detail). More precisely, we establish a double centralizer theorem for the actions of  $\mathfrak{gl}(m|n)$  and the symmetric group  $\mathfrak{S}_d$  in  $d$  letters on the tensor space  $(\mathbb{C}^{m|n})^{\otimes d}$ . We then provide an explicit multiplicity-free decomposition of the tensor space into a  $U(\mathfrak{gl}(m|n)) \otimes \mathbb{C}\mathfrak{S}_d$ -modules. We further present a simple formula obtained in our latest work with Lam [CLW] for extremal weights in a simple polynomial  $\mathfrak{gl}(m|n)$ -module with respect to all Borel subalgebras, which has an explicit diagrammatic interpretation from a Young diagram.

### 0.5

Howe's theory of reductive dual pairs [H1, H2] can be viewed as a representation theoretic reformulation and extension of the classical invariant theory (see Weyl [We]). For example, the first fundamental theorem on invariants for classical groups are reformulated in terms of double centralizer properties of two classical Lie groups/algebras. One advantage of Howe duality is that it allows natural generalizations to classical Lie groups/algebras (and superalgebras) other than type  $A$ .

We mainly use two examples of dual pairs to illustrate the main ideas of Howe duality and the new phenomena of superalgebra generalizations. For more detailed case study of Howe duality for Lie superalgebras, we refer to the original papers [BPT, CW1, CW2, CW3, CL1, CLZ, CZ2, CKW, LZ, Sv2]. In Section 4, we formulate the  $(\mathfrak{gl}(m|n), \mathfrak{gl}(d))$ -Howe duality and find the highest weight vectors for each isotypical component in the corresponding multiplicity-free decomposition. In Section 5 we present the  $(\mathrm{Sp}(d), \mathfrak{osp}(2m|2n))$ -Howe duality and its multiplicity-free decomposition. The application of the Howe duality to irreducible characters over Lie superalgebras follows the simpler approach in our work with Kwon [CKW] (which uses Howe duality for infinite-dimensional Lie algebras [Wa]).

## 0.6

We recall some truly super phenomena that have been the main obstacles towards a better understanding of super representation theory:

1. There exist odd roots as well as non-conjugate Borel subalgebras for a Lie superalgebra. A homomorphism between Verma modules may not be injective.
2. The linkage in category  $\mathcal{O}$  of modules for a Lie superalgebra is NOT controlled by the Weyl group of  $\mathfrak{g}_{\bar{0}}$ ; see e.g.  $\mathfrak{gl}(1|1)$ .
3. There is no uniform Weyl-type irreducible finite-dimensional character formula for Lie superalgebras.
4. The super geometry behind super representation theory is still inadequately developed.

In light of these super phenomena, it was a rather unexpected discovery [CWZ, CW4], which was partly inspired by Brundan [Br1], that there exists a (conjectural) equivalence of categories between Lie algebras and Lie superalgebras of type  $A$  (at a certain suitable limit at infinity), which was termed *Super Duality*. This conjecture in the full generality of [CW4] has been proved in [CL2], which in particular offers an elementary and conceptual solution to the character problem for all finite-dimensional simple modules and for a large class of infinite-dimensional simple highest weight modules over Lie superalgebras of type  $A$ .

Super duality has been subsequently formulated and established between various Lie superalgebras of type  $\mathfrak{osp}$  and the corresponding classical Lie algebras in our very recent work with Lam [CLW]. This in particular offers a conceptual solution of the irreducible character problem for a wide class of modules, which include all finite-dimensional irreducibles, of Lie superalgebras of type  $\mathfrak{osp}$  in terms of Kazhdan-Lusztig polynomials for classical Lie algebras [KL, BB, BK] (for more on Kazhdan-Lusztig theory see Tanisaki's lectures [Ta]). In addition, it follows easily from the approach of [CL2, CLW] that the  $\mathfrak{u}$ -homology groups (or Kazhdan-Lusztig polynomials in the sense of Vogan [Vo]) match perfectly between classical Lie superalgebras and the corresponding classical Lie algebras. This generalizes earlier partial results in this direction from Schur or Howe duality approach [CZ1, CK, CKW]. The super duality as outlined above is explained in Section 6.

Let us put the super duality work explained above in perspective. *Finite-dimensional* irreducible characters for  $\mathfrak{gl}(m|n)$  have been also obtained earlier in two totally different approaches by [Sva] and [Br1]. The mixed algebraic and geometric approach of Serganova has been extended very recently in [GS] to obtain all irreducible finite-dimensional  $\mathfrak{osp}$ -characters. Brundan and Stroppel [BrS] also provided another approach to the main results of [Br1] and independently proved a special case of the super duality conjecture in type  $A$  as formulated in [CWZ]. All these approaches have brought new and different insights into super representation theory. Our super duality approach has the advantages of explaining

the connection with classical Lie algebras and their Kazhdan-Lusztig polynomials, covering infinite-dimensional irreducible characters, and being extendable to general Kac-Moody Lie superalgebras.

A list of symbols is added at the end of the paper to facilitate the reading.

## 0.7

Let us end the Introduction with some remarks on the interrelations among the three dualities.

The  $(\mathfrak{gl}(d), \mathfrak{gl}(n))$ -Howe duality is equivalent to Schur duality. It follows from the Schur-Sergeev duality that the characters for irreducible polynomial  $\mathfrak{gl}(m|n)$ -modules are given by the so-called hook Schur functions. On the other hand, the irreducible character formulas for Lie superalgebras of types  $\mathfrak{gl}$  or  $\mathfrak{osp}$  obtained from Howe duality can be expressed in terms of infinite classical Weyl groups. The appearance of hook Schur functions and infinite Weyl groups in these formulas are conceptually explained from the viewpoint of super duality.

Super duality can be informally interpreted as a categorification of the standard involution on the ring of symmetric functions. It is well known that the ring of symmetric functions in infinitely many variables admits symmetries which are not observed in finitely many variables. Super duality is formulated precisely at the infinite rank limit. On the level of combinatorial parameterizations of highest weights, super duality manifests itself through (variation of) the conjugate of partitions.

Partly due to the time constraint of the lectures, we have left out many interesting topics on super representation theory. We refer to [BL, J] (and more recently [SZ]) for finite-dimensional irreducible characters of atypicality one, to [BKN, DS, Ma, Mu, Pe, PS] for geometric approaches, to [Br2, CWZ2] for further development of the Fock space approach of Brundan for the queer Lie superalgebra  $\mathfrak{q}(n)$  and for  $\mathfrak{osp}(2|2n)$ , to [CK, CKW, CZ1, Ger, San, Sva, Zou] for some cohomological aspects, to [BrK, SW, WZ] for prime characteristic, to [JHKT, Su] for related combinatorial structures; also see [Gor, KW, Naz] for additional work on Lie superalgebras.

**Acknowledgment.** This paper is a modified and expanded written account of the 8 lectures given by the second author at the summer school of East China Normal University (ECNU), Shanghai, July 2009. We are grateful to Ngau Lam for his collaboration and insight. We thank Bin Shu at ECNU for hospitality and an enjoyable summer school.

# 1 Lie superalgebra ABC

## 1.1

A vector superspace  $V$  is understood as a  $\mathbb{Z}_2$ -graded vector space  $V = V_0 \oplus V_1$ . An element  $a \in V_i$  has parity  $|a| = i$ , and an element in  $V_0$  (respectively,  $V_1$ ) is called even (respectively, odd).

**Definition 1.1.** A Lie superalgebra is a vector superspace  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  equipped with a bilinear bracket operation  $[\cdot, \cdot]$  satisfying  $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ ,  $i, j \in \mathbb{Z}_2$ , and the following two axioms: for  $\mathbb{Z}_2$ -homogeneous  $a, b, c \in \mathfrak{g}$ ,

1. (Skew-supersymmetry)  $[a, b] = -(-1)^{|a|\cdot|b|}[b, a]$ .
2. (Super Jacobi identity)  $[a, [b, c]] = [[a, b], c] + (-1)^{|a|\cdot|b|}[b, [a, c]]$ .

*Remark 1.2.* 1. For a Lie superalgebra  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ ,  $\mathfrak{g}_0$  is a Lie algebra and  $\mathfrak{g}_1$  is a  $\mathfrak{g}_0$ -module under the adjoint action.

2. (Sign Rule) As explained in Manin's book [Ma], there is a general heuristic sign rule for superalgebras as follows. *If in some formula for usual algebra there are monomials with interchanged terms, then in the corresponding formula for superalgebra every interchange of neighboring terms, say  $a$  and  $b$ , is accompanied by the multiplication of the monomials by the factor  $(-1)^{|a|\cdot|b|}$ .* This is already manifest in the definition of Lie superalgebra and will persist throughout the paper.

**Example 1.3.** 1. Let  $A = A_0 \oplus A_1$  be an associative superalgebra (i.e.  $\mathbb{Z}_2$ -graded). Then  $(A, [\cdot, \cdot])$  is a Lie superalgebra, where for homogeneous elements  $a, b \in A$ , we define

$$[a, b] = ab - (-1)^{|a|\cdot|b|}ba.$$

2. A Lie superalgebra  $\mathfrak{g}$  with  $\mathfrak{g}_1 = 0$  is just a usual Lie algebra. A Lie superalgebra  $\mathfrak{g}$  with purely odd part (i.e.  $\mathfrak{g}_0 = 0$ ) has to be *abelian*, i.e.  $[\mathfrak{g}, \mathfrak{g}] = 0$ .

## 1.2 Lie superalgebras of type $A$ and the supertrace

Let  $V = V_0 \oplus V_1$  be a vector superspace. Then  $\text{End}(V)$  is naturally an associative superalgebra. The Lie superalgebra  $\mathfrak{gl}(V) := (\text{End}(V), [\cdot, \cdot])$  from Example 1.3 (1) is called a *general linear Lie superalgebra*. If  $V_0 = \mathbb{C}^m$  and  $V_1 = \mathbb{C}^n$ , we denote  $V$  by  $\mathbb{C}^{m|n}$ , and  $\mathfrak{gl}(V)$  by  $\mathfrak{gl}(m|n)$ . Note that both  $\mathfrak{gl}(m|0) \cong \mathfrak{gl}(0|m)$  are isomorphic to the usual Lie algebra  $\mathfrak{gl}(m)$ .

The Lie superalgebra  $\mathfrak{gl}(m|n)$  consists of block matrices of size  $m|n$ :

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (1.1)$$

Throughout the paper, we choose to parameterize the rows and columns of the matrices by the set

$$I(m|n) = \{\bar{1}, \dots, \bar{m}; 1, \dots, n\}$$

with a total order

$$\bar{1} < \dots < \bar{m} < 0 < 1 < \dots < n \quad (1.2)$$

(where 0 is inserted for later convenience). Its even subalgebra consists of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

and is isomorphic to  $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ .

**Example 1.4.** For  $\mathfrak{g} = \mathfrak{gl}(1|1)$ , let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$[e, f] = h_1 + h_2 \text{ (the identity matrix).}$$

The *supertrace*, denoted by  $\text{str}$ , of (1.1) is defined to be

$$\text{str}(g) = \text{tr}(a) - \text{tr}(d).$$

The *special linear Lie superalgebra* is

$$\mathfrak{sl}(m|n) = \{x \in \mathfrak{gl}(m|n) \mid \text{str}(x) = 0\}.$$

The definitions of supertrace and of the Lie superalgebra  $\mathfrak{sl}$  are justified by the following.

**Exercise 1.5.** Show that  $\mathfrak{sl}(m|n) = [\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)]$  and in particular  $\mathfrak{sl}(m|n)$  is a Lie subalgebra of  $\mathfrak{gl}(m|n)$ .

The notion of simple Lie superalgebras is defined in the same way as for Lie algebras. We note that  $\mathfrak{sl}(n|n)$  is not a simple Lie superalgebra, as it contains a nontrivial center  $\mathbb{C}I_{2n}$ .

### 1.3 The bilinear form

Let  $\mathfrak{h}$  denote the Cartan subalgebra of  $\mathfrak{gl}(m|n)$  consisting of all diagonal matrices. Note that  $\mathfrak{h}$  is an even subalgebra of  $\mathfrak{gl}(m|n)$ .

Let  $E_{ij}$ , for  $i, j \in I(m|n)$ , denote the standard basis for  $\mathfrak{gl}(m|n)$ . We define a bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{g}$  by letting

$$(a, b) = \text{str}(ab), \quad a, b \in \mathfrak{g}.$$

This restricts to a nondegenerate symmetric bilinear form on  $\mathfrak{h}$ : for  $i, j \in I(m|n)$ ,

$$(E_{ii}, E_{jj}) = \begin{cases} 1 & \text{if } \bar{1} \leq i = j \leq \bar{m}, \\ -1 & \text{if } 1 \leq i = j \leq n, \\ 0 & \text{if } i \neq j. \end{cases}$$

Denote by  $\{\delta_i, \epsilon_j\}_{i,j}$  the basis of  $\mathfrak{h}^*$  dual to  $\{E_{\bar{i}\bar{i}}, E_{jj}\}_{i,j}$ , where  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . Under the bilinear form  $(\cdot, \cdot)$ , we have the identification  $\delta_i = (E_{\bar{i}\bar{i}}, \cdot)$  and  $\epsilon_j = -(E_{jj}, \cdot)$ . Whenever it is convenient we also use the notation

$$\epsilon_{\bar{i}} := \delta_i, \quad \text{for } 1 \leq i \leq m. \quad (1.3)$$

The form  $(\cdot, \cdot)$  on  $\mathfrak{h}$  induces a non-degenerate bilinear form on  $\mathfrak{h}^*$ , which will be denoted by the same notation, as follows: for  $i, j \in I(m|n)$ ,

$$(\epsilon_i, \epsilon_j) = \begin{cases} 1 & \text{if } \bar{1} \leq i = j \leq \bar{m}, \\ -1 & \text{if } 1 \leq i = j \leq n, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1.4)$$

## 1.4 The root system

For the Lie superalgebra  $\mathfrak{gl}(m|n)$ , we define the root space decomposition, a root system  $\Phi$ , a set  $\Phi^+$  (respectively,  $\Phi^-$ ) of positive (respectively, negative) roots, a set  $\Pi$  of simple roots (in  $\Phi^+$ ), etc. As this can be done in the same way as for semisimple Lie algebras or  $\mathfrak{gl}(m)$ , we will merely write down the statements for later use.

Now let us make the super phenomenon explicit. A root  $\alpha$  is *even* if  $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{0}}$ , and it is *odd* if  $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{1}}$ . Denote by  $\Phi_{\bar{0}}$  (respectively,  $\Phi_{\bar{1}}$ ) the set of all even (respectively, odd) roots in  $\Phi$ . Denote

$$\Phi_i^\pm = \Phi_i \cap \Phi^\pm, \quad \Pi_i = \Phi_i \cap \Pi, \quad i \in \mathbb{Z}_2.$$

With respect to the Cartan subalgebra  $\mathfrak{h}$  the Lie superalgebra  $\mathfrak{gl}(m|n)$  admits a root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

with a root system

$$\Phi = \{\epsilon_i - \epsilon_j \mid i, j \in I(m|n), i \neq j\}.$$

The standard set of simple roots is taken to be  $\Pi = \Pi_{\bar{0}} \cup \Pi_{\bar{1}}$ , where

$$\Pi_{\bar{0}} = \{\epsilon_{\bar{i}} - \epsilon_{\overline{i+1}}\}_{1 \leq i \leq m-1} \cup \{\epsilon_i - \epsilon_{i+1}\}_{1 \leq i \leq n-1}, \quad \Pi_{\bar{1}} = \{\epsilon_{\bar{m}} - \epsilon_1\},$$

and the associated standard set of positive roots is

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i, j \in I(m|n), i < j\},$$

where the odd roots are  $\epsilon_i - \epsilon_j$  with indices  $i < 0 < j$ . Clearly,  $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij}$ . It follows by (1.4) that

$$(\delta_i - \epsilon_j, \delta_i - \epsilon_j) = 0,$$

for all the odd roots  $\delta_i - \epsilon_j$ , where  $1 \leq i \leq m, 1 \leq j \leq n$ . An odd root  $\alpha$  with  $(\alpha, \alpha) = 0$  is called *isotropic*. The standard Dynkin diagram is:

$$\begin{array}{ccccccc} & & & \delta_m - \epsilon_1 & & & \\ & & & \otimes & & & \\ \circ & \cdots & \circ & \text{---} & \circ & \cdots & \circ \\ \delta_1 - \delta_2 & & \delta_{m-1} - \delta_m & & \epsilon_1 - \epsilon_2 & & \epsilon_{n-1} - \epsilon_n \end{array}$$

where we have used  $\otimes$  to denote an isotropic odd simple root.

*Remark 1.6.* The notion of root systems and Dynkin diagrams makes sense for all the basic classical Lie superalgebras, which consist of  $\mathfrak{gl}(m|n)$ ,  $\mathfrak{sl}(m|n)$ ,  $\mathfrak{osp}(m|2n)$  and three exceptional ones (besides the simple Lie algebras).

## 1.5 Non-conjugate Borel subalgebras and $\epsilon\delta$ -sequences

As we have seen above, the bilinear form on the real subspace  $\mathfrak{h}_{\mathbb{R}}^*$  spanned by the  $\epsilon_i$ 's is not positive-definite (due to the supertrace), and moreover, there exist isotropic odd roots.

Another distinguished feature of Lie superalgebras is the existence of non-conjugate Borel subalgebras or non-isomorphic Dynkin diagrams (under the Weyl group action).

**Lemma 1.7.** *Let  $\mathfrak{g}$  be a Lie superalgebra with triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ , which corresponds to the root system  $\Phi = \Phi^+ \cup \Phi^-$ . Let  $\alpha$  be an odd isotropic simple root. Let  $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ . Then,  $\Phi(\alpha)^+ := (\Phi^+ \setminus \{\alpha\}) \cup \{-\alpha\}$  is a new system of positive roots, whose corresponding set of simple roots is*

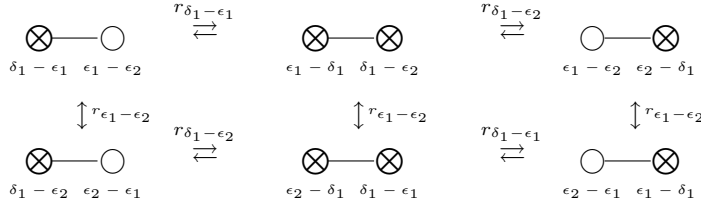
$$\Pi(\alpha) = \{\beta \in \Pi \mid (\beta, \alpha) = 0, \beta \neq \alpha\} \cup \{\beta + \alpha \mid \beta \in \Pi, (\beta, \alpha) \neq 0\} \cup \{-\alpha\}.$$

The new Borel subalgebra corresponding to  $\Pi(\alpha)$  will be denoted by  $\mathfrak{b}(\alpha)$ .

*Proof.* Follows from a straightforward verification.  $\square$

The process of obtaining  $\Pi(\alpha)$  from  $\Pi$  above will be referred to as an *odd reflection*, and will be denoted by  $r_\alpha$ , in accordance with the usual notion of real reflections.

**Example 1.8.** Associated to  $\mathfrak{gl}(1|2)$ , we have  $\Phi_{\bar{0}} = \{\pm(\epsilon_1 - \epsilon_2)\}$ , and  $\Phi_{\bar{1}} = \{\pm(\delta_1 - \epsilon_1), \pm(\delta_1 - \epsilon_2)\}$ . There are 6 sets of simple roots, that are related by the real and odd reflections as follows. There are three conjugacy classes of Borel subalgebras, and each vertical pair corresponds to such a conjugacy class.



One convenient way to parameterize the conjugacy classes of Borel subalgebras of  $\mathfrak{gl}(m|n)$  is via the notion of  $\epsilon\delta$ -sequences. Keeping (1.3) in mind, we list the simple roots associated to a given Borel subalgebra  $\mathfrak{b}$  in order as  $\epsilon_{i_1 - \epsilon_{i_2}}, \epsilon_{i_2 - \epsilon_{i_3}}, \dots, \epsilon_{i_{m+n-1} - \epsilon_{i_{m+n}}}$ , where  $\{i_1, i_2, \dots, i_{m+n}\} = I(m|n)$ . Switching the ordered sequence  $\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_{m+n}}$  to the  $\epsilon\delta$ -notation by (1.3) and then dropping the indices give us the  $\epsilon\delta$ -sequence associated to  $\mathfrak{b}$ . Note that the total number of  $\delta$ 's (respectively,  $\epsilon$ 's) is  $m$  (respectively,  $n$ ).

For example, the three conjugacy classes of Borels for  $\mathfrak{gl}(1|2)$  above correspond to the three sequences  $\delta\epsilon\epsilon$ ,  $\epsilon\delta\epsilon$ ,  $\epsilon\epsilon\delta$ , respectively. In more detail, the first sequence  $\delta\epsilon\epsilon$  is obtained by removing the indices of  $\delta_1\epsilon_1\epsilon_2$  (read off from the upper-left diagram above) or  $\delta_1\epsilon_2\epsilon_1$  (from the lower-left diagram above). Also the standard Borel of  $\mathfrak{gl}(m|n)$  corresponds to the sequence  $\underbrace{\delta \cdots \delta}_m \underbrace{\epsilon \cdots \epsilon}_n$  while the opposite Borel to the standard one corresponds to  $\underbrace{\epsilon \cdots \epsilon}_n \underbrace{\delta \cdots \delta}_m$ .

**Exercise 1.9.** Let  $\Phi$  be the roots of  $\mathfrak{gl}(m|n)$  with respect to the Cartan subalgebra  $\mathfrak{h}$ . Prove that the sets of simple roots in  $\Phi$  are in one-to-one correspondence with the  $\epsilon\delta$ -sequences with  $m$   $\delta$ 's and  $n$   $\epsilon$ 's in total. In particular, there are  $\binom{m+n}{m}$  conjugacy classes of Borel subalgebras.

## 1.6

Let  $B$  be a non-degenerate even supersymmetric bilinear form on a vector superspace  $V = V_0 \oplus V_1$ . Here  $B$  is *even* if  $B(V_i, V_j) = 0$  unless  $i = j \in \mathbb{Z}_2$ , and  $B$  is *supersymmetric* if  $B|_{V_0 \times V_0}$  is symmetric while  $B|_{V_1 \times V_1}$  is skew-symmetric (and hence  $\dim V_1$  is necessarily even).

For  $s \in \mathbb{Z}_2$ , let

$$\begin{aligned} \mathfrak{osp}(V)_s &= \{g \in \mathfrak{gl}(V)_s \mid B(g(x), y) = -(-1)^{s|x|}B(x, g(y)), \forall x, y \in V\}, \\ \mathfrak{osp}(V) &= \mathfrak{osp}_0(V) \oplus \mathfrak{osp}_1(V). \end{aligned}$$

One checks that  $\mathfrak{osp}(V)$  is a Lie superalgebra, whose even subalgebra is isomorphic to  $\mathfrak{so}(V_0) \oplus \mathfrak{sp}(V_1)$ . When  $V = \mathbb{C}^{\ell|2n}$ , we write  $\mathfrak{osp}(V) = \mathfrak{osp}(\ell|2n)$ .

*Remark 1.10.* One can also define the Lie superalgebra  $\mathfrak{spo}(V)$  as the subalgebra of  $\mathfrak{gl}(V)$  which preserves a non-degenerate skew-supersymmetric bilinear form on  $V$  (here  $\dim V_0$  has to be even). When  $V = \mathbb{C}^{2n|\ell}$ , we write  $\mathfrak{spo}(V) = \mathfrak{spo}(2n|\ell)$ .

**Exercise 1.11.** Show that Lie superalgebras  $\mathfrak{osp}(\ell|2n)$  and  $\mathfrak{spo}(2n|\ell)$  are isomorphic.

## 1.7

Define the *super transpose*  $st$  as follows: for a matrix in the block form (1.1), we let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{st} = \begin{pmatrix} a^t & c^t \\ -b^t & d^t \end{pmatrix},$$

where  $x^t$  denotes the usual transpose of the matrix  $x$ .

Define the  $(2n+2m+1) \times (2n+2m+1)$  matrix in the  $(n|n|m|m|1)$ -block form

$$\mathfrak{J}_{2n|2m+1} := \begin{pmatrix} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.5)$$

where  $I_n$  is the  $n \times n$  identity matrix. Let  $\mathfrak{J}_{2n|2m}$  denote the  $(2n+2m) \times (2n+2m)$  matrix obtained from  $\mathfrak{J}_{2n|2m+1}$  by deleting the last row and column. Then, for  $\ell = 2m$  or  $2m+1$ , by definition  $\mathfrak{spo}(2n|\ell)$  is the subalgebra of  $\mathfrak{gl}(2n|\ell)$  which preserves the bilinear form on  $\mathbb{C}^{2n|\ell}$  with matrix  $\mathfrak{J}_{2n|\ell}$  relative to the standard basis of  $\mathbb{C}^{2n|\ell}$ , and hence

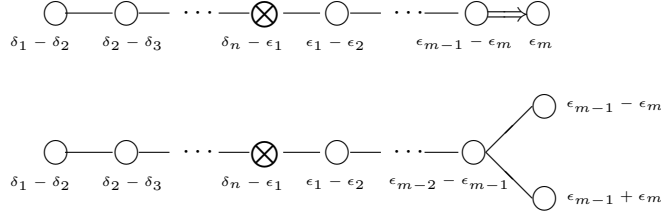
$$\mathfrak{spo}(2n|\ell) = \{g \in \mathfrak{gl}(2n|\ell) \mid g^{st}\mathfrak{J}_{2n|\ell} + \mathfrak{J}_{2n|\ell}g = 0\}.$$

By a direct computation,  $\mathfrak{spo}(2n|2m+1)$  consists of the  $(2n+2m+1) \times (2n+2m+1)$  matrices of the following  $(n|n|m|m|1)$ -block form

$$\begin{pmatrix} d & e & y_1^t & x_1^t & z_1^t \\ f & -d^t & -y^t & -x^t & -z^t \\ x & x_1 & a & b & -v^t \\ y & y_1 & c & -a^t & -u^t \\ z & z_1 & u & v & 0 \end{pmatrix}, \quad b, c \text{ skew-symmetric, } e, f \text{ symmetric.} \quad (1.6)$$

The Lie superalgebra  $\mathfrak{spo}(2n|2m)$  consists of matrices (1.6) with the last row and column removed. Here and below, the rows and columns of the matrices  $\mathfrak{J}_{2n|\ell}$  and (1.6) (or its modification) are indexed by the finite set  $I(2n|\ell)$ .

The standard Dynkin diagrams of  $\mathfrak{spo}(2n|2m+1)$  and  $\mathfrak{spo}(2n|2m)$  are given respectively as follows:



**Example 1.12.** The Lie superalgebra  $\mathfrak{g} = \mathfrak{osp}(1|2)$  has even subalgebra  $\mathfrak{g}_0 \cong \mathfrak{sl}(2) = \mathbb{C}\langle e, h, f \rangle$  and  $\mathfrak{g}_1$  isomorphic to the 2-dimensional natural  $\mathfrak{sl}(2)$ -module  $\mathbb{C}\langle E, F \rangle$ . Moreover,  $[E, E] = 2e$ ,  $[F, F] = -2f$ ,  $[E, F] = h$ .

The simple root consists of a (unique) odd non-isotropic root  $\delta$ , twice of which is an even root. The Dynkin diagram of  $\mathfrak{osp}(1|2)$  is denoted by  $\bullet$  (in order to distinguish from an odd simple isotropic root  $\otimes$ ).

**Exercise 1.13.** Prove the following identities in  $U(\mathfrak{osp}(1|2))$  ( $n \in \mathbb{Z}_+$ ):

- (i)  $[E, F^{2n}] = -nF^{2n-1}$ ,
- (ii)  $[E, F^{2n+1}] = F^{2n}(h - n)$ .

**Exercise 1.14.** Use Exercise 1.13 to show that the finite-dimensional irreducible representations of  $\mathfrak{osp}(1|2)$  are parameterized by the set of highest weights  $\{n\delta | n \in \mathbb{Z}_+\}$ . Denoting the irreducible module corresponding to  $n\delta$  by  $L(n\delta)$ , show that

$$\text{ch}L(n\delta) = \frac{e^{(n+\frac{1}{2})\delta} - e^{-(n+\frac{1}{2})\delta}}{e^{\frac{1}{2}\delta} - e^{-\frac{1}{2}\delta}}.$$

**Exercise 1.15.** Consider the element

$$\Omega := 2h^2 + 2h + FE + 4fe \in U(\mathfrak{osp}(1|2)).$$

Use Exercise 1.13 to prove that  $[F, \Omega] = [E, \Omega] = 0$ , and hence  $\Omega$  is in the center of  $U(\mathfrak{osp}(1|2))$ . Conclude from Exercise 1.14 that  $\Omega$  acts as different scalars on different finite-dimensional irreducible representations and hence that every finite-dimensional  $\mathfrak{osp}(1|2)$ -module is completely reducible.

## 2 Finite-dimensional modules of Lie superalgebras

### 2.1 PBW theorem

The *universal enveloping algebra*  $U(\mathfrak{g})$  of a Lie superalgebra  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  is an associative superalgebra characterized by a universal property exactly as for Lie algebras.

Assume that  $\{x_1, \dots, x_r\}$  is a basis for  $\mathfrak{g}_{\bar{0}}$  and  $\{y_1, \dots, y_s\}$  is a basis for  $\mathfrak{g}_{\bar{1}}$ . Then the universal enveloping algebra  $U(\mathfrak{g})$  admits a PBW basis

$$x_1^{a_1} \cdots x_r^{a_r} y_1^{b_1} \cdots y_s^{b_s}, \quad \forall a_i \in \mathbb{Z}_{\geq 0}, \forall b_j \in \{0, 1\}. \quad (2.1)$$

Alternatively, if we define the standard PBW filtration  $\{F^d U(\mathfrak{g})\}$  on  $U(\mathfrak{g})$  by letting  $F^d U(\mathfrak{g})$  be the span of elements (2.1) with  $\sum_i a_i + \sum_j b_j \leq d$ , then we have the following isomorphism for the associated graded of  $U(\mathfrak{g})$  in terms of the symmetric algebra  $S(\mathfrak{g}_{\bar{0}})$  and exterior algebra  $\wedge(\mathfrak{g}_{\bar{1}})$ :

$$\mathrm{gr}^F U(\mathfrak{g}) \cong S(\mathfrak{g}_{\bar{0}}) \otimes \wedge(\mathfrak{g}_{\bar{1}}).$$

**Example 2.1.** Let  $\mathfrak{g} = \mathfrak{g}_{\bar{1}}$  be a purely odd Lie superalgebra. Then its universal enveloping algebra is isomorphic to the exterior algebra  $\wedge(\mathfrak{g}_{\bar{1}})$ .

## 2.2 Representations of $\mathfrak{gl}(1|1)$

Recall the basis  $\{e, h_1, h_2, f\}$  for  $\mathfrak{g} = \mathfrak{gl}(1|1)$  from Example 1.4. Note that the even subalgebra  $\mathfrak{gl}(1|1)_{\bar{0}} = \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)$  coincides with the Cartan subalgebra  $\mathfrak{h}$ , and the Weyl group is trivial in this case.

Given  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ , we denote by  $\mathbb{C}_\lambda := \mathbb{C}v_\lambda$  the one-dimensional  $\mathfrak{h}$ -module by letting  $h_i v_\lambda = \lambda_i v_\lambda$ , for  $i = 1, 2$ . Regarding  $\mathbb{C}_\lambda$  as a module over the Borel subalgebra  $\mathfrak{h} + \mathbb{C}e$ , on which  $e$  acts trivially, we define the Verma modules (which coincide with Kac modules defined below for  $\mathfrak{gl}(1|1)$ ) over  $\mathfrak{g}$ :

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} + \mathbb{C}e)} \mathbb{C}_\lambda.$$

By the PBW theorem  $K(\lambda)$  can be identified with the following two-dimensional space (where by abuse of notation  $1 \otimes v_\lambda$  is denoted by  $v_\lambda$ ):

$$K(\lambda) = \mathbb{C}\langle v_\lambda, f v_\lambda \rangle.$$

The  $\mathfrak{g}$ -module  $K(\lambda)$  has a unique simple quotient, denoted by  $L(\lambda)$ .

The  $\mathfrak{g}$ -module  $K(\lambda)$  is simple if and only if  $\lambda_1 \neq -\lambda_2$ . This follows from

$$e f v_\lambda = (h_1 + h_2) v_\lambda - f e v_\lambda = (\lambda_1 + \lambda_2) v_\lambda.$$

If  $\lambda_1 \neq -\lambda_2$ , then  $K(\lambda_1, \lambda_2)$  is the unique simple object in its block (in the category of finite-dimensional  $\mathfrak{gl}(1|1)$ -modules). This block is semisimple.

For  $a \in \mathbb{C}$ , we have a non-split short exact sequence of  $\mathfrak{g}$ -modules:

$$0 \rightarrow L(a-1, 1-a) \rightarrow K(a, -a) \rightarrow L(a, -a) \rightarrow 0.$$

The block containing  $L(a, -a)$  is not semisimple, and it contains infinitely many simple objects  $L(b, -b)$  for  $b \in a + \mathbb{Z}$  (the underlying algebra can be described in terms of the  $A_\infty$ -quiver).

### 2.3 Finite dimensional simple $\mathfrak{gl}(m|n)$ -modules

Let  $\mathfrak{g} = \mathfrak{gl}(m|n)$  in this subsection.

Let  $\mathfrak{n}^+$  (and respectively,  $\mathfrak{n}^-$ ) be the subalgebra of strictly upper (and respectively, lower) triangular matrices of  $\mathfrak{gl}(m|n)$  with respect to the standard basis. Then we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The even subalgebra admits a compatible triangular decomposition

$$\mathfrak{g}_{\bar{0}} = \mathfrak{n}_{\bar{0}}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\bar{0}}^+,$$

where  $\mathfrak{n}_{\bar{0}}^{\pm} = \mathfrak{g}_{\bar{0}} \cap \mathfrak{n}^{\pm}$ .

Moreover, the Lie superalgebra  $\mathfrak{g}$  admits a  $\mathbb{Z}$ -grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1,$$

where  $\mathfrak{g}_{-1}$  (respectively,  $\mathfrak{g}_1$ ) is spanned by all  $E_{ij}$  with  $i > 0 > j$  (respectively,  $i < 0 < j$ ). Note that this  $\mathbb{Z}$ -grading is compatible with the  $\mathbb{Z}_2$ -grading, i.e., the degree zero subspace coincides with the  $\mathbb{Z}_2$ -degree zero subspace. This is equivalent to the fact that  $\mathfrak{g}_{\bar{0}}$  is a Levi subalgebra of  $\mathfrak{g}$  (corresponding to the removal of the odd simple root from the standard Dynkin diagram of  $\mathfrak{g}$ ).

For  $\lambda \in \mathfrak{h}^*$ , let  $L(\lambda)$  (respectively,  $L^0(\lambda)$ ) be the simple module of  $\mathfrak{g}$  (respectively,  $\mathfrak{g}_{\bar{0}}$ ) of highest weight  $\lambda$ . Define the *Kac module* over  $\mathfrak{g}$  by

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1)} L^0(\lambda),$$

which can be identified by the PBW theorem with

$$K(\lambda) = \wedge(\mathfrak{g}_{-1}) \otimes L^0(\lambda). \quad (2.2)$$

**Proposition 2.2.** *There exists a surjective  $\mathfrak{g}$ -module homomorphism (unique up to a scalar multiple)  $K(\lambda) \twoheadrightarrow L(\lambda)$ . Moreover, the following are equivalent:*

1.  $L(\lambda)$  is finite-dimensional.
2.  $L^0(\lambda)$  is finite-dimensional.
3.  $K(\lambda)$  is finite-dimensional.

*Proof.* The surjective homomorphism follows from the fact that  $K(\lambda)$  is a highest weight  $\mathfrak{g}$ -module of highest weight  $\lambda$ .

(1)  $\Rightarrow$  (2). Note that  $L^0(\lambda)$  is a quotient of  $L(\lambda)$  regarded as  $\mathfrak{g}_{\bar{0}}$ -module.

(2)  $\Rightarrow$  (3). Follows from (2.2).

(3)  $\Rightarrow$  (1). Follows from the surjective map  $K(\lambda) \twoheadrightarrow L(\lambda)$ .  $\square$

Arguing as for Lie algebras, every finite-dimensional simple  $\mathfrak{g}$ -module is a highest weight module  $L(\lambda)$  for some  $\lambda$ , and moreover,  $L(\lambda) \not\cong L(\mu)$  if  $\lambda \neq \mu$ . Hence, the above proposition gives a classification of finite-dimensional simple  $\mathfrak{g}$ -modules.

## 2.4 Typical irreducible characters

Let  $\mathfrak{g} = \mathfrak{gl}(m|n)$  in this subsection.

We will denote by  $P^+$  the set of weights  $\lambda \in \mathfrak{h}^*$  such that  $L^0(\lambda)$  is finite-dimensional. Recall that the character of  $L^0(\lambda)$  for  $\lambda \in P^+$  is given by Weyl's formula

$$\text{ch}L^0(\lambda) = \frac{\sum_{\sigma \in W} \text{sgn}(\sigma) e^{\sigma(\lambda + \rho_0) - \rho_0}}{\prod_{\alpha \in \Phi_0^+} (1 - e^{-\alpha})}.$$

Here  $W (\cong \mathfrak{S}_m \times \mathfrak{S}_n)$  denotes the Weyl group of  $\mathfrak{g}_0$ ,  $\text{sgn}$  denotes the sign representation of  $W$ , and  $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Phi_0^+} \alpha$ .

Also denote by

$$\rho_1 = \frac{1}{2} \sum_{\alpha \in \Phi_1^+} \alpha, \quad \rho = \rho_0 - \rho_1.$$

**Lemma 2.3.** *We have*

1.  $(\rho, \beta) = \frac{1}{2}(\beta, \beta), \forall \beta \in \Pi$ .
2.  $\sigma(\rho_1) = \rho_1, \forall \sigma \in W$ .

*Proof.* Part 1 can be verified directly. Part 2 follows from the fact that  $\mathfrak{g}_1$  is preserved by the adjoint group  $G_0$  associated to  $\mathfrak{g}_0$  under the adjoint action.  $\square$

**Corollary 2.4.** *We have*

$$\text{ch}K(\lambda) = \frac{\prod_{\alpha \in \Phi_1^+} (1 + e^{-\alpha})}{\prod_{\alpha \in \Phi_0^+} (1 - e^{-\alpha})} \sum_{\sigma \in W} \text{sgn}(\sigma) e^{\sigma(\lambda + \rho) - \rho}. \quad (2.3)$$

*Proof.* By (2.2),  $\text{ch}K(\lambda) = \text{ch}L^0(\lambda) \prod_{\alpha \in \Phi_1^+} (1 + e^{-\alpha})$ . Now by Weyl's character formula for  $L^0(\lambda)$ , we have

$$\text{ch}K(\lambda) = \frac{\prod_{\alpha \in \Phi_1^+} (1 + e^{-\alpha})}{\prod_{\alpha \in \Phi_0^+} (1 - e^{-\alpha})} \sum_{\sigma \in W} \text{sgn}(\sigma) e^{\sigma(\lambda + \rho_0) - \rho_0}.$$

This is equivalent to the formula in (2.3) by the second part of Lemma 2.3.  $\square$

**Exercise 2.5.** Show that

$$\rho_1 = \frac{n}{2} \sum_{i=1}^m \delta_i - \frac{m}{2} \sum_{j=1}^n \epsilon_j.$$

Again it follows that  $\sigma(\rho_1) = \rho_1, \forall \sigma \in W$ .

**Definition 2.6.** A weight  $\lambda \in \mathfrak{h}^*$  is called *typical*, if  $\prod_{\alpha \in \Phi_1^+} (\lambda + \rho, \alpha) \neq 0$ . It is called *atypical* if it is not typical.

**Theorem 2.7** (Kac). *Let  $\lambda \in P^+$ . Then  $\lambda$  is typical if and only if  $\text{ch}L(\lambda)$  is given by the right hand side of (2.3), if and only if  $K(\lambda)$  is irreducible.*

The example of  $\mathfrak{gl}(1|1)$  in Section 2.2 fits well with this theorem.

*Sketch of a proof.* [K2] Let  $v_\lambda \in K(\lambda)$  be a highest weight vector. Denote by  $e_\alpha, f_\alpha$  the generators of  $\mathfrak{g}_{\pm\alpha}$ , where  $\alpha \in \Phi^+$ .

Assume that  $S$  is a nonzero  $\mathfrak{g}$ -submodule of  $K(\lambda)$ . One first shows easily that

$$(i) \prod_{\alpha \in \Phi_1^+} f_\alpha v_\lambda \in S.$$

It follows that

$$(ii) \prod_{\beta \in \Phi_1^+} e_\beta \prod_{\alpha \in \Phi_1^+} f_\alpha v_\lambda \in S.$$

Then one further shows that (up to a nonzero scalar multiple)

$$(iii) \prod_{\beta \in \Phi_1^+} e_\beta \prod_{\alpha \in \Phi_1^+} f_\alpha v_\lambda = \prod_{\alpha \in \Phi_1^+} (\lambda + \rho, \alpha) v_\lambda.$$

Step (iii) uses the  $W$ -invariance of  $\Phi_1^+$  and a degree counting argument among others.

From (i) and (iii) it follows that if  $\lambda$  is atypical, then  $K(\lambda)$  is irreducible.

Now suppose that  $K(\lambda)$  is irreducible. The  $\text{ad } \mathfrak{g}_0$ -invariance of  $\mathfrak{g}_{\pm 1}$  implies that if  $v_\lambda \in U(\mathfrak{g}) \prod_{\alpha \in \Phi_1^+} f_\alpha v_\lambda$ , then  $v_\lambda$  is a scalar multiple of  $\prod_{\beta \in \Phi_1^+} e_\beta \prod_{\alpha \in \Phi_1^+} f_\alpha v_\lambda$ . Thus by (iii)  $\lambda$  is typical.  $\square$

## 2.5 Odd reflections

As usual, we let  $\{e_\beta, h_\beta, f_\beta\}$  denote the Chevalley generators for  $\beta \in \Phi^+$ . The following simple and fundamental lemma for odd reflections has been used by many authors (e.g. [LSS, Appendix], [PS, Lemma 0.3] and [KW, Lemma 1.4]; also compare [DP, (2.12)] for an unconventional definition). Recall the new Borel  $\mathfrak{b}(\alpha)$  for a simple isotropic odd root  $\alpha$  from Lemma 1.7.

**Lemma 2.8.** *Let  $L$  be a simple  $\mathfrak{g}$ -module of  $\mathfrak{b}$ -highest weight  $\lambda$  and let  $v$  be a  $\mathfrak{b}$ -highest weight vector of  $L$ . Let  $\alpha$  be a simple isotropic odd root.*

1. *If  $\langle \lambda, h_\alpha \rangle = 0$ , then  $L$  is a  $\mathfrak{g}$ -module of  $\mathfrak{b}(\alpha)$ -highest weight  $\lambda$  and  $v$  is a  $\mathfrak{b}(\alpha)$ -highest weight vector.*
2. *If  $\langle \lambda, h_\alpha \rangle \neq 0$ , then  $L$  is a  $\mathfrak{g}$ -module of  $\mathfrak{b}(\alpha)$ -highest weight  $\lambda - \alpha$  and  $f_\alpha v$  is a  $\mathfrak{b}(\alpha)$ -highest weight vector.*

*Proof.* We first observe three simple identities:

$$(i) e_\alpha f_\alpha v = [e_\alpha, f_\alpha]v = h_\alpha v = \langle \lambda, h_\alpha \rangle v.$$

(ii)  $e_\beta f_\alpha v = [e_\beta, f_\alpha]v = 0$  for any  $\beta \in \Phi^+ \cap \Phi(\alpha)^+$ , since either  $\beta - \alpha$  is not a root or it belongs to  $\Phi^+ \cap \Phi(\alpha)^+$ .

$$(iii) f_\alpha^2 v = 0, \text{ since } \alpha \text{ is an isotropic odd root.}$$

Now, we consider the two cases separately.

(1) Assume that  $\langle \lambda, h_\alpha \rangle = 0$ . Then we must have  $f_\alpha v = 0$ , otherwise  $f_\alpha v$  would be a  $\mathfrak{b}$ -singular vector in the simple  $\mathfrak{g}$ -module  $V$  by (i) and (ii). This together with Lemma 1.7 implies that  $v$  is a  $\mathfrak{b}(\alpha)$ -highest weight vector of weight  $\lambda$  in the  $\mathfrak{g}$ -module  $V$ .

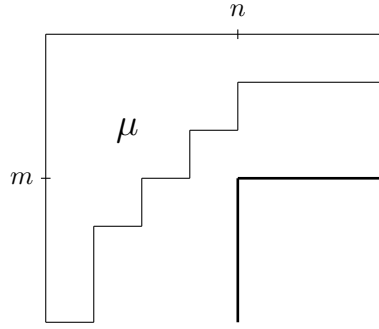
(2) Assume that  $\langle \lambda, h_\alpha \rangle \neq 0$ . Then (i), (ii), (iii) and Lemma 1.7 imply that  $f_\alpha v$  is nonzero and it is a  $\mathfrak{b}(\alpha)$ -highest weight vector of weight  $\lambda - \alpha$  in  $V$ .  $\square$

## 2.6 Finite-dimensional simple $\mathfrak{osp}$ -modules

We will treat the case of  $\mathfrak{g} = \mathfrak{spo}(2m|2n+1)$  in detail (while the case of  $\mathfrak{g} = \mathfrak{spo}(2m|2n)$  is similar). As usual, we have the triangular decomposition of  $\mathfrak{g}$  (with respect to the standard Borel)  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ , which allows us to define the Verma module  $\Delta(\lambda)$  associated to  $\lambda = \sum_{i=1}^m \lambda_i \delta_i + \sum_{j=1}^n \lambda_j \epsilon_j \in \mathfrak{h}^*$ . Then  $\Delta(\lambda)$  admits a unique simple quotient  $\mathfrak{g}$ -module, denoted by  $L(\lambda)$ .

As for Lie algebras, a finite-dimensional simple  $\mathfrak{g}$ -module has to be a highest weight module (with respect to any Borel), and hence is isomorphic to some  $L(\lambda)$ , and  $L(\lambda) \not\cong L(\mu)$  if  $\lambda \neq \mu$ . However, the classification of finite-dimensional simple  $\mathfrak{g}$ -modules is non-trivial, partly because the even subalgebra of  $\mathfrak{g}$  is not a Levi subalgebra. Clearly a necessary condition for  $L(\lambda)$  to be finite-dimensional is that  $\lambda$  is dominant integral with respect to the even subalgebra  $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2m) \oplus \mathfrak{so}(2n+1)$ .

**Definition 2.9.** A partition  $\mu = (\mu_1, \mu_2, \dots)$  is called an  $(m|n)$ -hook partition, or simply a hook partition when  $m, n$  are implicitly understood, if  $\mu_{m+1} \leq n$ .



Given an  $(m|n)$ -hook partition  $\mu$ , we denote by  $\mu^+ = (\mu_{m+1}, \mu_{m+2}, \dots)$  and write its transpose, which is necessarily of length  $\leq n$ , as  $\nu = (\mu^+)' = (\nu_1, \dots, \nu_n)$ . We define the weights

$$\begin{aligned} \mu^{\natural} &= \mu_1 \delta_1 + \dots + \mu_m \delta_m + \nu_1 \epsilon_1 + \dots + \nu_{n-1} \epsilon_{n-1} + \nu_n \epsilon_n \\ \mu^{\natural}_- &= \mu_1 \delta_1 + \dots + \mu_m \delta_m + \nu_1 \epsilon_1 + \dots + \nu_{n-1} \epsilon_{n-1} - \nu_n \epsilon_n. \end{aligned} \quad (2.4)$$

( $\mu^{\natural}_-$  is only used for  $\mathfrak{spo}(2m|2n)$  below).

**Theorem 2.10.** *Given any  $(m|n)$ -hook partition  $\mu$ , the simple  $\mathfrak{spo}(2m|2n+1)$ -module  $L(\mu^{\natural})$  (with respect to the standard Borel) is finite-dimensional. Moreover, these modules form a complete list of non-isomorphic finite-dimensional simple  $\mathfrak{spo}(2m|2n+1)$ -modules.*

The above theorem is due to Kac [K2] who formulated the conditions for finite-dimensional highest weight simple  $\mathfrak{g}$ -modules in terms of Dynkin labels (instead of hook partitions). A different proof using odd reflections is given by Shu and Wang (which also works in characteristic  $p > 0$ ). Let us sketch the idea of a proof following [SW], as the argument therein bears some similarity to the argument used later on for super duality. The same type of argument works for Theorem 2.12 below for  $\mathfrak{spo}(2m|2n)$ .

*Sketch of a proof.* Let  $\mu$  be an  $(m|n)$ -hook partition. We observe that the simple  $\mathfrak{spo}(2M|2n+1)$ -module  $V$  of highest weight  $\mu$  (with respect to the standard Borel) is finite-dimensional for  $M \geq \ell(\mu)$ , since it appears as a subquotient of a suitable tensor product of simple modules of fundamental weights. Then we use odd reflections to change the standard Borel of  $\mathfrak{spo}(2M|2n+1)$  to a Borel which is compatible with the standard Borel of  $\mathfrak{spo}(2m|2n+1)$  (which is regarded as a subalgebra of  $\mathfrak{spo}(2M|2n+1)$ ). We can show that the highest weight of  $V$  with respect to the new Borel is  $\mu^\natural$ , hence by restriction to  $\mathfrak{spo}(2m|2n+1)$  we have proved the first part of the theorem.

A highest weight for a simple finite-dimensional  $\mathfrak{g}$ -module is necessarily of the form  $\mu_1\delta_1 + \dots + \mu_m\delta_m + \nu_1\epsilon_1 + \dots + \nu_n\epsilon_n$ , where  $(\mu_1, \dots, \mu_m)$  and  $(\nu_1, \dots, \nu_n)$  are partitions by the dominance condition on the even subalgebra  $\mathfrak{g}_0$ . To prove the remaining condition  $\nu'_1 \leq \mu_m$ , it suffices to prove it for  $\mathfrak{spo}(2|2n+1)$  (which is a subalgebra of  $\mathfrak{spo}(2m|2n+1)$ ). Let  $V$  be a finite-dimensional simple  $\mathfrak{spo}(2|2n+1)$ -module. Via the sequence of odd reflections  $\delta_1 - \epsilon_1, \delta_1 - \epsilon_2, \dots, \delta_1 - \epsilon_n$  we change the standard Borel to a Borel with an odd non-isotropic simple root  $\delta_1$ . The dominance condition on the simple root  $\delta_1$  (or rather on the even root  $2\delta_1$ ), which is imposed by the finite-dimensionality of  $V$ , provides the desired necessary condition.  $\square$

**Exercise 2.11.** Complete the details in the proof of Theorem 2.10.

Similarly, we classify the finite-dimensional simple  $\mathfrak{spo}(2m|2n)$ -modules.

**Theorem 2.12.** [K2] (cf. [SW]) *The modules  $L(\mu^\natural)$  and  $L(\mu_-^\natural)$  (with respect to the standard Borel), where  $\mu$  runs over all  $(m|n)$ -hook partitions, are all the non-isomorphic finite-dimensional simple  $\mathfrak{spo}(2m|2n)$ -modules.*

Since  $\mathfrak{g}_0$  is not a Levi subalgebra of  $\mathfrak{g}$ , the above definition of the Kac module for  $\mathfrak{gl}(m|n)$ -module is no longer valid for  $\mathfrak{g}$ . Nevertheless, the notion of “typical weights” in Definition 2.6 still makes sense for  $\mathfrak{osp}$  (or any basic classical Lie superalgebra). The following is the  $\mathfrak{osp}$  counterpart of Theorem 2.7, whose much more involved proof will be skipped.

**Theorem 2.13.** [K2] *Let  $\mathfrak{g} = \mathfrak{osp}(\ell|2n)$ , for  $\ell = 2m$  or  $2m+1$ . If the weight  $\lambda = \mu^\natural \in \mathfrak{h}^*$  (and in addition  $\lambda = \mu_-^\natural$  when  $\ell = 2m$ ) for an  $(m|n)$ -hook partition  $\mu$  is typical, then  $chL(\lambda)$  is given by the right hand side of (2.3).*

**Exercise 2.14.** Recall that  $\mathfrak{so}(\ell)$  may be identified with  $\wedge^2(\mathbb{C}^\ell)$ , and more generally,  $\mathfrak{osp}(\ell|2n)$  may be identified with  $\wedge^2(\mathbb{C}^{\ell|2n}) = \wedge^2(\mathbb{C}^\ell) \oplus (\mathbb{C}^\ell \otimes \mathbb{C}^{2n}) \oplus S^2(\mathbb{C}^{2n})$ . Now using the invariant bilinear form to identify  $\mathbb{C}^{\ell|2n}$  with  $\mathbb{C}^{\ell|0} \oplus \mathbb{C}^{0|n} \oplus \mathbb{C}^{0|n*}$ , we define a  $\mathbb{Z}$ -gradation on  $\mathbb{C}^{\ell|2n}$  by setting  $\deg v = 0$  for  $v \in \mathbb{C}^{\ell|0}$ ,  $\deg v = 1$  for  $v \in \mathbb{C}^{0|n}$ , and  $\deg v = -1$  for  $v \in \mathbb{C}^{0|n*}$ . This induces a  $\mathbb{Z}$ -gradation on  $\wedge^2(\mathbb{C}^{\ell|2n})$  and hence on  $\mathfrak{osp}(\ell|2n)$ . Prove that

$$\mathfrak{osp}(\ell|2n) = \bigoplus_{i=-2}^2 \mathfrak{osp}(\ell|2n)_i,$$

where  $\mathfrak{osp}(\ell|2n)_0 \cong \mathfrak{so}(\ell) \oplus \mathfrak{gl}(n)$ . Furthermore, as  $\mathfrak{osp}(\ell|2n)_0$ -modules we have  $\mathfrak{osp}(\ell|2n)_1 \cong \mathbb{C}^\ell \otimes \mathbb{C}^n$  and  $\mathfrak{osp}(\ell|2n)_2 \cong \mathbb{C} \otimes S^2(\mathbb{C}^n)$ .

### 3 Schur-Sergeev duality

#### 3.1 The formulation

Let  $\mathfrak{g} = \mathfrak{gl}(m|n)$ . Let  $\{e_i | i \in I(m|n)\}$  be the standard basis for the natural  $\mathfrak{g}$ -module  $V = \mathbb{C}^{m|n}$ .

Then  $V^{\otimes d}$  is naturally a  $\mathfrak{g}$ -module by letting

$$\begin{aligned} \Phi_d(g).(v_1 \otimes v_2 \otimes \dots \otimes v_d) = & g.v_1 \otimes \dots \otimes v_d + (-1)^{|g| \cdot |v_1|} v_1 \otimes g.v_2 \otimes \dots \otimes v_d \\ & + \dots + (-1)^{|g| \cdot (|v_1| + \dots + |v_{d-1}|)} v_1 \otimes v_2 \otimes \dots \otimes g.v_d, \end{aligned}$$

where  $g \in \mathfrak{g}$  and  $v_i \in V$  are assumed to be  $\mathbb{Z}_2$ -homogeneous.

On the other hand, the action of the symmetric group  $\mathfrak{S}_d$  on  $V^{\otimes d}$  is determined by

$$\begin{aligned} \Psi_d((i, i+1)).v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d \\ = (-1)^{|v_i| \cdot |v_{i+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_d, \quad 1 \leq i \leq d-1, \end{aligned}$$

where  $(i, j)$  denotes a transposition in  $\mathfrak{S}_d$  and  $v_i, v_{i+1}$  are  $\mathbb{Z}_2$ -homogeneous.

**Lemma 3.1.** *The actions of  $(\mathfrak{g}, \Phi_d)$  and  $(\mathfrak{S}_d, \Psi_d)$  on  $V^{\otimes d}$  commute with each other. Symbolically, we write*

$$\mathfrak{g} \overset{\Phi_d}{\curvearrowright} V^{\otimes d} \overset{\Psi_d}{\curvearrowleft} \mathfrak{S}_d.$$

**Exercise 3.2.** Verify Lemma 3.1.

We write  $\lambda \vdash d$  for a partition  $\lambda$  of  $d$ . Denote by  $\mathcal{P}(d, m|n)$  the set of all  $(m|n)$ -hook partitions of size  $d$  and let

$$\mathcal{P}(m|n) = \cup_{d \geq 0} \mathcal{P}(d, m|n) = \{\lambda \mid \lambda \text{ are partitions with } \lambda_{m+1} \leq n\}.$$

Also set  $\mathcal{P}(d, m) = \mathcal{P}(d, m|0)$ . We denote by  $L(\lambda^\natural)$  for  $\lambda \in \mathcal{P}(m|n)$  the simple  $\mathfrak{g}$ -module of highest weight  $\lambda^\natural$  with respect to the standard Borel subalgebra. For a partition  $\lambda \vdash d$ , we denote by  $S^\lambda$  the Specht module of  $\mathfrak{S}_d$ . For example,  $S^{(d)}$  is the trivial module, and  $S^{(1^d)}$  is the sign module  $\text{sgn}$ . The following superalgebra analog of Schur duality was due to Sergeev [Sv1] (also cf. [BeR]).

**Theorem 3.3** (Schur-Sergeev duality). *1. The images  $\Phi_d$  and  $\Psi_d$ ,  $\Phi_d(U(\mathfrak{g}))$  and  $\Psi_d(\mathbb{C}\mathfrak{S}_d)$ , satisfy the double centralizer property, i.e.*

$$\begin{aligned} \Phi_d(U(\mathfrak{g})) &= \text{End}_{\mathfrak{S}_d}(V^{\otimes d}), \\ \text{End}_{U(\mathfrak{g})}(V^{\otimes d}) &= \Psi_d(\mathbb{C}\mathfrak{S}_d). \end{aligned}$$

*2. As  $U(\mathfrak{gl}(m|n)) \otimes \mathbb{C}\mathfrak{S}_d$ -module, we have*

$$V^{\otimes d} \cong \bigoplus_{\lambda \in \mathcal{P}(d, m|n)} L(\lambda^\natural) \otimes S^\lambda.$$

We refer to a  $U(\mathfrak{gl}(m|n)) \otimes \mathbb{C}\mathfrak{S}_d$ -module as a  $(\mathfrak{gl}(m|n), \mathfrak{S}_d)$ -module. Similar conventions also apply to similar setups below.

*Remark 3.4.* 1. For  $n = 0$ , the above Sergeev duality reduces to the usual Schur duality. If in addition  $d = 2$ ,  $(\mathbb{C}^m)^{\otimes 2} = S^2(\mathbb{C}^m) \oplus \wedge^2(\mathbb{C}^m)$ . This fits well with the well-known fact that as  $\mathfrak{gl}(m)$ -modules  $S^2(\mathbb{C}^m)$  and  $\wedge^2(\mathbb{C}^m)$  are, respectively, irreducible of highest weights  $2\Lambda_1$  and  $\Lambda_2$ , where  $\Lambda_i$  denotes the  $i$ th fundamental weight.

2. For  $m = 0$ ,  $\lambda^\natural = \lambda'$  (the conjugate partition), and  $S^\lambda = S^{\lambda'} \otimes \text{sgn}$ . In this case, the Sergeev duality reduces to the version of Schur duality twisted by the sign representation of  $\mathfrak{S}_d$ , i.e., as  $(\mathfrak{gl}(m), \mathfrak{S}_d)$ -module,

$$(\mathbb{C}^m)^{\otimes d} \cong \bigoplus_{\mu \in \mathcal{P}(d, m)} L(\mu) \otimes S^{\mu'}.$$

3. If  $d \leq mn + m + n$ , then  $\mathcal{P}(d, m|n) = \{\lambda \vdash d\}$ , and every simple  $\mathfrak{S}_d$ -module appears in the Sergeev duality decomposition.

4. For  $d = 2$ , the Sergeev duality reduces to the decomposition  $(\mathbb{C}^{m|n})^{\otimes 2} = S^2(\mathbb{C}^{m|n}) \oplus \wedge^2(\mathbb{C}^{m|n})$ , where  $S^2$  and  $\wedge^2$  are understood in the super sense. In particular, as an ordinary vector spaces,

$$S^2(\mathbb{C}^{m|n}) = S^2(\mathbb{C}^m) \oplus (\mathbb{C}^m \otimes \mathbb{C}^n) \oplus \wedge^2(\mathbb{C}^n).$$

### 3.2 Proof of Theorem 3.3

Set  $\mathcal{A} := \Phi_d(U(\mathfrak{g}))$  and  $\mathcal{B} := \Psi(\mathbb{C}\mathfrak{S}_d)$ . It is clear that

$$\mathcal{A} \subset \text{End}_{\mathfrak{S}_d}(V^{\otimes d}) = (\text{End}(V^{\otimes d}))^{\mathfrak{S}_d} \cong S^d(\text{End}V),$$

where  $S^d(-)$  denotes the  $d$ th symmetric tensor. With extra work, one proves that  $S^d(\text{End}V) \subset \mathcal{A}$  (the superalgebra generalization does not cause any extra difficulty). Hence  $\Phi_d(U(\mathfrak{g})) = \text{End}_{\mathfrak{S}_d}(V^{\otimes d})$ .

Since  $\mathcal{B} = \mathbb{C}\mathfrak{S}_d$  is a semisimple algebra, it follows (cf. [GW, Theorem 3.3.7]) that  $\text{End}_{U(\mathfrak{g})}(V^{\otimes d}) = \Psi_d(\mathbb{C}\mathfrak{S}_d)$ . This proves (1).

Let  $W = V^{\otimes d}$ . It follows from the double centralizer property and the semisimplicity of  $\mathbb{C}\mathfrak{S}_d$  that we have a multiplicity-free decomposition of the  $(\mathfrak{gl}(m|n), \mathfrak{S}_d)$ -module  $W$ :

$$W \cong V^{\otimes d} \cong \bigoplus_{\lambda \in \mathfrak{P}(d, m|n)} L^{[\lambda]} \otimes S^\lambda,$$

where  $L^{[\lambda]}$  is some simple  $\mathfrak{gl}(m|n)$ -module associated to  $\lambda$ , whose highest weight (with respect to the standard Borel) is to be determined. Also to be determined is the index set  $\mathfrak{P}(d, m|n) = \{\lambda \vdash d \mid L^{[\lambda]} \neq 0\}$ .

First we need to prepare some notations.

Let  $\text{Cp}(m|n)$  be the set of pairs  $\nu|\mu$  of compositions  $\nu = (\nu_1, \dots, \nu_m)$  of length  $\leq m$  and  $\mu = (\mu_1, \dots, \mu_n)$  of length  $\leq n$ , and let

$$\text{Cp}(d, m|n) = \{\nu|\mu \in \text{Cp}(m|n) \mid \sum_i \nu_i + \sum_j \mu_j = d\}.$$

We have the following weight space decomposition (with respect to the Cartan subalgebra of diagonal matrices  $\mathfrak{h} \subset \mathfrak{gl}(m|n)$ ):

$$W = \bigoplus_{\nu|\mu \in \text{Cp}(d, m|n)} W_{\nu|\mu},$$

where  $W_{\nu|\mu}$  has a linear basis  $e_{i_1} \otimes \dots \otimes e_{i_d}$ , with the indices satisfying the following equality of sets:

$$\{i_1, \dots, i_d\} = \{\underbrace{\bar{1}, \dots, \bar{1}}_{\nu_1}, \dots, \underbrace{\bar{m}, \dots, \bar{m}}_{\nu_m}; \underbrace{1, \dots, 1}_{\mu_1}, \dots, \underbrace{n, \dots, n}_{\mu_n}\}. \quad (3.1)$$

Let  $\mathfrak{S}_{\nu|\mu} := \mathfrak{S}_{\nu_1} \times \dots \times \mathfrak{S}_{\nu_m} \times \mathfrak{S}_{\mu_1} \times \dots \times \mathfrak{S}_{\mu_n}$ . The span of the vector  $e_{\nu|\mu} := e_{\bar{1}}^{\otimes \nu_1} \otimes \dots \otimes e_{\bar{m}}^{\otimes \nu_m} \otimes e_1^{\otimes \mu_1} \otimes \dots \otimes e_n^{\otimes \mu_n}$  can be identified with the  $\mathfrak{S}_{\nu|\mu}$ -module  $1_\nu \otimes \text{sgn}_\mu$ . Since  $\mathfrak{S}_d e_{\nu|\mu}$  spans  $W_{\nu|\mu}$  we have a surjective  $\mathfrak{S}_d$ -homomorphism from  $\text{Ind}_{\mathfrak{S}_{\nu|\mu}}^{\mathfrak{S}_d} (1_\nu \otimes \text{sgn}_\mu)$  onto  $W_{\nu|\mu}$  by Frobenius Reciprocity. By counting the dimensions we have an  $\mathfrak{S}_d$ -isomorphism:

$$W_{\nu|\mu} \cong \text{Ind}_{\mathfrak{S}_{\nu|\mu}}^{\mathfrak{S}_d} (1_\nu \otimes \text{sgn}_\mu).$$

Let us denote the decomposition of  $W_{\nu|\mu}$  into irreducibles by

$$W_{\nu|\mu} = \bigoplus_{\lambda} K_{\lambda, \nu|\mu} S^\lambda, \quad \text{for } K_{\lambda, \nu|\mu} \in \mathbb{Z}_+.$$

Let  $\lambda$  be a partition which is identified with its Young diagram. Recall  $I(m|n)$  is totally ordered by (1.2). A *hook tableau*  $T$  of shape  $\lambda$ , or an *hook  $\lambda$ -tableau*  $T$ , is an assignment of an element in  $I(m|n)$  to each box of the Young diagram  $\lambda$  satisfying the following conditions:

1. The numbers are weakly increasing along each row and column.
2. The numbers from  $\{\bar{1}, \dots, \bar{m}\}$  are strictly increasing along each column.
3. The numbers from  $\{1, \dots, n\}$  are strictly increasing along each row.

Such a  $T$  is said to have *content*  $\nu|\mu \in \text{Cp}(m|n)$  if  $\bar{i} \in I(m|0)$  appears  $\nu_i$  times and  $j \in I(0|n)$  appears  $\mu_j$  times. Denote by  $\mathcal{HT}(\lambda, \nu|\mu)$  the set of hook  $\lambda$ -tableaux of content  $\nu|\mu$ .

**Lemma 3.5.** *We have  $K_{\lambda, \nu|\mu} = \#\mathcal{HT}(\lambda, \nu|\mu)$ .*

*Proof.* Recall that  $\text{Ind}_{\mathfrak{S}_{\nu|\mu}}^{\mathfrak{S}_d} (1_\nu \otimes \text{sgn}_\mu) \cong \bigoplus_{\lambda} K_{\lambda, \nu|\mu} S^\lambda$ .

First assume that  $\mu = \emptyset$ , and we prove the formula by induction on the length  $r = \ell(\nu)$ . A hook (=semistandard) tableau  $T$  of shape  $\lambda$  and content  $\nu$  gives rise to a sequence of partitions  $\emptyset = \lambda^0 \subset \lambda^1 \subset \dots \subset \lambda^r = \lambda$  such that  $\lambda^i$  has the shape given by the parts of  $T$  with entries  $\leq i$ , and  $\lambda^i / \lambda^{i-1}$  has  $\nu_i$  boxes for each  $i$ . This sets up a bijection between  $\mathcal{HT}(\lambda, \nu)$  and the set of such sequences of partitions. Denote  $d_1 = d - \nu_r$  and  $\tilde{\nu} = (\nu_1, \dots, \nu_{r-1})$ . We have  $\text{Ind}_{\mathfrak{S}_{\tilde{\nu}}}^{\mathfrak{S}_{d_1}} 1_{\mathfrak{S}_{\tilde{\nu}}} \cong \bigoplus_{\rho \vdash d_1} K_{\rho, \tilde{\nu}} S^\rho$ ,

where  $K_{\rho, \tilde{\nu}} = \#\mathcal{HT}(\rho, \tilde{\nu})$  by induction hypothesis. Now the induction step is simply the Piere's rule (c.f. [Mac, Chapter 1 (5.16)]): for a partition  $\rho \vdash d_1$ ,

$$\text{Ind}_{\mathfrak{S}_{d_1} \times \mathfrak{S}_{\nu_r}}^{\mathfrak{S}_d} (S^\rho \otimes 1_{\nu_r}) \cong \bigoplus_{\lambda} S^\lambda,$$

where  $\lambda$  is such that  $\lambda/\rho$  is a horizontal strip of  $\nu_r$  boxes.

Then using the above special case as the initial step, we complete the proof in the general case by induction on the length of  $\mu$ , in which the induction step is exactly the conjugated Piere's rule.  $\square$

**Lemma 3.6.** *Let  $\lambda \vdash d$  and  $\nu|\mu \in \text{Cp}(d, m|n)$ . Then  $K_{\lambda, \nu|\mu} = 0$  unless  $\lambda \in \mathcal{P}(d, m|n)$ .*

*Proof.* By the identity  $K_{\lambda, \nu|\mu} = \#\mathcal{HT}(\lambda, \nu|\mu)$ , it suffices to prove that if a hook  $\lambda$ -tableau  $T$  of content  $\nu|\mu$  exists, then  $\lambda_{m+1} \leq n$ .

By applying the hook tableau condition (2) to the first column of  $T$ , we see that the first entry  $k \in I(m|n)$  in row  $(m+1)$  satisfies  $k > 0$ . Applying the hook tableau condition (3) to the  $(m+1)$ st row, we conclude that  $\lambda_{m+1} \leq n$ .  $\square$

Lemma 3.6 implies that  $\mathfrak{P}(d, m|n) \subset \mathcal{P}(d, m|n)$ . On the other hand, given  $\lambda \in \mathcal{P}(d, m|n)$ , clearly a hook  $\lambda$ -tableau exists, e.g., we can fill in the numbers  $\bar{1}, \dots, \bar{m}$  on the first  $m$  rows of  $\lambda$  row by row downward, and then for the (possibly) remaining rows of  $\lambda$ , we fill in the numbers  $1, \dots, n$  column by column from left to right. This distinguished  $\lambda$ -tableau will be denoted by  $T_\lambda$ . Hence, we have proved that

$$\mathfrak{P}(d, m|n) = \mathcal{P}(d, m|n).$$

For a given  $\lambda \in \mathcal{P}(d, m|n)$ , we have  $L^{[\lambda]} = \bigoplus_{\nu|\mu \in \text{Cp}(d, m|n)} L_{\nu|\mu}^{[\lambda]}$ . Among all the contents of hook  $\lambda$ -tableaux, the one for  $T_\lambda$  corresponds to a highest weight (by the three hook tableau conditions). Hence, we conclude that  $L^{[\lambda]} = L(\lambda^\natural)$ , the simple  $\mathfrak{g}$ -module of highest weight  $\lambda^\natural$ . This completes the proof of Theorem 3.3.

### 3.3

Clearly, the character of  $L(\lambda^\natural)$

$$\text{ch}L(\lambda^\natural) = \sum_{\nu|\mu \in \text{Cp}(m|n)} \dim L(\lambda^\natural)_{\nu|\mu} \prod_{i,j} x_i^{\nu_i} y_j^{\mu_j}$$

is a polynomial which is symmetric in  $\underline{x} = (x_1, \dots, x_m)$  and  $\underline{y} = (y_1, \dots, y_n)$ , respectively. Denote by  $m_\nu(\underline{x})$  the monomial symmetric polynomial associated to a partition  $\nu$ .

The following can be read off from the above proof of Theorem 3.3.

**Corollary 3.7.** *The character of  $L(\lambda^\natural)$  is given by:*

$$\text{ch}L(\lambda^\natural) = \sum_{\nu|\mu \in \text{Cp}(d, m|n)} \#\mathcal{HT}(\lambda, \nu|\mu) m_\nu(\underline{x}) m_\mu(\underline{y}),$$

where  $\nu$  and  $\mu$  above are partitions.

*Remark 3.8.* The standard basis vectors for  $S^2(\mathbb{C}^{m|n})$  are  $e_i \otimes e_j + (-1)^{|i||j|} e_j \otimes e_i$ , where  $i, j \in I(m|n)$  satisfy  $i \leq j < 0$ ,  $i < 0 < j$ , or  $0 < i < j$  (cf. Remark 3.4 (4)). They are in bijection with the hook tableaux of shape  $\lambda = (2)$ :

$$\begin{array}{|c|c|} \hline i & j \\ \hline \end{array}$$

This is compatible with the identification  $S^2(\mathbb{C}^{m|n}) \cong L(\mathfrak{gl}(m|n), 2\delta_1)$ .

*Remark 3.9.* We sketch below a more standard argument for the standard Schur duality on  $(\mathbb{C}^n)^{\otimes d}$ , which emphasizes the decomposition of  $W$  as a  $\mathfrak{g}$ -module.

Given a composition (or a partition)  $\mu$  of  $d$  of length  $\leq n$ , we denote by  $W_\mu$  the  $\mu$ -weight space of the  $\mathfrak{gl}(n)$ -module. Clearly  $W_\mu$  has a basis

$$e_{i_1} \otimes \cdots \otimes e_{i_d}, \quad \text{where } \{i_1, \dots, i_d\} = \underbrace{\{1, \dots, 1\}}_{\mu_1}, \dots, \underbrace{\{n, \dots, n\}}_{\mu_n}. \quad (3.2)$$

As a  $(\mathfrak{gl}(n), \mathfrak{S}_d)$ -module,

$$W \cong \bigoplus_{\lambda} L(\lambda) \otimes U^\lambda,$$

where  $U^\lambda := \text{Hom}_{\mathfrak{gl}(n)}(L(\lambda), W) \cong W_\lambda^{n^+}$  (the space of highest weight vectors in  $W$  of weight  $\lambda$ ). Only  $\lambda \in \mathcal{P}(d, n)$  can be as highest weights for  $\mathfrak{gl}(n)$  which appears in the decomposition of  $(\mathbb{C}^n)^{\otimes d}$ , and every such  $\lambda$  indeed appears as it appears as a summand of  $\wedge^{\lambda'_1}(\mathbb{C}^n) \otimes \wedge^{\lambda'_2}(\mathbb{C}^n) \otimes \cdots$ .

Note that  $\mathcal{P}(d, n)$  has two interpretations: one as the polynomial weights for  $\mathfrak{gl}(n)$  and the other as compositions of  $d$ . A remarkable fact is that the partial order on weights induced by the positive roots of  $\mathfrak{gl}(n)$  coincides with the dominance partial order  $\succeq$  on compositions.

Since  $W = \bigoplus_{\mu} W_\mu = \bigoplus_{\mu, \lambda: \lambda \succeq \mu} L(\lambda)_\mu \otimes U^\lambda$ , we conclude that, as  $\mathfrak{S}_d$ -module,

$$W_\mu \cong \bigoplus_{\lambda \succeq \mu} \dim L(\lambda)_\mu U^\lambda.$$

On the other hand,  $\mathfrak{S}_d$  acts on the basis of  $W_\mu$  (3.2) transitively and the stabilizer of the basis element  $e_1^{\otimes \mu_1} \otimes \cdots \otimes e_n^{\otimes \mu_n}$  is the Young subgroup  $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_n}$ . Therefore we have

$$W_\mu \cong \text{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_d} 1_\mu = \bigoplus_{\lambda \succeq \mu} K_{\lambda\mu} S^\lambda,$$

where  $K_{\lambda\mu}$  is the Kostka number which satisfies  $K_{\lambda\lambda} = 1$ .

By the double centralizer property,  $U^\lambda$  has to be an irreducible  $\mathfrak{S}_d$ -module for each  $\lambda$ . We compare the above two interpretations of  $W_\mu$  in the special case when  $\mu$  is dominant (i.e., a partition). One by one downward along the dominance order, this provides the identification  $U^\mu = S^\mu$  for every  $\mu$ , and moreover, we obtain the well-known equality

$$\dim L(\lambda)_\mu = K_{\lambda\mu}.$$

**Definition 3.10.** Let  $\mu \in \mathfrak{h}^*$ . The *degree of atypicality* of  $\mu$  is the maximum cardinality of a set of pairwise orthogonal  $\alpha \in \Phi_{\mathfrak{I}}^+$  such that  $(\mu + \rho, \alpha) = 0$ .

The following observation seems to be new.

**Proposition 3.11.** *Let  $\lambda \in \mathcal{P}(m|n)$ . The degree of atypicality of  $\lambda^{\mathfrak{b}}$  equals the minimal number  $i$  such that  $\lambda$  contains a partition of rectangular shape  $((m-i)^{n-i})$  with  $0 \leq i \leq \min\{m, n\}$ .*

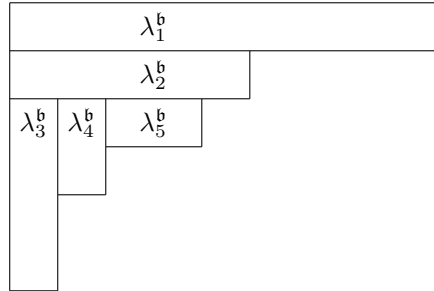
As a corollary,  $\lambda^{\mathfrak{b}}$  is typical if and only if  $\lambda_m \geq n$ , as observed earlier in [BeR].

**Exercise 3.12.** Prove Proposition 3.11.

*Remark 3.13.* There exists another interesting generalization of Schur duality for the queer Lie superalgebras due to Sergeev [Sv1].

### 3.4

Assume that the  $\epsilon\delta$ -sequence associated to a Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{gl}(m|n)$  (cf. Section 1.5) is given by a sequence of  $d_1$   $\delta$ 's,  $e_1$   $\epsilon$ 's,  $d_2$   $\delta$ 's,  $e_2$   $\epsilon$ 's,  $\dots$ ,  $d_r$   $\delta$ 's,  $e_r$   $\epsilon$ 's (possibly  $d_1 = 0$  or  $e_r = 0$ ). Associated to an  $(m|n)$ -hook Young diagram  $\lambda$ , we define a weight  $\lambda^{\mathfrak{b}} \in \mathfrak{h}^*$  as follows. Take the first  $d_1$  row numbers of  $\lambda$  as the coefficients of the first  $d_1$   $\delta$ 's. Denote by  $\lambda^1$  the Young diagram obtained from  $\lambda$  with the first  $d_1$  rows of  $\lambda$  removed. Take the first  $e_1$  column numbers of  $\lambda^1$  as the coefficients of the first  $e_1$   $\epsilon$ 's. Denote by  $\lambda^2$  the Young diagram obtained from  $\lambda^1$  with the first  $e_1$  columns of  $\lambda^1$  removed. Then take the first  $d_2$  row numbers of  $\lambda^2$  as the coefficients of the following  $d_2$   $\delta$ 's, and so on, until we reach the empty partition. The resulting weight is denoted by  $\lambda^{\mathfrak{b}}$ . Below is an example of  $\lambda_{\mathfrak{b}}$  for  $d_1 = e_1 = 2$  and  $d_2 = 1$ .



Recall that by convention  $L(\lambda^{\mathfrak{b}})$  is a highest weight  $\mathfrak{g}$ -module with respect to the standard Borel subalgebra  $\mathfrak{b}^{\text{st}}$ .

**Theorem 3.14.** [CLW] *Let  $\lambda$  be an  $(m|n)$ -hook partition. Let  $\mathfrak{b}$  be an arbitrary Borel subalgebra of  $\mathfrak{gl}(m|n)$ . Then, the  $\mathfrak{b}$ -highest weight of the simple  $\mathfrak{gl}(m|n)$ -module  $L(\lambda^{\mathfrak{b}})$  is  $\lambda^{\mathfrak{b}}$ .*

*Proof.* Let us consider an odd reflection which changes a Borel subalgebra  $\mathfrak{b}_1$  to  $\mathfrak{b}_2$ . Assume the theorem holds for  $\mathfrak{b}_1$ . We observe by Lemma 2.8 that the statement of the theorem for  $\mathfrak{b}_2$  follows from the validity of the theorem for  $\mathfrak{b}_1$ . The statement of the theorem is apparently consistent with a change of Borel subalgebras induced from a real reflection, and all Borel subalgebras of  $\mathfrak{gl}(m|n)$

are linked by a sequence of real and odd reflections. Hence, once we know the theorem holds for one particular Borel subalgebra, it holds for all. We finally note that the theorem holds for the standard Borel subalgebra  $\mathfrak{b}^{\text{st}}$ , which corresponds to the sequence of  $m$   $\delta$ 's followed by  $n$   $\epsilon$ 's. It is clear that  $\lambda^{\mathfrak{b}^{\text{st}}} = \lambda^{\natural}$ .  $\square$

*Remark 3.15.* A variant of Theorem 3.14 holds for Lie superalgebras of type  $\mathfrak{osp}$ , see [CLW].

**Example 3.16.** Let us describe the highest weights in Theorem 3.14 with respect to the three Borel subalgebras of special interest.

1. As seen above,  $\lambda^{\mathfrak{b}^{\text{st}}} = \lambda^{\natural}$ .
2. If we take the opposite Borel subalgebra  $\mathfrak{b}^{\text{op}}$  corresponding to a sequence of  $n$   $\epsilon$ 's followed by  $m$   $\delta$ 's, then  $\lambda^{\mathfrak{b}^{\text{op}}} = (\lambda')^{\natural}$ , where  $\natural$  applies to a  $(n|m)$ -hook partition (instead of  $(m|n)$ -hook partition). This is as expected from Schur-Sergeev duality Theorem 3.3.
3. In case when  $|m - n| \leq 1$ , we may take a Borel subalgebra  $\mathfrak{b}^{\circ}$  whose simple roots are all odd (or equivalently, the corresponding  $\epsilon\delta$ - sequence is alternating between  $\epsilon$  and  $\delta$ ):

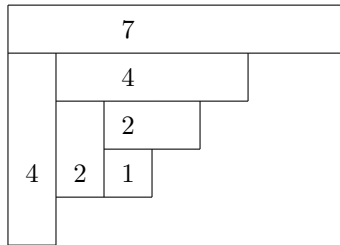


In this case Theorem 3.14 reduces to [CW3, Theorem 7.1], which states the coefficients of  $\delta$  and  $\epsilon$  in  $\lambda^{\mathfrak{b}^{\circ}}$  are given by the *modified Frobenius coordinates*  $(p_i|q_i)_{i \geq 1}$  of the partition  $\lambda$  (respectively,  $\lambda'$ ), when the first simple root is of the form  $\delta - \epsilon$  (respectively,  $\epsilon - \delta$ ). Here by modified Frobenius coordinates we mean

$$p_i = \max\{\lambda_i - i + 1, 0\}, \quad q_i = \max\{\lambda'_i - i, 0\}$$

so that  $\sum_i (p_i + q_i) = |\lambda|$ ; “modified” here refers to a shift by 1 from the  $p_i$  coordinates defined in [Mac, Chapter 1, Page 3].

For  $\lambda = (7, 5, 4, 3, 1)$ , we have  $(p_1, p_2, p_3|q_1, q_2, q_3) = (7, 4, 2|4, 2, 1)$ .



**Exercise 3.17.** Let  $\lambda = (7, 2, 2, 1, 1)$  be an  $(1|2)$ -hook partition (i.e.  $\lambda_2 \leq 2$ ). Write down all the extremal weights with respect to the six Borel subalgebras of  $\mathfrak{gl}(1|2)$  from Example 1.8 for  $L(\mathfrak{gl}(1|2), \lambda^{\natural})$ .

## 4 Howe duality for Lie superalgebras of type $\mathfrak{gl}$

### 4.1 The $(\mathfrak{gl}(m|n), \mathfrak{gl}(d))$ -Howe duality

Let  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  be a vector superspace. The symmetric algebra  $S(V)$  is understood as  $S(V) = S(V_{\bar{0}}) \otimes \wedge(V_{\bar{1}})$ . It follows that  $S^k(V) = \bigoplus_{\ell=0}^k S^\ell(V_{\bar{0}}) \otimes \wedge^{k-\ell}(V_{\bar{1}})$ .

The commuting action of  $\mathfrak{gl}(m|n)$  and  $\mathfrak{gl}(d)$  on the super space  $\mathbb{C}^d \otimes \mathbb{C}^{m|n}$  induces a corresponding commuting action on its symmetric algebra  $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ .

**Theorem 4.1.** [H1, CW1, Sv2]

1. The images of  $U(\mathfrak{gl}(d))$  and  $U(\mathfrak{gl}(m|n))$  in  $\text{End}(S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}))$  satisfy the double centralizer property.
2. As a  $(\mathfrak{gl}(d), \mathfrak{gl}(m|n))$ -module,

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) = \sum_{\lambda \in \mathcal{P}(m|n), \ell(\lambda) \leq d} L_d(\lambda) \otimes L_{m|n}(\lambda^\natural).$$

Here and below we use subscripts  $m|n$  and  $d$  to indicate the (super)algebras under consideration.

The part (2) of the theorem is equivalent to

$$S^k(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) = \sum_{\substack{\lambda \vdash k \\ \lambda_{m+1} \leq n, \ell(\lambda) \leq d}} L_d(\lambda) \otimes L_{m|n}(\lambda^\natural), \quad k \in \mathbb{N}.$$

*Proof.* Let us prove the equivalent version for  $S^k(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$  for (2). The argument below is well-known (see [H2]).

By the definition of the  $k$ -th supersymmetric algebra we have

$$S^k(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \cong \left( (\mathbb{C}^d)^{\otimes k} \otimes (\mathbb{C}^{m|n})^{\otimes k} \right)^{\Delta_k},$$

where  $\Delta_k$  is the diagonal subgroup of  $\mathfrak{S}_k \times \mathfrak{S}_k$  and  $(-)^{\Delta_k}$  denotes the  $\Delta_k$ -invariant subspace. By Theorem 3.3 we have therefore

$$\begin{aligned} S^k(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) &\cong \left( \left( \sum_{|\mu|=k} L_d(\mu) \otimes S^\mu \right) \otimes \left( \sum_{|\lambda|=k} L_{m|n}(\lambda^\natural) \otimes S^\lambda \right) \right)^{\Delta_k} \\ &\cong \sum_{|\lambda|=|\mu|=k} (L_d(\mu) \otimes L_{m|n}(\lambda^\natural)) \otimes (S^\mu \otimes S^\lambda)^{\Delta_k} \\ &\cong \sum_{|\lambda|=k} L_d(\lambda) \otimes L_{m|n}(\lambda^\natural). \end{aligned}$$

The last equality follows since the Specht module  $S^\lambda$  is self-contragredient.

Part (1) follows from the decomposition in (2).  $\square$

*Remark 4.2.* In case  $n = 0$ , we have  $\lambda^\natural = \lambda$ , and Theorem 4.1 reduces to the classical  $(\mathfrak{gl}(d), \mathfrak{gl}(m))$ -Howe duality on a symmetric tensor space (see [H2, Section 2.1.2]).

In case  $m = 0$ , we have  $\lambda^\natural = \lambda'$ , and Theorem 4.1 reduces to the classical  $(\mathfrak{gl}(d), \mathfrak{gl}(n))$ -Howe duality on an exterior space  $\wedge(\mathbb{C}^d \otimes \mathbb{C}^n)$  (see [H2, Section 4.1.1]).

*Remark 4.3.* Similarly, using Sergeev's duality for queer Lie superalgebra  $\mathfrak{q}(n)$  (see Remark 3.13), we can establish a  $(\mathfrak{q}(m), \mathfrak{q}(n))$ -Howe duality, see [CW2, Naz, Sv2].

## 4.2 Hook Schur polynomials as $\mathfrak{gl}(m|n)$ -characters

Let  $\underline{x} = (x_1, \dots, x_m)$  and  $\underline{y} = (y_1, \dots, y_n)$  be formal variables. For two partitions  $\lambda$  and  $\mu$  with  $\mu \subseteq \lambda$  denote by  $s_{\lambda/\mu}(\underline{x})$  the skew Schur polynomial associated to the skew Young diagram  $\lambda/\mu$  such that  $\ell(\lambda) \leq m$ . The skew Schur polynomials specialize to the (usual) Schur polynomials for  $\mu = \emptyset$ :  $s_{\lambda/\emptyset}(\underline{x}) = s_\lambda(\underline{x})$ . We refer to Macdonald [Mac] for more on symmetric functions.

The *hook Schur polynomials*  $hs_\lambda(\underline{x}, \underline{y})$  in variables  $\underline{x}$  and  $\underline{y}$  are defined as

$$hs_\lambda(\underline{x}, \underline{y}) = \sum_{\mu \subseteq \lambda} s_\mu(\underline{x}) s_{\lambda'/\mu'}(\underline{y}), \quad \lambda \in \mathcal{P}(m|n). \quad (4.1)$$

This is one of the several equivalent definitions, see [BeR]. We should compare this definition with the classical identity for (skew) Schur functions

$$s_\lambda(\underline{x}, \underline{y}) = \sum_{\mu \subseteq \lambda} s_\mu(\underline{x}) s_{\lambda/\mu}(\underline{y}). \quad (4.2)$$

The character of the  $\mathfrak{gl}(m|n)$ -module  $L_{m|n}(\lambda^\natural)$  is by definition the trace of the action of the diagonal matrix  $\text{diag}(x_1, \dots, x_m; y_1, \dots, y_n)$  on  $L_{m|n}(\lambda^\natural)$ . The following theorem was obtained in [BeR], where the notion of hook Schur polynomials was formulated, and it is actually equivalent to the combinatorial formula given in Corollary 3.7. We offer a different proof below using Howe duality (cf. [CL1]).

**Theorem 4.4.** *Let  $\lambda \in \mathcal{P}(m|n)$ . The character of  $L_{m|n}(\lambda^\natural)$  is given by*

$$\text{ch}L_{m|n}(\lambda^\natural) = hs_\lambda(\underline{x}, \underline{y}).$$

*Proof.* Let  $\underline{u} = (u_1, \dots, u_d)$  be another set of formal variables. It is well known that  $s_\lambda(\underline{u})$  is the character of the  $\mathfrak{gl}(d)$ -module  $L_d(\lambda)$ , i.e. the trace of the diagonal matrix  $\text{diag}(u_1, \dots, u_d)$ . Thus, comparing the characters of both sides of Theorem 4.1 (2), we obtain the following combinatorial identity:

$$\sum_{\lambda \in \mathcal{P}(m|n), \ell(\lambda) \leq d} s_\lambda(\underline{u}) \text{ch}L_{m|n}(\lambda^\natural) = \prod_{i \leq m, j \leq n, k \leq d} (1 - x_i u_k)^{-1} (1 + y_j u_k). \quad (4.3)$$

Let  $X = (x_1, x_2, \dots)$  and  $Y = (y_1, y_2, \dots)$  denote two sets of infinitely many variables. Recall the classical Cauchy identity

$$\sum_{\ell(\lambda) \leq d} s_\lambda(\underline{u}) s_\lambda(X) = \prod_{k \leq d} \prod_{i \geq 1} (1 - x_i u_k)^{-1}, \quad (4.4)$$

which readily follows from Howe duality Theorem 4.1 (by setting  $n = 0$  and letting  $m \rightarrow \infty$ ).

Consider the involution  $\omega$  on the ring of symmetric functions in the set of variables  $(x_{m+1}, x_{m+2}, \dots)$ . Applying  $\omega$  to (4.4), replacing  $(x_{m+1}, x_{m+2}, \dots)$  by  $Y$ , and using (4.2), we obtain

$$\sum_{\ell(\lambda) \leq d} s_\lambda(\underline{u}) h s_\lambda(\underline{x}, Y) = \prod_{i \leq m, k \leq d, j \geq 1} (1 - x_i u_k)^{-1} (1 + y_j u_k).$$

Now setting  $y_{n+1} = y_{n+2} = \dots = 0$  in the above identity, we obtain that

$$\sum_{\ell(\lambda) \leq d} s_\lambda(\underline{u}) h s_\lambda(\underline{x}, \underline{y}) = \prod_{i \leq m, j \leq n, k \leq d} (1 - x_i u_k)^{-1} (1 + y_j u_k).$$

By comparing with (4.3) and noting the linear independence of  $\{s_\lambda(\underline{u})\}$  for  $d = \infty$ , we have proved the theorem.  $\square$

*Remark 4.5.* It follows that

$$h s_\lambda(X, \emptyset) = s_\lambda(X), \quad h s_\lambda(\emptyset, Y) = s_{\lambda'}(Y), \quad h s_\lambda(Y, X) = h s_{\lambda'}(X, Y).$$

Both the usual Cauchy identity and its dual version are obtained from specializations of the corresponding character identity of Theorem 4.1.

### 4.3 Formulas for highest weight vectors

We let  $e^1, \dots, e^d$  denote the standard basis for the natural  $\mathfrak{gl}(d)$ -module  $\mathbb{C}^d$ , and recall that  $e_i, i \in I(m|n)$ , denote the standard basis for the natural  $\mathfrak{gl}(m|n)$ -module  $\mathbb{C}^{m|n}$ . We set

$$x_a^i := e^i \otimes e_{\bar{a}} \quad (1 \leq a \leq m); \quad \eta_b^i := e^i \otimes e_b \quad (1 \leq b \leq n). \quad (4.5)$$

We will denote by  $\mathbb{C}[\underline{x}, \underline{\eta}]$  the polynomial superalgebra generated by (4.5). The commuting actions on  $\mathbb{C}[\underline{x}, \underline{\eta}]$  of  $\mathfrak{gl}(d)$  and  $\mathfrak{gl}(m|n)$  may be realized in terms of first order differential operators (4.6) and (4.7) respectively ( $1 \leq i, i' \leq d$  and  $1 \leq s, s' \leq m; 1 \leq k, k' \leq n$ ):

$$E^{ii'} := \sum_{j=1}^m x_j^i \frac{\partial}{\partial x_j^{i'}} + \sum_{j=1}^n \eta_j^i \frac{\partial}{\partial \eta_j^{i'}}, \quad (4.6)$$

$$\sum_{j=1}^d x_s^j \frac{\partial}{\partial x_{s'}^j}, \quad \sum_{j=1}^d \eta_k^j \frac{\partial}{\partial \eta_{k'}^j}, \quad \sum_{j=1}^d x_s^j \frac{\partial}{\partial \eta_k^j}, \quad \sum_{j=1}^d \eta_k^j \frac{\partial}{\partial x_s^j}. \quad (4.7)$$

In particular, as  $(\mathfrak{gl}(d), \mathfrak{gl}(m|n))$ -module,  $S(\mathbb{C}^p \otimes \mathbb{C}^{m|n})$  is isomorphic to  $\mathbb{C}[\underline{x}, \underline{\eta}]$ . The root vectors corresponding to the simple roots of  $\mathfrak{gl}(d)$  and  $\mathfrak{gl}(m|n)$  are

$$\sum_{j=1}^m x_j^{i-1} \frac{\partial}{\partial x_j^i} + \sum_{j=1}^n \eta_j^{i-1} \frac{\partial}{\partial \eta_j^i}, \quad (4.8)$$

$$\sum_{j=1}^d x_{s-1}^j \frac{\partial}{\partial x_s^j}, \quad \sum_{j=1}^d \eta_{k-1}^j \frac{\partial}{\partial \eta_k^j}, \quad \sum_{j=1}^d x_m^j \frac{\partial}{\partial \eta_1^j}, \quad (4.9)$$

respectively.

Let  $\lambda$  be an  $(m|n)$ -hook partition of size  $k$  and of length at most  $d$ . We are looking for the joint highest weight vector (with respect to the standard Borel subalgebras), or equivalently the vector annihilated by (4.8) and (4.9), for the highest weight module  $L_d(\lambda) \otimes L_{m|n}(\lambda^\natural)$  of  $\mathfrak{gl}(d) \times \mathfrak{gl}(m|n)$  appearing in the decomposition of  $\mathbb{C}[\underline{x}, \underline{\eta}]$ . Such a vector is unique up to a scalar multiple, thanks to the multiplicity-free decomposition in Part 2 of Theorem 4.1.

For  $1 \leq r \leq m$ , define

$$\diamond_r := \det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_r^1 \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^r & x_2^r & \cdots & x_r^r \end{pmatrix}. \quad (4.10)$$

Let us assume for now that  $d > m$  (the case of  $d \leq m$  is trivially included with much simplification, in which case we will not need the definition of  $\diamond_{k,r}$  below). Under this assumption, the condition  $\lambda_{m+1} \leq n$  is no longer an empty condition. Recall  $\lambda' = (\lambda'_1, \lambda'_2, \dots)$  denotes the transposed partition of  $\lambda$ . We have  $d \geq \lambda'_1 \geq \lambda'_2 \geq \dots$  and  $m \geq \lambda'_{n+1}$ .

Recall that the *row determinant* of an  $r \times r$  matrix with possibly non-commuting entries  $A = [a_i^j]$  is defined to be

$$\text{rdet} A = \sum_{\sigma \in \mathfrak{S}_r} (-1)^{\ell(\sigma)} a_1^{\sigma(1)} a_2^{\sigma(2)} \cdots a_r^{\sigma(r)}.$$

For  $m \leq r \leq d$ , we introduce the following row determinant:

$$\diamond_{k,r} := \text{rdet} \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^r \\ x_2^1 & x_2^2 & \cdots & x_2^r \\ \vdots & \vdots & \cdots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^r \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \\ \vdots & \vdots & \cdots & \vdots \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \end{pmatrix}, \quad k = 1, \dots, n, \quad (4.11)$$

where the last  $(r - m)$  rows are filled with the same vector  $(\eta_k^1, \eta_k^2, \dots, \eta_k^r)$ .

*Remark 4.6.* The determinant (4.11) is always nonzero. It reduces to (4.10) when  $m = r$ , and, up to a scalar multiple, it reduces to  $\eta_k^1 \cdots \eta_k^r$  when  $m = 0$ .

Now let  $r$  be specified by the conditions  $\lambda'_r > m$  and  $\lambda'_{r+1} \leq m$ . Denote by  $\lambda_{\leq r}$  the subdiagram of the Young diagram  $\lambda$  which consists of the first  $r$  columns of  $\lambda$ , i.e., the columns of length  $> m$ . A formula for the highest weight vector associated to the Young diagram  $\lambda_{\leq r}$  is given by the following lemma.

**Lemma 4.7.** *The vector  $\prod_{k=1}^r \diamond_{k, \lambda'_k}$  is a highest weight vector for the highest weight module  $L_d(\lambda_{\leq r}) \otimes L_{m|n}(\lambda_{\leq r}^\natural)$  in the decomposition of  $\mathbb{C}[\underline{x}, \underline{\eta}]$ .*

*Proof.* Set  $\diamond = \prod_{k=1}^r \diamond_{k, \lambda'_k}$ , and  $\mu = \lambda_{\leq r}$ . We verify the following.

- (i)  $\diamond$  has weight  $(\mu, \mu^\natural)$  with respect to the action of  $\mathfrak{gl}(d) \times \mathfrak{gl}(m|n)$ .
- (ii)  $\diamond$  is non-zero.
- (iii)  $\diamond$  is annihilated by the operators in (4.8).

By Exercise 4.8 below, (i)-(iii) imply that  $\diamond$  is also a highest weight vector for  $\mathfrak{gl}(m|n)$ .  $\square$

**Exercise 4.8.** Assume that  $v \in \mathbb{C}[\underline{x}, \underline{\eta}]$  has weights  $\lambda$  and  $\lambda^\natural$  with respect to  $\mathfrak{gl}(d)$  and  $\mathfrak{gl}(m|n)$ , respectively. Prove that if  $v$  is a highest weight vector for  $\mathfrak{gl}(d)$ , then so is it for  $\mathfrak{gl}(m|n)$ .

**Theorem 4.9.** [CW1] Let  $\lambda$  be a Young diagram of length  $\leq d$  such that  $\lambda_{m+1} \leq n$ . Then, a highest weight vector of weight  $(\lambda, \lambda^\natural)$  in the  $\mathfrak{gl}(d) \times \mathfrak{gl}(m|n)$ -module  $\mathbb{C}[\underline{x}, \underline{\eta}]$  is given by

$$\prod_{k=1}^r \diamond_{k, \lambda'_k} \prod_{j=r+1}^{\lambda_1} \diamond_{\lambda'_j},$$

where  $r$  is defined by  $\lambda'_r > m$  and  $\lambda'_{r+1} \leq m$ .

*Proof.* Since the expression is a product of two highest weight vectors, it is a highest weight vector. Also it is easy to verify that it has the correct weight.  $\square$

## 5 Howe duality for Lie superalgebras of type $\mathfrak{osp}$

### 5.1 General ideas of dual pairs

Let  $V$  be a vector (super)space, and  $\mathcal{P}(V)$  the polynomial (super)algebra on  $V$ . Let  $\mathcal{D}(V)$  denote the Weyl (super)algebra of differential operators with polynomial coefficients on  $V$ . For example, if  $x_1, \dots, x_m, \xi_1, \dots, \xi_n$  are the even/odd coordinates on  $V$  relative to a homogeneous basis (so that  $V \cong \mathbb{C}^{m|n}$ ), then  $\mathcal{D}(V)$  is generated by the even generators  $x_i, \partial_i := \partial/\partial x_i$  and the odd generators  $\xi_j, \bar{\partial}_j := \partial/\partial \xi_j$  ( $1 \leq i \leq m, 1 \leq j \leq n$ ), subject to the only nontrivial (super) commutation relations among the generators:

$$\partial_i x_i - x_i \partial_i = 1, \quad \bar{\partial}_j \xi_j + \xi_j \bar{\partial}_j = 1.$$

Given a reductive group  $G$  acting on  $V$ , we have a multiplicity-free decomposition over  $(G, \mathcal{D}(V)^G)$ :

$$\mathcal{P}(V) = \bigoplus_{\lambda \in \mathfrak{P}} L(\lambda) \otimes U^\lambda,$$

where  $L(\lambda)$  are pairwise non-isomorphic irreducible  $G$ -modules and  $U^\lambda$  are pairwise non-isomorphic irreducible  $\mathcal{D}(V)^G$ -modules, as  $\lambda$  runs over some set  $\mathfrak{P}$  of parameters.

If there exists a generating set  $\{T_1, \dots, T_r\}$  for the (super)algebra  $\mathcal{D}(V)^G$ , such that  $\mathfrak{g}' := \mathbb{C}\text{-span}\langle T_1, \dots, T_r \rangle$  forms a Lie subalgebra of  $\mathcal{D}(V)^G$ , then the above  $U^\lambda$ 's are pairwise non-isomorphic irreducible  $\mathfrak{g}'$ -modules. We refer to [GW] for more detailed discussion.

*Remark 5.1.* For a suitable choice of  $V$  and  $G$ , there could exist a very canonical candidate for the dual partner  $\mathfrak{g}'$ , which is suggested by the First Fundamental Theorem of invariant theory [H1, GW]. We will refer to  $(G, \mathfrak{g}')$  as a *Howe dual pair* acting on  $S(V)$ .

Once  $G, \mathfrak{g}'$  are fixed, the basic problems in  $(G, \mathfrak{g}')$ -Howe duality include describing the  $\mathfrak{g}'$ -module  $U^\lambda$  explicitly (e.g. describing its highest weight if this makes sense) and determining the parameter set  $\mathfrak{P}$ .

For example, if we let  $V = (\mathbb{C}^d \otimes \mathbb{C}^n)^*$  and  $G = \mathrm{GL}(d)$ , then  $\mathcal{P}(V) = S(\mathbb{C}^d \otimes \mathbb{C}^n)$  and we can choose  $\mathfrak{g}' = \mathfrak{gl}(n)$ . This gives rise to the  $(\mathfrak{gl}(d), \mathfrak{gl}(n))$ -duality from Remark 4.2.

## 5.2

In the type  $A$  Howe dualities, we can replace the Lie group  $\mathrm{GL}(d)$  by its Lie algebra  $\mathfrak{gl}(d)$  and vice versa without loss of information. However for more general Howe dualities, it is essential to use Lie groups  $G$  as formulated in Section 5.1. Associated to a given Lie algebra  $\mathfrak{g}$  (e.g.  $\mathfrak{so}(d)$ ), one associates several different (possibly disconnected) groups  $G$ , e.g.  $\mathrm{SO}(d), \mathrm{O}(d), \mathrm{Spin}(d), \mathrm{Pin}(d)$ . The Howe duality formulation favors  $\mathrm{O}(d)$  and  $\mathrm{Pin}(d)$  in the sense of Remark 5.1.

To avoid the complication of describing the irreducible representations of different covering groups, we will choose to discuss a Howe duality involving the Lie group  $\mathrm{Sp}(d)$ , where  $d = 2\ell$  must be an even integer. In this case, the finite-dimensional simple modules over the group  $\mathrm{Sp}(d)$  and those over the Lie algebra  $\mathfrak{sp}(d)$  correspond bijectively, and it makes no difference if we use  $\mathfrak{sp}(d)$  to replace  $\mathrm{Sp}(d)$ . Besides, they admit a nice parametrization in terms of partitions. We denote the  $\mathrm{Sp}(d)$ -module corresponding to the partition  $\lambda$  by  $L(\mathrm{Sp}(d), \lambda)$ .

According to [H1], there exists a Howe dual pair  $(\mathrm{Sp}(d), \mathfrak{so}(2m))$  on  $S(\mathbb{C}^d \otimes \mathbb{C}^m)$ , and a Howe dual pair  $(\mathrm{Sp}(d), \mathfrak{sp}(2n))$  on  $\wedge(\mathbb{C}^d \otimes \mathbb{C}^n)$ . By fusing these two dual pairs together, we obtain a Howe dual pair  $(\mathrm{Sp}(d), \mathfrak{osp}(2m|2n))$  on  $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ . Let us explain the commuting actions below.

As before, we identify  $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) = \mathbb{C}[\underline{x}, \underline{\eta}]$ . The action of the Lie algebra  $\mathfrak{sp}(d)$  as the subalgebra of  $\mathfrak{gl}(d)$  on  $\mathbb{C}[\underline{x}, \underline{\eta}]$  lifts to an action of Lie group  $\mathrm{Sp}(d)$ . On the other hand, the following action of  $\mathfrak{gl}(m|n)$  on  $\mathbb{C}[\underline{x}, \underline{\eta}]$  is obtained from (4.7) by a shift of scalars on the diagonal matrices:

$$\begin{aligned} E_{is}^{xx} &= \sum_{j=1}^d x_i^j \frac{\partial}{\partial x_s^j} + \frac{d}{2} \delta_{is}, & E_{ik}^{x\eta} &= \sum_{j=1}^d x_i^j \frac{\partial}{\partial \eta_k^j}, \\ E_{ki}^{\eta x} &= \sum_{j=1}^d \eta_k^j \frac{\partial}{\partial x_i^j}, & E_{tk}^{\eta\eta} &= \sum_{j=1}^d \eta_t^j \frac{\partial}{\partial \eta_k^j} - \frac{d}{2} \delta_{ik}, \end{aligned} \quad (5.1)$$

where  $i, s = 1, \dots, m$  and  $k, t = 1, \dots, n$ . Introduce the following additional



**Theorem 5.3.** *Let  $d = 2\ell$ . As an  $(\mathrm{Sp}(d), \mathfrak{osp}(2m|2n))$ -module,*

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) = \bigoplus_{\lambda \in \mathcal{P}(m|n), \ell(\lambda) \leq \ell} L(\mathrm{Sp}(d), \lambda) \otimes L(\mathfrak{osp}(2m|2n), \lambda^{\natural} + \ell \mathbf{1}).$$

*Proof.* An element  $f \in \mathbb{C}[\underline{x}, \underline{\eta}]$  is called *harmonic*, if  $f$  is annihilated by the subalgebra  $\mathfrak{u}^+$ . The space of harmonics will be denoted by  ${}^{\mathrm{Sp}}H$  and it evidently admits an action of  $\mathrm{Sp}(d) \times \mathfrak{gl}(m|n)$ . Furthermore, since  $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$  is a completely reducible  $\mathfrak{gl}(m|n)$ -module,  $L(\mathfrak{osp}(2m|2n), \mu)^{\mathfrak{u}^+}$  is also completely reducible over  $\mathfrak{gl}(m|n)$ , for any irreducible  $\mathfrak{osp}(2m|2n)$ -module  $L(\mathfrak{osp}(2m|2n), \mu)$  that appears in  $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ . By irreducibility of  $L(\mathfrak{osp}(2m|2n), \mu)$  we must have

$$L(\mathfrak{osp}(2m|2n), \mu)^{\mathfrak{u}^+} \cong L_{m|n}(\mu).$$

So, by Theorem 5.2  $(\mathrm{Sp}(d), \mathfrak{gl}(m|n))$  forms a dual pair on the space of harmonics  ${}^{\mathrm{Sp}}H$ . Thus, proving the theorem is equivalent to establishing the following decomposition of  ${}^{\mathrm{Sp}}H$  as an  $\mathrm{Sp}(d) \times \mathfrak{gl}(m|n)$ -module:

$${}^{\mathrm{Sp}}H \cong \bigoplus_{\lambda \in \mathcal{P}(m|n), \ell(\lambda) \leq \ell} L(\mathrm{Sp}(d), \lambda) \otimes L_{m|n}(\lambda^{\natural} + \ell \mathbf{1}). \quad (5.3)$$

We first consider the limit case  $n = \infty$  with the space of harmonics denoted by  ${}^{\mathrm{Sp}}H^{\infty}$ . Here the only restriction on  $\lambda$  is  $\ell(\lambda) \leq \ell$ , and we observe that the vector given in Theorem 4.9 associated to such a partition  $\lambda$  is indeed annihilated by  $\mathfrak{u}^+$  and hence is a joint  $\mathrm{Sp}(d) \times \mathfrak{gl}(m|n)$ -highest weight vector of weight  $(\lambda, \lambda^{\natural} + \ell \mathbf{1})$ . Hence all the summands on the right hand side of (5.3) occur in the space of harmonics, and in particular, all irreducible representations of  $\mathrm{Sp}(d)$  occur. Therefore, we have established (5.3) in this case.

Now consider the finite  $n$  case. We may regard  $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \subseteq S(\mathbb{C}^d \otimes \mathbb{C}^{m|\infty})$  with compatible actions  $\mathfrak{osp}(2m|2n) \subseteq \mathfrak{osp}(2m|2\infty)$ . From the formulas of the  $\Delta$ -operators in (5.2) we see that  ${}^{\mathrm{Sp}}H \subseteq {}^{\mathrm{Sp}}H^{\infty}$ . Thus the space  ${}^{\mathrm{Sp}}H$  is obtained from  ${}^{\mathrm{Sp}}H^{\infty}$  by setting the variables  $\eta_k^i = 0$ , for  $k > n$ . However, it is clear, from the explicit formulas of the joint highest vectors in  ${}^{\mathrm{Sp}}H^{\infty}$ , that when setting the variables  $\eta_k^i = 0$ , for  $k > n$ , precisely those vectors corresponding to  $(m|n)$ -hook partitions will survive. This completes the proof.  $\square$

### 5.3 The Howe dual pair $(\mathrm{Sp}(d), \mathfrak{c}_{\infty})$

Set  $d = 2\ell$ . Let  $\mathbb{C}^{\infty}$  be the vector space over  $\mathbb{C}$  with basis  $\{e_i \mid i \in \mathbb{Z}\}$ . Let  $\mathfrak{gl}_{\infty}$  denote the Lie algebra consisting of matrices  $(a_{ij})_{i,j \in \mathbb{Z}}$  with finitely many non-zero  $a_{ij}$ 's, and  $\mathbb{C}^{\infty}$  is the natural  $\mathfrak{gl}_{\infty}$ -module. Then  $\mathfrak{gl}_{\infty}$  has a linear basis given by the matrix units  $E_{ij}$  ( $i, j \in \mathbb{Z}$ ). Denote by  $\widehat{\mathfrak{gl}}_{\infty} = \mathfrak{gl}_{\infty} \oplus \mathbb{C}K$  the central extension of  $\mathfrak{gl}_{\infty}$  by a one-dimensional center  $\mathbb{C}K$  given by the 2-cocycle

$$\tau(A, B) := \mathrm{Tr}([J, A]B),$$

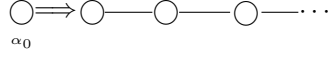
where  $J = \sum_{i \leq 0} E_{ii}$ . The Lie algebra  $\widehat{\mathfrak{gl}}_{\infty}$  admits a  $\mathbb{Z}$ -grading  $\widehat{\mathfrak{gl}}_{\infty} = \bigoplus_{i \in \mathbb{Z}} \widehat{\mathfrak{gl}}_{\infty, \pm i}$  by letting  $\deg E_{ij} = j - i$  and  $\deg K = 0$ , and this induces a triangular decomposition of  $\widehat{\mathfrak{gl}}_{\infty}$ , with  $\widehat{\mathfrak{gl}}_{\infty, \pm} = \bigoplus_{i > 0} \widehat{\mathfrak{gl}}_{\infty, \pm i}$ .

**Exercise 5.4.** Show that the cocycle  $\tau$  on  $\mathfrak{gl}_\infty$  is a coboundary and hence  $\widehat{\mathfrak{gl}}_\infty$  is isomorphic to  $\mathfrak{gl}_\infty \oplus \mathbb{C}K$  as Lie algebras.

Let  $\bar{\mathfrak{c}}_\infty$  be the subalgebra of  $\mathfrak{gl}_\infty$  preserving the following bilinear form on  $\mathbb{C}^\infty$ :

$$(e_i | e_j) = (-1)^i \delta_{i,1-j}, \quad i, j \in \mathbb{Z}.$$

Let  $\mathfrak{c}_\infty = \bar{\mathfrak{c}}_\infty \oplus \mathbb{C}K$  be the central extension of  $\bar{\mathfrak{c}}_\infty$  determined by the restriction of the two-cocycle  $\tau$  above. Then  $\mathfrak{c}_\infty$  is an infinite-rank affine Kac-Moody algebra with Dynkin diagram as follows:



The Lie algebra  $\mathfrak{c}_\infty$  has a natural triangular decomposition induced from  $\widehat{\mathfrak{gl}}_\infty$ :

$$\mathfrak{c}_\infty = \mathfrak{c}_{\infty,-} \oplus \mathfrak{h} \oplus \mathfrak{c}_{\infty,+},$$

where the Cartan subalgebra of  $\mathfrak{c}_\infty$  is spanned by  $K$  and

$$\tilde{E}_i := E_{ii} - E_{1-i,1-i}, \quad i \in \mathbb{N}.$$

Let  $\epsilon_i \in \mathfrak{h}^*$  be so that  $\langle \epsilon_i, \tilde{E}_j \rangle = \delta_{ij}$ , for  $i, j \in \mathbb{N}$ . For  $i \in \mathbb{Z}_+$ , we denote by  $\Lambda_i^\epsilon$  the  $i$ -th fundamental weight for  $\mathfrak{c}_\infty$ . Then  $\Lambda_0^\epsilon \in \mathfrak{h}^*$  is determined by  $\langle \Lambda_0^\epsilon, \tilde{E}_i \rangle = 0$  for  $i \in \mathbb{N}$  and  $\langle \Lambda_0^\epsilon, K \rangle = 1$ , and

$$\Lambda_i^\epsilon = \Lambda_0^\epsilon + \epsilon_1 + \dots + \epsilon_i, \quad i \geq 1.$$

Let  $L(\mathfrak{c}_\infty, \mu)$  denote the irreducible highest weight module with respect to the Borel subalgebra  $\mathfrak{h} \oplus \mathfrak{c}_{\infty,+}$  of highest weight  $\mu$ .

**Theorem 5.5.** [Wa, Theorem 3.4] *Let  $d = 2\ell$ . There exists a Howe dual pair  $(\mathrm{Sp}(d), \mathfrak{c}_\infty)$  acting on the fermionic Fock space  $\mathfrak{F}^\ell$  of  $\ell$  pairs of complex fermions. Moreover, as an  $(\mathrm{Sp}(d), \mathfrak{c}_\infty)$ -module,*

$$\mathfrak{F}^\ell \cong \bigoplus_{\ell(\lambda) \leq \ell} L(\mathrm{Sp}(d), \lambda) \otimes L(\mathfrak{c}_\infty, \Lambda^\epsilon(\lambda)), \quad (5.4)$$

where  $\Lambda^\epsilon(\lambda) := \frac{d}{2} \Lambda_0^\epsilon + \sum_{k \geq 1} \lambda'_k \epsilon_k = \sum_{k=1}^{\frac{d}{2}} \Lambda_{\lambda_k}^\epsilon$ .

As all we need later on is the combinatorial identity (5.5) below, we will skip the precise definitions of the fermionic Fock space  $\mathfrak{F}^\ell$  and the commuting actions of  $\mathrm{Sp}(d)$  and  $\mathfrak{c}_\infty$  on  $\mathfrak{F}^\ell$  (see [Wa] for detail).

Recalling (4.6) we set  $\tilde{E}^i := E^{ii} - E^{d-i+1, d-i+1}$ , for  $i = 1, \dots, \frac{d}{2}$ . Computing the trace of  $\prod_{n \in \mathbb{N}} x_n^{\tilde{E}^n} \prod_{i=1}^{\frac{d}{2}} z_i^{\tilde{E}^i}$  on both sides of (5.4), we have

$$\prod_{i=1}^{\frac{d}{2}} \prod_{n \in \mathbb{N}} (1 + x_n z_i)(1 + x_n z_i^{-1}) = \sum_{\ell(\lambda) \leq \ell} \mathrm{ch} L(\mathrm{Sp}(d), \lambda) \mathrm{ch} L(\mathfrak{c}_\infty, \Lambda^\epsilon(\lambda)). \quad (5.5)$$

## 5.4 Irreducible $\mathfrak{c}_\infty$ -characters

Let  $W$  (respectively,  $W_0$ ) be the Weyl group of  $\mathfrak{c}_\infty$  (respectively, of the Levi subalgebra of  $\mathfrak{c}_\infty$  corresponding to the removal of the simple root  $\alpha_0$ ). Let  $W_k^0$  be the set of the minimal length representatives of the right coset space  $W_0 \backslash W$  of length  $k$ . Then it is standard to write  $W = W_0 W^0$  with  $W^0 = \bigsqcup_{k \geq 0} W_k^0$ . It follows that, for each  $w \in W_k^0$ , we may find a partition  $\lambda_w = ((\lambda_w)_1, (\lambda_w)_2, \dots)$  such that

$$w(\Lambda^\mathfrak{c}(\lambda) + \rho) - \rho = \frac{d}{2} \Lambda_0^\mathfrak{c} + \sum_{j > 0} (\lambda_w)_j \epsilon_j. \quad (5.6)$$

The following is obtained by applying the Kostant homology formula for integrable modules over Kac-Moody algebras and the Euler-Poincaré principle (cf. [CKW, Proposition 2.5]).

**Proposition 5.6.** *We have the following character formula:*

$$\text{ch}L(\mathfrak{c}_\infty, \Lambda^\mathfrak{c}(\lambda)) = \frac{1}{\prod_{1 \leq i < j} (1 - x_i x_j)} \sum_{k=0}^{\infty} \sum_{w \in W_k^0} (-1)^k s_{\lambda_w}(x_1, x_2, \dots).$$

**Exercise 5.7.** Prove the existence of  $\lambda_w$  for each  $w \in W^0$  in (5.6).

## 5.5 The irreducible $\mathfrak{osp}(2m|2n)$ -characters

The following character formula for  $L(\mathfrak{osp}(2m|2n), \lambda^\natural + \ell \mathbf{1})$  in a different form was first obtained in [CZ2]. The proof here follows [CKW] and is simpler.

**Theorem 5.8.** *For  $\lambda \in \mathcal{P}(m|n)$  such that  $\ell(\lambda) \leq \ell$ , we have*

$$\text{ch}L(\mathfrak{osp}(2m|2n), \lambda^\natural + \ell \mathbf{1}) = \left( \frac{y_1 \cdots y_m}{x_1 \cdots x_n} \right)^\ell \frac{\prod_{1 \leq i \leq m} \prod_{1 \leq s \leq n} (1 + y_i x_s)}{\prod_{1 \leq i < j \leq m} \prod_{1 \leq s \leq t \leq n} (1 - y_i y_j)(1 - x_s x_t)} \sum_{k=0}^{\infty} \sum_{w \in W_k^0} (-1)^k h_{s_{\lambda_w}}(\underline{x}, \underline{y}).$$

*Proof.* Computing the trace of the operator  $\prod_{i,j} y_i^{\tilde{E}_i} x_j^{\tilde{E}_j} \prod_{k=1}^{\frac{d}{2}} z_k^{\tilde{E}_k}$  on both sides of the isomorphism in Theorem 5.3, where  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , we obtain

$$\begin{aligned} & \prod_{k=1}^{\frac{d}{2}} \prod_{i,j} \frac{(1 + x_i z_k^{-1})(1 + x_i z_k)}{(1 - y_j z_k^{-1})(1 - y_j z_k)} \\ &= \left( \frac{y_1 \cdots y_m}{x_1 \cdots x_n} \right)^{-\ell} \sum_{\substack{\lambda \in \mathcal{P}(m|n) \\ \ell(\lambda) \leq \ell}} \text{ch}L(\text{Sp}(d), \lambda) \text{ch}L(\mathfrak{osp}(2m|2n), \lambda^\natural + \ell \mathbf{1}). \end{aligned} \quad (5.7)$$

Replacing  $\text{ch}L(\mathfrak{c}_\infty, \Lambda^\mathfrak{c}(\lambda))$  in (5.5) by the expression in Proposition 5.6, we obtain an identity of symmetric functions in variables  $x_1, x_2, \dots$ . Next, we replace  $x_{n+i}$

by  $y_i$  for  $i \in \mathbb{N}$ , and then apply to this identity the involution on the ring of symmetric functions in  $y_1, y_2, \dots$ . Finally, we put  $y_j = 0$  for  $j \geq m + 1$ . Under the composition of those maps, we obtain a new identity which shares the same left hand side as (5.7). Comparing the right hand sides of this new identity and of (5.7), we obtain the result thanks to the linear independence of  $\text{ch}L(\text{Sp}(d), \lambda)$ .  $\square$

*Remark 5.9.* The above irreducible character formula can be understood as an alternating sum of the characters of parabolic Vermas over a certain infinite Weyl group. A similar observation was made for type  $A$  in [CZ1, Corollary 4.15].

The Kostant  $\mathfrak{u}$ -cohomology groups for these  $\mathfrak{g}$ -modules can be further computed and described in terms of the infinite Weyl groups, and in particular, they are multiplicity-free as modules over Levi subalgebras [CKW] (also cf. [CK]). In other words, the corresponding Kazhdan-Lusztig polynomials via  $\mathfrak{u}$ -cohomology in the sense of Vogan [Vo] are monomials.

## 6 Super duality

The goal of the remaining part of the lectures is to formulate precisely a direct connection between representation theories of Lie algebras and Lie superalgebras, following [CLW] (cf. [CWZ, CW4, CL2]). This is in part supported by the following.

1. The  $\mathfrak{gl}(1|1)$ -principal block can be understood in terms of  $A_\infty$ -quivers, which is classical in its own way.
2. There exist intimate connections between finite-dimensional classical Lie superalgebras and infinite Weyl groups (cf. Remark 5.9).
3. By a purely algebraic approach [Br1], Brundan showed that the category of finite-dimensional  $\mathfrak{gl}(m|n)$ -modules categorifies the Fock space  $\wedge^m \mathbb{V} \otimes \wedge^n \mathbb{V}^*$ , where  $\mathbb{V}$  denotes the natural  $U_q(\mathfrak{gl}(\infty))$ -module.
4. The Brundan-Kazhdan-Lusztig polynomials for some suitable parabolic category  $\mathcal{O}$  of super type  $A$  are shown in [CWZ, CW4] to coincide with the usual Kazhdan-Lusztig polynomials of type  $A$ . For the so-called polynomial representations this was already observed in [CZ1]. Moreover, the super duality conjecture [CW4] (generalizing [CWZ]) on a category equivalence between Lie superalgebras and Lie algebras of type  $A$  has been established recently in [CL2].

### 6.1 Parabolic category $\mathcal{O}$

Assume that  $\mathfrak{k}$  is a (possibly Kac-Moody, and or possibly infinite rank) Lie (super)algebra, with Cartan subalgebra  $\mathfrak{h}$  and root system  $\Phi$ . Let  $I$  be a set of simple roots, and  $\mathfrak{b}$  the corresponding Borel subalgebra. Let  $Y$  be a subset of  $I$  and  $\mathfrak{l}_Y$  be the associated Levi subalgebra of  $\mathfrak{k}$ . Let  $\mathfrak{k} = \mathfrak{u}^- \oplus \mathfrak{l}_Y \oplus \mathfrak{u}^+$  be the generalized triangular decomposition. Denote by  $P$  the weight lattice for  $\mathfrak{k}$ , and by  $P_Y \subseteq P$  the subset of weights which are  $Y$ -dominant.

Given  $\lambda \in \mathfrak{h}^*$ , we denote by  $L(\mathfrak{l}_Y, \lambda)$  the irreducible  $\mathfrak{l}_Y$ -module of highest weight  $\lambda$  with respect to  $\mathfrak{l}_Y \cap \mathfrak{b}$ , which extends to a  $(\mathfrak{l}_Y + \mathfrak{u}^+)$ -module with a trivial action of  $\mathfrak{u}^+$ . Define the parabolic Verma  $\mathfrak{k}$ -module

$$\Delta(\lambda) := U(\mathfrak{k}) \otimes_{U(\mathfrak{l}_Y + \mathfrak{u}^+)} L(\mathfrak{l}_Y, \lambda).$$

Let  $L(\lambda)$  be the irreducible quotient  $\mathfrak{k}$ -module of  $\Delta(\lambda)$ .

Let  $\mathcal{O} = \mathcal{O}(\mathfrak{k}, Y, P_Y)$  be the category of  $\mathfrak{g}$ -modules  $M$  such that  $M$  is a semisimple  $\mathfrak{h}$ -module with finite-dimensional weight subspaces  $M_\gamma$ ,  $\gamma \in \mathfrak{h}^*$ , satisfying

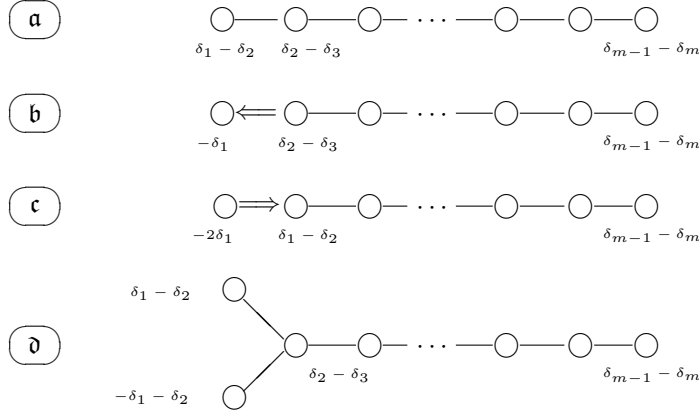
- (i)  $M$  decomposes over  $\mathfrak{l}_Y$  into a direct sum of  $L(\mathfrak{l}_Y, \mu)$  for  $\mu \in P_Y$ .
- (ii) There exist finitely many weights  $\lambda_1, \lambda_2, \dots, \lambda_k \in P_Y$  (depending on  $M$ ) such that if  $\gamma$  is a weight in  $M$ , then  $\gamma \in \lambda_i - \sum_{\alpha \in I} \mathbb{Z}_+ \alpha$ , for some  $i$ .

The choice of  $P_Y$  should be natural and general enough so that  $L(\lambda), \Delta(\lambda) \in \mathcal{O}(\mathfrak{k}, Y, P_Y)$  for every  $\lambda \in P_Y$ . This will be spelled out clearly in the cases of interest later on.

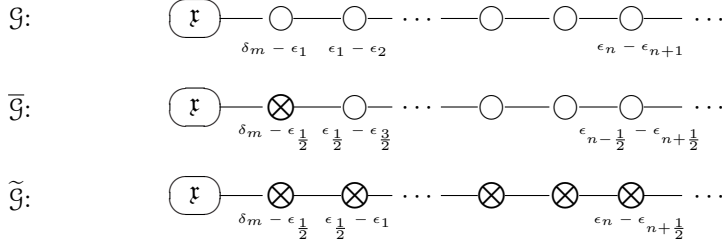
*Remark 6.1.* In the case when  $\mathfrak{k}$  is a finite-dimensional semisimple Lie algebra, or  $\mathfrak{gl}(n)$ , or of the four classical infinite-rank affine Kac-Moody Lie algebras  $\mathfrak{a}_\infty^+$ ,  $\mathfrak{b}_\infty$ ,  $\mathfrak{c}_\infty$ ,  $\mathfrak{d}_\infty$ , each block is controlled by the Weyl group  $W$  of  $\mathfrak{k}$ . For the principal block (in the full category  $\mathcal{O}$ ), the transition matrix between  $[L(\lambda)]$  and  $[\Delta(\mu)]$  is given by the Kazhdan-Lusztig polynomials associated to  $W$  (see Tanisaki's lectures [Ta]). Other blocks and parabolic cases are reduced to the principal block.

## 6.2 The master Lie (super)algebras

Let  $\mathfrak{f}$  denote one of the following four classical Dynkin diagrams of types  $\mathfrak{a}$ ,  $\mathfrak{b}$ ,  $\mathfrak{c}$ ,  $\mathfrak{d}$ , which will be referred to as a *head diagram*:



A head diagram  $\mathfrak{f}$  is connected with one of the three tail diagrams to produce three “master” Dynkin diagrams as below, whose corresponding Lie (super)algebras will be denoted by  $\mathcal{G} = \mathcal{G}^{\mathfrak{f}}$ ,  $\overline{\mathcal{G}} = \overline{\mathcal{G}}^{\mathfrak{f}}$ , and  $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}^{\mathfrak{f}}$ , respectively.



For example, the resulting master Dynkin diagrams for  $\mathcal{G}$  are the four infinite-rank classical Lie algebras  $\mathfrak{a}_\infty^+$  (one-sided infinity),  $\mathfrak{b}_\infty$ ,  $\mathfrak{c}_\infty$ ,  $\mathfrak{d}_\infty$ .

### 6.3 The categories $\mathcal{O}$ , $\bar{\mathcal{O}}$ , $\tilde{\mathcal{O}}$

Let  $\Pi^{\mathfrak{f}}$  denote the standard set of simple roots associated to the diagram  $\textcircled{\mathfrak{f}}$ , and fix  $Y^0 \subset \Pi^{\mathfrak{f}}$ . Let  $\mathfrak{T}^1$ ,  $\bar{\mathfrak{T}}^1$  and  $\tilde{\mathfrak{T}}^1$  be the subsets of the three tail diagrams with the first simple root removed. Let

$$Y = Y^0 \cup \mathfrak{T}^1, \quad \bar{Y} = Y^0 \cup \bar{\mathfrak{T}}^1, \quad \tilde{Y} = Y^0 \cup \tilde{\mathfrak{T}}^1.$$

Let  $P$  (respectively,  $\bar{P}$ ,  $\tilde{P}$ ) be the lattice which consists of the integral span of  $\epsilon_i$ , where  $i$  runs over  $I(m|\mathbb{N})$  (respectively,  $I(m|\frac{1}{2} + \mathbb{N})$ ,  $I(m|\frac{1}{2}\mathbb{N})$ ). Here  $I(m|\mathbb{N}) = \{\bar{1}, \dots, \bar{m}\} \cup \mathbb{N}$ , and the other two sets are similarly defined.

Define  $P_Y = \cup_{d \in \mathbb{Z}} P_Y^d$ , where

$$P_Y^d = \left\{ \lambda = \sum_{i \in I(m|\mathbb{N})} \lambda_i \epsilon_i \in P \mid \lambda \text{ is } Y\text{-dominant}; \lambda_i = d \text{ for } i \gg 0 \right\}. \quad (6.1)$$

We define  $P_{\bar{Y}}^d$  by changing  $I(m|\mathbb{N})$  in the definition of  $P_Y^d$  to  $I(m|\frac{1}{2} + \mathbb{N})$  and changing  $\lambda_i = d$  to  $\lambda_i = -d$  for  $i \gg 0$ , and let  $P_{\tilde{Y}}^d = \cup_{d \in \mathbb{Z}} P_{\tilde{Y}}^d$ .

Then, for  $\lambda \in P_Y^d$ ,  $\lambda^+ := (\lambda_1 - d, \lambda_2 - d, \dots)$  is a partition. Also, for  $\lambda \in P_{\bar{Y}}^d$ ,  $\lambda^+ := (\lambda_{1/2} + d, \lambda_{3/2} + d, \dots)$  is a partition. We have a bijection of sets:

$$\natural : P_Y^d \longrightarrow P_{\bar{Y}}^d, \quad \lambda \mapsto \lambda^\natural,$$

which is determined by the conditions

$$\lambda_i = \lambda_i^\natural, \quad \forall i \in I(m|0); \quad (\lambda^\natural)^+ = (\lambda^+)'.$$

This further induces a bijection  $\natural : P_Y \rightarrow P_{\bar{Y}}$ .

We define another map

$$\theta : P_Y^d \rightarrow P_{\tilde{Y}}^d, \quad \lambda \mapsto \lambda^\theta = \sum_{i \in I(m|0)} \lambda_i \epsilon_i + \sum_{i \in \mathbb{N}} ((p_i - d)\epsilon_{i-\frac{1}{2}} + (q_i + d)\epsilon_i),$$

where the image  $\lambda^\theta$  is determined by the conditions  $\lambda_i = \lambda_i^\theta$  for all  $i \in I(m|0)$  and  $(p_i|q_i)_{i \geq 1}$  is the modified Frobenius coordinates of the partition  $(\lambda^+)'$  (as defined

in Example 3.16 (3)). The map  $\theta$  is clearly injective, and so we obtain a bijection  $\theta : P_Y^d \rightarrow P_{\tilde{Y}}^d := \theta(P_Y^d)$  for each  $d \in \mathbb{Z}$ . Letting  $P_{\tilde{Y}} = \cup_{d \in \mathbb{Z}} P_{\tilde{Y}}^d$  we also obtain a bijection  $\theta : P_Y \rightarrow P_{\tilde{Y}}$ .

Following the construction of a parabolic category  $\mathcal{O}$  in Section 6.1, we obtain the categories  $\mathcal{O} := \mathcal{O}(\mathcal{G}, Y, P_Y)$ ,  $\overline{\mathcal{O}} := \mathcal{O}(\overline{\mathcal{G}}, \overline{Y}, \overline{P_{\tilde{Y}}})$ , and  $\tilde{\mathcal{O}} := \mathcal{O}(\tilde{\mathcal{G}}, \tilde{Y}, \tilde{P_{\tilde{Y}}})$ . We observe the following direct sum decompositions of categories:

$$\mathcal{O} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}^d, \quad \overline{\mathcal{O}} = \bigoplus_{d \in \mathbb{Z}} \overline{\mathcal{O}}^d, \quad \tilde{\mathcal{O}} = \bigoplus_{d \in \mathbb{Z}} \tilde{\mathcal{O}}^d,$$

where the index  $d$  indicates additional weight constraints:  $(\lambda, \epsilon_i) = d$  for  $i \in \mathbb{Z}, i \gg 0$  for  $\lambda \in P_Y$ ;  $(\lambda, \epsilon_i) = -d$  for  $i \in \frac{1}{2} + \mathbb{Z}, i \gg 0$  for  $\lambda \in P_{\tilde{Y}}$ ;  $(\lambda, \epsilon_i) = (-1)^{2i}d$  for  $i \in \frac{1}{2}\mathbb{Z}, i \gg 0$  for  $\lambda \in P_{\tilde{Y}}$ , respectively.

*Remark 6.2.* A more conceptual formulation (see [CLW] for details) is to introduce central extensions of  $\mathcal{G}, \overline{\mathcal{G}}, \tilde{\mathcal{G}}$  respectively by a one-dimensional center, whose module categories at level  $d$  are equivalent to  $\mathcal{O}^d, \overline{\mathcal{O}}^d, \tilde{\mathcal{O}}^d$  above, respectively. In this way, the additional weight constraints above are transferred to be  $(\lambda, \epsilon_i) = 0$  for  $i \gg 0$ , which is independent of  $d$ .

## 6.4 The equivalence of categories

We start with two simple observations:

(i) The Lie (super)algebras  $\mathcal{G}$  and  $\overline{\mathcal{G}}$  are naturally Lie subalgebras (though not Levi subalgebras) of  $\tilde{\mathcal{G}}$ , and the standard triangular decompositions of the three Lie (super)algebras are compatible with the inclusions  $\mathcal{G} \subset \tilde{\mathcal{G}}$  and  $\overline{\mathcal{G}} \subset \tilde{\mathcal{G}}$ . This follows by examining the simple roots of the three algebras.

(ii) We may naturally regard the lattices  $P$  and  $\overline{P}$  as sublattices of  $\tilde{P}$ , by definition of the lattices  $P, \overline{P}, \tilde{P}$ .

Given a  $\tilde{\mathcal{G}}$ -module with weight space decomposition  $\tilde{M} = \bigoplus_{\mu \in \tilde{P}} \tilde{M}_\mu$ , we define

$$T(\tilde{M}) := \bigoplus_{\mu \in P} \tilde{M}_\mu, \quad \text{and} \quad \overline{T}(\tilde{M}) := \bigoplus_{\mu \in \overline{P}} \tilde{M}_\mu.$$

Clearly,  $T(\tilde{M})$  is a  $\mathcal{G}$ -module and  $\overline{T}(\tilde{M})$  is a  $\overline{\mathcal{G}}$ -module. We can check that  $T$  and  $\overline{T}$  send objects in category  $\tilde{\mathcal{O}}$  to objects in category  $\mathcal{O}$  and  $\overline{\mathcal{O}}$ , respectively, and hence we obtain functors

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \xrightarrow{\tilde{T}} & \mathcal{O} \\ \downarrow \overline{T} & & \\ \overline{\mathcal{O}} & & \end{array}$$

More precisely, we have  $T : \tilde{\mathcal{O}}^d \rightarrow \mathcal{O}^d$  and  $\overline{T} : \tilde{\mathcal{O}}^d \rightarrow \overline{\mathcal{O}}^d$ .

We will use  $\overline{\Delta}(\lambda)$  and  $\tilde{\Delta}(\lambda)$  to denote the parabolic Verma modules in the categories  $\overline{\mathcal{O}}$  and  $\tilde{\mathcal{O}}$  of highest weights  $\lambda$ , respectively. Similarly, we use  $\overline{L}(\lambda)$  and  $\tilde{L}(\lambda)$  to denote the corresponding irreducible modules. We are ready to formulate the main result of [CLW].

**Theorem 6.3.** [CLW] *We have*

$$\begin{aligned} T(\tilde{\Delta}(\lambda^\theta)) &= \Delta(\lambda), & \bar{T}(\tilde{\Delta}(\lambda^\theta)) &= \bar{\Delta}(\lambda^\natural), \\ T(\tilde{L}(\lambda^\theta)) &= L(\lambda), & \bar{T}(\tilde{L}(\lambda^\theta)) &= \bar{L}(\lambda^\natural). \end{aligned}$$

Moreover,  $T : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$  and  $\bar{T} : \tilde{\mathcal{O}} \rightarrow \bar{\mathcal{O}}$  are equivalences of categories.

As a consequence,  $T : \tilde{\mathcal{O}}^d \rightarrow \mathcal{O}^d$  and  $\bar{T} : \tilde{\mathcal{O}}^d \rightarrow \bar{\mathcal{O}}^d$  are equivalences of categories.

*Idea of a proof.* The proof consists of several major steps:

(i)  $T(L(\mathfrak{l}_{\tilde{\mathcal{Y}}}, \lambda^\theta)) = L(\mathfrak{l}_Y, \lambda)$ . This follows by locating a highest weight  $\lambda$  for the module  $L(\mathfrak{l}_{\tilde{\mathcal{Y}}}, \lambda^\theta)$  with respect to a new Borel of  $\mathfrak{l}_{\tilde{\mathcal{Y}}}$  which is compatible with the imbedding  $\mathfrak{l}_Y \subseteq \mathfrak{l}_{\tilde{\mathcal{Y}}}$ . Then we check the equality on the character level (which boils down to the hook-Schur functions.)

(ii)  $T(\tilde{\Delta}(\lambda^\theta))$  is a highest weight module of highest weight  $\lambda$  with respect to the standard Borel of  $\mathcal{G}$ . This follows from a weight argument.

(iii)  $T(\tilde{\Delta}(\lambda^\theta)) = \Delta(\lambda)$ . Check on the character level, and use (i) and (ii).

(iv)  $T(\tilde{L}(\lambda^\theta)) = L(\lambda)$ . Follows by showing that there is no singular vector in  $T(\tilde{L}(\lambda^\theta))$  of weight different from  $\lambda$ .

This proves the equality for  $T$  in the theorem. The proof of the equality for  $\bar{T}$  is similar. More work is needed to establish the equivalence of categories.  $\square$

**Corollary 6.4** (Super Duality). *The categories  $\mathcal{O}$  and  $\bar{\mathcal{O}}$  are equivalent.*

*Remark 6.5* (History of Super Duality). First for type  $A$ .

- (1) For  $Y^0 = \Pi^a$ , the super duality was conjectured in [CWZ, Conjecture 6.10]. This was motivated by Brundan's work [Br1] on finite-dimensional simple  $\mathfrak{gl}(m|n)$ -characters.
- (2) For arbitrary Levi subalgebra associated to  $Y^0 \subseteq \Pi^a$ , the super duality conjecture was formulated in [CW4, Conjecture 4.18]. It is shown in [CWZ, CW4] that the Brundan-Kazhdan-Lusztig polynomials coincide with the usual Kazhdan-Lusztig polynomials in type  $A$ .
- (3) The conjecture of [CWZ] is proved in [BrS], where the underlying algebras for the two categories were computed and shown to be isomorphic.
- (4) The more general conjecture of [CW4] is independently proved in [CL2] essentially in the form of Theorem 6.3. In particular, this provides a new proof of the main result of [Br1].

Then for type  $\mathfrak{osp}$ .

- (5) A first supporting evidence for the super duality was provided by the computation in [CKW] of the Kostant  $\mathfrak{u}$ -homology groups with coefficients in the modules of classical Lie superalgebras appearing in Howe duality decompositions.

(6) Theorem 6.3 is formulated and established in [CLW].

Note that very little has been known about representation theory of infinite-dimensional Lie superalgebras, such as affine superalgebras, with the exception of work of Kac and Wakimoto [KW].

(7) The super duality approach applies to and sheds new insight on more general Lie superalgebras, including affine superalgebras.

## 6.5 Irreducible $\overline{\mathfrak{G}}$ -characters

By Remark 6.1, the irreducible character problem for the category  $\mathcal{O}$  is solved by the Kazhdan-Lusztig conjectures [KL] (theorem of Beilinson-Bernstein [BB] and Brylinski-Kashiwara [BK]). Hence by Theorem 6.3, the irreducible character problem for the module category  $\overline{\mathcal{O}}$  for Lie superalgebras is solved by the same classical Kazhdan-Lusztig polynomials.

Recall for  $\mathfrak{r} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$ , we have defined a Lie superalgebra  $\overline{\mathfrak{G}} = \overline{\mathfrak{G}}^{\mathfrak{r}}$  whose Dynkin diagram is obtained by connecting  $\overline{\mathfrak{r}}$  with the type  $\mathfrak{gl}(1|\infty)$  tail diagram  $\overline{\mathfrak{T}}$ . For  $n \in \mathbb{N}$ , we will also consider a Lie superalgebra  $\overline{\mathfrak{G}}_n = \overline{\mathfrak{G}}_n^{\mathfrak{r}}$  whose Dynkin diagram (see below) is obtained by connecting  $\overline{\mathfrak{r}}$  with a tail diagram  $\overline{\mathfrak{T}}_n$  of type  $\mathfrak{gl}(1|n)$ :

$$\overline{\mathfrak{G}}_n^{\mathfrak{r}}: \quad \overline{\mathfrak{r}} \text{---} \otimes \text{---} \circ \text{---} \cdots \text{---} \circ \quad (6.2)$$

$\delta_m - \epsilon_{\frac{1}{2}} \quad \epsilon_{\frac{1}{2}} - \epsilon_{\frac{3}{2}} \quad \epsilon_{n-\frac{3}{2}} - \epsilon_{n-\frac{1}{2}}$

As  $\overline{\mathfrak{r}}$  is taken to be one of the four classical Dynkin diagrams,  $\overline{\mathfrak{G}}_n$  is a classical finite-dimensional Lie superalgebra of type either  $\mathfrak{gl}$  or  $\mathfrak{osp}$ .

Let  $\overline{\mathfrak{T}}_n^1$  be the subset of  $\overline{\mathfrak{T}}_n$ , which is obtained from  $\overline{\mathfrak{T}}_n$  by the removal of the first simple root. Let  $\overline{Y}_n = Y^0 \cup \overline{\mathfrak{T}}_n^1$ , and let

$$P_{\overline{Y}_n}^d = \left\{ \lambda = \sum_{i \in I(m|n-\frac{1}{2})} \lambda_i \epsilon_i \mid \lambda \text{ is } \overline{Y}_n\text{-dominant; } \lambda_{n-\frac{1}{2}} \geq -d \right\}.$$

For  $d \in \mathbb{Z}$ , we introduce a category  $\overline{\mathcal{O}}_n^d := \mathcal{O}(\overline{\mathfrak{G}}_n, \overline{Y}_n, P_{\overline{Y}_n}^d)$  of  $\overline{\mathfrak{G}}_n$ -modules. Note that  $\overline{\mathcal{O}}_n^d$  is a full subcategory of the (more standard) category  $\overline{\mathcal{O}}_n := \mathcal{O}(\overline{\mathfrak{G}}_n, \overline{Y}_n, P_{\overline{Y}_n})$ , where

$$P_{\overline{Y}_n} = \left\{ \lambda = \sum_{i \in I(m|n-\frac{1}{2})} \lambda_i \epsilon_i \mid \lambda \text{ is } \overline{Y}_n\text{-dominant} \right\}.$$

Moreover,  $\overline{\mathcal{O}}_n^d \subset \overline{\mathcal{O}}_n^{d+1}$  for all  $d$ , and  $\overline{\mathcal{O}}_n = \cup_{d \in \mathbb{Z}} \overline{\mathcal{O}}_n^d$ .

We also introduce truncation functors  $\overline{\mathfrak{T}}_n : \overline{\mathcal{O}}^d \rightarrow \overline{\mathcal{O}}_n^d$  as follows: Let  $M \in \overline{\mathcal{O}}^d$  such that  $M = \bigoplus_{\mu} M_{\mu}$ . We define

$$\overline{\mathfrak{T}}_n(M) = \bigoplus_{\{\mu \mid \mu_{n+\frac{1}{2}} = -d\}} M_{\mu},$$

which is clearly a  $\overline{\mathfrak{G}}_n$ -module. Note that  $\mu_{n+\frac{1}{2}} = -d$  ensures  $\mu_{k+\frac{1}{2}} = -d$  for all  $k \geq n$ . Such a truncation functor to a Levi subalgebra (except that we drop an abelian direct summand of the Levi) has its analogue in algebraic group setting studied by Donkin (cf. e.g. [CW4] for a formulation in type  $A$ ). The functors  $\overline{\mathfrak{T}}_n : \overline{\mathcal{O}}^d \rightarrow \overline{\mathcal{O}}_n^d$  have the following key property.

**Proposition 6.6.** *For  $X = \Delta$  or  $L$ , we have*

$$\overline{\mathfrak{T}}_n(X(\lambda)) = \begin{cases} X_n(\lambda^{(n)}), & \text{if } \lambda_{n+\frac{1}{2}} = -d, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lambda^{(n)}$  is obtained from  $\lambda$  by ignoring  $\lambda_j$ , for  $j > n - \frac{1}{2}$ .

*Remark 6.7.* By Proposition 6.6 and Theorem 6.3, the classical Kazhdan-Lusztig solution of the irreducible character problem of  $\mathcal{O}$  induces a complete solution of the irreducible character problem of  $\overline{\mathcal{O}}_n$  for each finite  $n$ . It is not hard to show [CLW] that every finite-dimensional irreducible modules of every ortho-symplectic Lie superalgebra appears in some  $\overline{\mathcal{O}}_n$ .

We end the paper with a list of symbols.

Symbol	Meaning
$ v $	$\mathbb{Z}_2$ -parity of a homogeneous vector $v$ in a vector superspace
$\mathbb{C}^{m n}$	complex vector superspace of super dimension $m n$
$\mathfrak{gl}(m n)$	the general Lie superalgebra
$I(m n)$	the totally order set $\{\overline{1} < \overline{2} < \dots < \overline{m} < 1 < \dots < n\}$
$\mathfrak{sl}(m n)$	the special linear Lie superalgebra
$\mathfrak{h}$	Cartan subalgebra (of diagonal matrices)
$\delta_i$	dual basis element corresponding to $E_{\overline{ii}}$
$\epsilon_j$	dual basis element corresponding to $E_{jj}$
$\Phi, \Phi_{\overline{0}},$ or $\Phi_{\overline{1}}$	sets of all, even, or odd roots
$\Pi$	set of simple roots
$\Phi_{\epsilon}^{\pm}$	set of positive/negative roots of parity $\epsilon = \overline{0}, \overline{1}$
$\mathfrak{osp}(m n)$	ortho-symplectic Lie superalgebra preserving a non-degenerate supersymmetric bilinear form on $\mathbb{C}^{m n}$
$\mathfrak{spo}(m n)$	symplectic-orthogonal Lie superalgebra preserving a non-degenerate skew-supersymmetric bilinear form on $\mathbb{C}^{m n}$
$U(\mathfrak{g})$	universal enveloping algebra of a Lie (super)algebra $\mathfrak{g}$
$K(\lambda)$	Kac module of highest weight $\lambda$
$L(\mathfrak{g}, \lambda)$ or $L(\lambda)$	irreducible $\mathfrak{g}$ -module of highest weight $\lambda$
$\text{ch}M$	character of the $\mathfrak{h}$ -semisimple module $M$
$\rho_0, \rho_1$	half sums of positive even and odd roots, respectively
$\rho$	$\rho_0 - \rho_1$
$\mathfrak{S}_d$	symmetric group in $d$ letters

Symbol	Meaning
$S^\lambda$	Specht module corresponding to the partition $\lambda$
$\Delta(\lambda)$	(parabolic) Verma $\mathfrak{g}$ -module of highest weight $\lambda$
$\mu'$	conjugate partition of the partition $\mu$
$\mathcal{P}(d, m n)$	set of $(m n)$ -hook partitions of size $d$ (Definition 2.9)
$\mathcal{P}(m n)$	$\bigcup_{d \geq 0} \mathcal{P}(d, m n)$
$s_\lambda(\underline{x})$	Schur function in $\underline{x}$ for the partition $\lambda$
$hs_\lambda(\underline{x}, \underline{y})$	hook Schur function for the hook partition $\lambda$ (4.1)
$\diamond_r, \diamond_{k,r}$	row determinants given in (4.10) and (4.11), respectively
$\mathfrak{a}_\infty^+$	(one-sided) infinite-rank Kac-Moody Lie algebras of type $A$
$\mathfrak{b}_\infty, \mathfrak{c}_\infty, \mathfrak{d}_\infty$	infinite-rank Kac-Moody Lie algebras of types $B, C, D$
$\Lambda_i^{\mathfrak{c}}$	$i$ th fundamental weight of $\mathfrak{c}_\infty$
$\lambda_w$	see (5.6)
$\mathfrak{g}^{\mathfrak{r}}, \overline{\mathfrak{G}}^{\mathfrak{r}}, \widetilde{\mathfrak{G}}^{\mathfrak{r}}$	Lie (super)algebras corresponding to master diagrams of Section 6.2 of type $\mathfrak{r}$ ; also denoted by $\mathfrak{G}, \overline{\mathfrak{G}}, \widetilde{\mathfrak{G}}$ , respectively
$\mathcal{O}, \overline{\mathcal{O}}, \widetilde{\mathcal{O}}$	parabolic categories of $\mathfrak{G}$ -, $\overline{\mathfrak{G}}$ - and $\widetilde{\mathfrak{G}}$ -modules, respectively (Sections 6.1 and 6.3)
$\overline{\Delta}(\lambda), \widetilde{\Delta}(\lambda)$	parabolic Verma modules in $\overline{\mathcal{O}}, \widetilde{\mathcal{O}}$ (Section 6.4)
$\overline{L}(\lambda), \widetilde{L}(\lambda)$	irreducible modules in $\overline{\mathcal{O}}, \widetilde{\mathcal{O}}$ (Section 6.4)
$\mathfrak{r}$	head Dynkin diagrams of types $\mathfrak{r} = \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ (Section 6.2)
$P_Y^d, P_{\overline{Y}}^d, P_{\widetilde{Y}}^d$	set of dominant weights for $\mathfrak{G}, \overline{\mathfrak{G}}, \widetilde{\mathfrak{G}}$ , respectively (Section 6.3)
$\mathfrak{h}, \theta$	bijections from $P_Y^d$ to $P_{\overline{Y}}^d$ and $P_{\widetilde{Y}}^d$ (Section 6.3), respectively
$T, \text{ or } \overline{T}$	functors from $\widetilde{\mathcal{O}}$ to $\mathcal{O}$ , or to $\overline{\mathcal{O}}$ , respectively (Section 6.4)
$\overline{\mathcal{O}}_n$	parabolic category of a finite-dimensional Lie (super)algebra (Section 6.5)
$\mathfrak{T}\mathfrak{r}_n$	truncation functor from $\overline{\mathcal{O}}$ to $\overline{\mathcal{O}}_n$ (Section 6.5)

## References

- [BB] A. Beilinson and J. Bernstein, *Localisation de  $\mathfrak{g}$ -modules*, C.R. Acad. Sci. Paris Ser. I Math. **292** (1981), 15–18.
- [BL] I.N. Bernstein and D.A. Leites, *A formula for the characters of the irreducible finite-dimensional representations of Lie superalgebras of series  $gl$  and  $sl$* , C.R. Aca. Bulg. Sci. **33** (1980), 1049–51 [in Russian].
- [BeR] A. Berele and A. Regev, *Hook Young Diagrams with Applications to Combinatorics and to Representations of Lie Superalgebras*, Adv. in Math. **64** (1987), 118–175.
- [BKN] B. Boe, J. Kujawa and D. Nakano, *Cohomology and support varieties for Lie superalgebras*, Trans. AMS (to appear), arXiv:math.RT/0609363.
- [BK] J.L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387–410.

- [BPT] A. Brini, A. Palareti, and A. Teolis, *Gordan-Capelli series in superalgebras*, Proc. Natl. Acad. Sci. USA **85** (1988), 1330–1333.
- [Br1] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra  $\mathfrak{gl}(m|n)$* , J. Amer. Math. Soc. **16** (2003), 185–231.
- [Br2] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra  $\mathfrak{q}(n)$* , Adv. Math. **182** (2004), 28–77.
- [BrK] J. Brundan and J. Kujawa, *A new proof of the Mullineux conjecture*, J. Alg. Combin. **18** (2003), 13–39.
- [BrS] J. Brundan and C. Stroppel, *Highest Weight Categories Arising from Khovanov’s Diagram Algebras IV: the general linear supergroup*, arXiv:0907.2543.
- [CK] S.-J. Cheng and J.-H. Kwon, *Howe duality and Kostant’s homology formula for infinite-dimensional Lie superalgebras*, IMRN (2008), Art. ID rnn 085, 52 pp.
- [CKL] S.-J. Cheng and J.-H. Kwon and N. Lam, *A BGG-type resolution for tensor modules over general linear superalgebra*, Lett. Math. Phys. **84** (2008), 75–87.
- [CKW] S.-J. Cheng and J.-H. Kwon and W. Wang, *Kostant homology formulas for oscillator modules of Lie superalgebras*, Adv. Math. (2010), doi:10.1016/j.aim.2010.01.002.
- [CL1] S.-J. Cheng and N. Lam, *Infinite-dimensional Lie superalgebras and hook Schur functions*, Commun. Math. Phys. **238** (2003), 95–118.
- [CL2] S.-J. Cheng and N. Lam, *Irreducible characters of general linear superalgebra and super duality*, arXiv:0905.0332.
- [CLW] S.-J. Cheng, N. Lam and W. Wang, *Super duality and irreducible characters of ortho-symplectic Lie superalgebras*, arXiv:0911.0129.
- [CLZ] S.-J. Cheng, N. Lam and R.B. Zhang, *Character formula for infinite dimensional unitarizable modules of the general linear superalgebra*, J. Algebra **273** (2004), 780–805.
- [CW1] S.-J. Cheng and W. Wang, *Howe duality for Lie superalgebras*, Compositio Math. **128** (2001), 55–94.
- [CW2] S.-J. Cheng and W. Wang, *Remarks on Schur-Howe-Sergeev duality*, Lett. Math. Phys. **52** (2000), 143–153.
- [CW3] S.-J. Cheng and W. Wang, *Lie subalgebras of differential operators on the super circle*, Publ. Res. Inst. Math. Sci. **39** (2003), 545–600.
- [CW4] S.-J. Cheng and W. Wang, *Brundan-Kazhdan-Lusztig and Super Duality Conjectures*, Publ. Res. Inst. Math. Sci. **44** (2008), 1219–1272.

- [CW5] S.-J. Cheng and W. Wang, *Remarks on modules of the ortho-symplectic Lie superalgebras*, Bull. Inst. Math. Acad. Sin. (N.S.) **3** (2008), 353–372.
- [CWZ] S.-J. Cheng, W. Wang and R.B. Zhang, *Super duality and Kazhdan-Lusztig polynomials*, Trans. Amer. Math. Soc. **360** (2008) 5883–5924.
- [CWZ2] S.-J. Cheng, W. Wang and R.B. Zhang, *A Fock space approach to representation theory of  $\mathfrak{osp}(2|2n)$* , Transformation Groups **12** (2007), 209–225.
- [CZ1] S.-J. Cheng and R.B. Zhang, *Analogue of Kostant’s  $\mathfrak{u}$ -homology formula for general linear Superalgebras*, International Mathematics Research Notices **2004**, no. 1, 31-53.
- [CZ2] S.-J. Cheng and R.B. Zhang, *Howe duality and combinatorial character formula for orthosymplectic Lie superalgebras*, Adv. Math. **182** (2004), 124–172.
- [J] J. Van der Jeugt, *Character formulae for Lie superalgebra  $C(n)$* , Comm. Algebra **19**, (1991), 199–222.
- [JHKT] J. Van der Jeugt, J.W.B. Hughes, R. C. King and J. Thierry-Mieg, *Character formulas for irreducible modules of the Lie superalgebras  $\mathfrak{sl}(m/n)$* , J. Math. Phys. **31** (1990), 2278–2304.
- [DP] V. Dobrev and V. Petkova, *Group-theoretical approach to extended conformal supersymmetry: function space realization and invariant differential operators*, Fortschr. Phys. **35** (1987), 537–572.
- [DS] M. Duflo and V. Serganova, *On associated variety for Lie superalgebras*, Preprint 2005, arXiv:math/0507198.
- [Ger] J. Germoni, *Indecomposable representations of  $\mathfrak{osp}(3, 2)$ ,  $D(2, 1; \alpha)$  and  $G(3)$* . Colloquium on Homology and Representation Theory (Spanish) (Vaquerias, 1998). Bol. Acad. Nac. Cienc. (Cordoba) **65** (2000), 147–163.
- [GW] R. Goodman and N. Wallach, *Representations and invariants of the classical groups*, Encyclopedia of Math. and Appl. **68**, Cambridge Univ. Press, 1998.
- [Gor] M. Gorelik, *Annihilation theorem and separation theorem for basic classical Lie superalgebras*, J. AMS **15** (2001), 113–165.
- [GS] C. Gruson and V. Serganova, *Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras*, arXiv:0906.0918.
- [H1] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570.
- [H2] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The Schur Lectures, Israel Math. Conf. Proc. **8**, Tel Aviv (1992), 1–182.

- [K1] V. Kac, *Lie superalgebras*, Adv. in Math. **26** (1977), 8–96.
- [K2] V. Kac, *Representations of classical Lie superalgebras*, Lect. Notes in Math. **676**, pp. 597–626, Springer Verlag, 1978.
- [KW] V. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, Lie theory and geometry, 415–456, Progr. Math., **123**, Birkhauser, Boston, MA, 1994.
- [KL] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.
- [LSS] D. Leites, M. Saveliev and V. Serganova, *Embeddings of  $\mathfrak{osp}(N/2)$  and the associated nonlinear supersymmetric equations*. Group theoretical methods in physics, Vol. I (Yurmala, 1985), 255–297, VNU Sci. Press, Utrecht, 1986.
- [LZ] N. Lam and R.B. Zhang, *Quasi-finite modules for Lie superalgebras of infinite rank*, Trans. Amer. Math. Soc. **358** (2006), 403–439.
- [Ma] Yu. Manin, *Gauge field theory and complex geometry*, Grundlehren der mathematischen Wissenschaften **289**, Second Edition, Springer, 1997.
- [Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second Edition, Clarendon Press, Oxford, 1995.
- [Mu] I. Musson, *Lie superalgebras, Clifford algebras, induced modules and nilpotent orbits*, Adv. Math. **207** (2006), 39–72.
- [Naz] M. Nazarov, *Capelli identities for Lie superalgebras*, Ann. Sci. École Norm. Sup. (4) **30** (1997), 847–872.
- [Pe] I. Penkov, *Borel-Weil-Bott theory for classical Lie supergroups*, J. Soviet Math. **51** (1990), 2108–2140.
- [PS] I. Penkov and V. Serganova, *Representations of classical Lie superalgebras of type I*, Indag. Math. **3** (1992) 419–466.
- [San] J. Santos, *Foncteurs de Zuckermann pour les superalgèbres de Lie*, J. Lie Theory **9** (1999), 69–112.
- [SNR] M. Scheunert, W. Nahm, and V. Rittenberg, *Classification of all simple graded Lie algebras whose Lie algebra is reductive I, II. Construction of the exceptional algebras*, J. Math. Phys. **17** (1976), 1626–1639, & 1640–1644.
- [Sva] V. Serganova, *Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra  $\mathfrak{gl}(m|n)$* , Selecta Math. (N.S.) **2** (1996), 607–651.
- [Sv1] A. Sergeev, *The tensor algebra of the identity representation as a module over the Lie superalgebras  $\mathfrak{gl}(n, m)$  and  $Q(n)$* , Math. USSR Sbornik **51** (1985), 419–427.

- [Sv2] A. Sergeev, *An Analog of the Classical Invariant Theory, I, II*, Michigan J. Math. **49** (2001), 113–146, 147–168.
- [SW] B. Shu and W. Wang, *Modular representations of the ortho-symplectic supergroups*, Proc. London Math. Soc. **96** (2008), 251–271.
- [Su] Y. Su, *Composition factors of Kac modules for the general linear Lie superalgebras*, Math. Z. **252** (2006), 731–754.
- [SZ] Y. Su and R.B. Zhang, *Generalised Verma modules for the orthosymplectic Lie superalgebra  $\mathfrak{osp}(k|2)$* , arXiv:0911.0735.
- [Ta] T. Tanisaki, *Lectures on D-modules and Representation Theory*, this volume.
- [Vo] D. Vogan, *Irreducible characters of semisimple Lie Groups II: The Kazhdan-Lusztig Conjectures*, Duke Math. J. **46** (1979), 805–859.
- [Wa] W. Wang, *Duality in infinite dimensional Fock representations*, Commun. Contem. Math. **1** (1999), 155–199.
- [We] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Second Edition (15th Printing), Princeton University Press, Princeton, 1997.
- [WZ] W. Wang and L. Zhao, *Representations of Lie superalgebras in prime characteristic I*, Proc. London Math. Soc. **99** (2009), 145–167.
- [Zou] Y. Zou, *Categories of finite-dimensional weight modules over type I classical Lie superalgebras*, J. Algebra **180** (1996), 459–482.

\*chengsj@math.sinica.edu.tw

†ww9c@virginia.edu