Universal coefficient theorems for unital extensions of $C^*$-algebras and the BDF-theory

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Abstract In this paper certain universal coefficient theorems for unital extensions of $C^*$-algebras are proved under moderate assumptions. We also give some connections between the six-term exact sequences with base points and the equivalence relations of unital extensions. Using these results we give a generalization of the classic BDF-theory and classify certain classes of unital extensions up to unitary equivalence or isomorphism.

Key Words UCT, unital extension, Ext-group

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1. Introduction

Since Brown, Douglas and Fillmore [4] began their remarkable work on $C^*$-algebra extensions, the research of universal coefficient theorems (UCT) has been placed in the center position of $K$-theory for $C^*$-algebras. Firstly, Brown, Douglas and Fillmore gave a special case of the UCT for $C(X)$ which solved completely the classification question of essential normal operators. Secondly, Brown [2] proved the general case of the UCT for extensions of commutative $C^*$-algebras by the compact operators $\mathcal{K}$. Due to Kasparov’s $KK$-groups [9], $K$-theory and extension theory were unified into $KK$-theory. Subsequently, Rosenberg and Schochet [25] proved the most popular UCT

$$0 \to \text{Ext}(K_*(A), K_*(B)) \to \text{Ext}(A, B) \to \text{Hom}(K_*(A), K_{*+1}(B)) \to 0.$$  

Since the UCT is an inevitable tool to compute $\text{Ext}$-groups and $\text{KK}$-groups which are important invariants for classifying $C^*$-algebras and their extensions, it plays a crucial role in $KK$-theory of $C^*$-algebras. But unlike the original BDF-theory, the group $\text{Ext}(A, B)$ in $KK$-theory classifies extensions under stable unitary equivalence. Hence $\text{Ext}(A, B)$ provides little
information for classifications of $C^*$-algebras and unital extensions. So Brown and Dadarlat [3] proved a UCT for unital extensions in the case $B = K$. Using this UCT, they gave a complete invariant for certain extension algebras with stable rank one. Based on their work, we [28] proved two UCTs for unital extensions under the condition, $B$ has corona factorization property, due to using group action. In order to obtain more applications, we have to remove this restriction. This is one purpose of this paper (see 3.6, 3.15 and 3.16).

On the other hand, classifications of $C^*$-algebras have been studied deeply (see [7, 14-16, 21]). In order to classify unital extension algebras, one needs to study the classification of unital extensions up to isomorphism. The natural invariant for unital extensions is the six-term exact sequence with the equivalence class of units. As we mentioned above, $KK$-theory and the UCT of Rosenberg and Schochet deal with nonunital extensions and stable unitary equivalence effectively, but they almost do nothing to unital extensions and unitary equivalence. In the unital case, the key step is to establish connections between the six-term exact sequences with base points and the equivalence relations of unital extensions. Here we prove certain results towards this direction (see 3.10-3.12). Our approach is lifting unitaries in the corona algebra to partial isometries and constructing the so-called ”subextensions”. Applying these results, we give a classification theorem for unital quasidiagonal extensions of $AT$-algebras in [31].

As applications, in Section 4 we give a classification theorem for certain class of unital extensions in the original sense of the BDF-theory (Theorem 4.1). By the UCT for unital extensions, Lin’s some results on classification of unital extensions with $B$ stable are immediately follows. Finally, we prove that $(K(e), [1]_0)$ is a complete invariant for certain unital extensions of $C(X)$ up to isomorphism (Theorem 4.8).

2. Preliminaries

In this section, we recall some notations and invariants for $C^*$-algebras and their extensions. One can see [1, 23, 24, 30, 31, etc.] for details.

2.1. Let $A$ and $B$ be $C^*$-algebras. Let $e : 0 \to B \to E \to A \to 0$ be an extension of $A$ by $B$ with Busby invariant $\tau : A \to Q(B)$, where $Q(B) = M(B)/B$ is the corona algebra of $B$ with quotient map $\pi : M(B) \to Q(B)$. The above extension $e$ is called trivial if the exact sequence splits.

We call $e$ essential, if its Busby invariant $\tau$ is an injective homomorphism. Denote by $\mathcal{E}xt(A, B)$ the set of essential extensions of $A$ by $B$. In this paper we only consider essential extensions.

Suppose that $e : 0 \to B \to E \xrightarrow{\psi} A \to 0$ is an essential extension with Busby invariant
Let \( E' = \pi^{-1}(\tau(A)) \) and \( \psi' = \tau^{-1}|_{\tau(A)} \circ \pi|_{E'} \). Then we have another essential extension \( e' : 0 \to B \to E' \xrightarrow{\psi'} A \to 0 \) with the same Busby invariant as \( e \) which is called standard form of \( e \) in [23]. Moreover, there is a commutative diagram

\[
\begin{array}{c}
0 \to B \to E \xrightarrow{\psi} A \to 0 \\
\downarrow \cong \quad \quad \quad \quad \quad \downarrow \cong \\
0 \to B \to E' \xrightarrow{\psi'} A \to 0.
\end{array}
\]

We will identify \( e \) with its standard form.

If \( A \) is unital and the Busby invariant is also unital, then \( e \) is called unital. Denote by \( \mathcal{E}xt^u(A,B) \) the set of unital essential extensions of \( A \) by \( B \) if \( A \) is unital.

2.2. Suppose that \( A \) and \( B \) are \( C^* \)-algebras. There are several equivalence relations of extensions of \( A \) by \( B \). Let \( e_i : 0 \to B \to E_i \to A \to 0 \) be two extensions with Busby invariants \( \tau_i \) for \( i = 1, 2 \).

Two extensions \( e_1 \) and \( e_2 \) are called (strongly) unitarily equivalent, denoted by \( e_1 \cong e_2 \), if there exists a unitary \( u \in M(B) \) such that \( \tau_2(a) = \pi(u)\tau_1(a)\pi(u)^* \) for all \( a \in A \). Denote by \( \text{Ext}(A,B) \) or \( \text{Ext}_s(A,B) \) the set of (strong) unitary equivalence classes of extensions of \( A \) by \( B \). If \( A \) is unital, we denote by \( \text{Ext}^u_s(A,B) \) the set of unitary equivalence classes of unital essential extensions of \( A \) by \( B \).

Two extensions \( e_1 \) and \( e_2 \) are called weakly unitarily equivalent, denoted by \( e_1 \sim e_2 \), if there exists a unitary \( v \in Q(B) \) such that \( \tau_2(a) = v\tau_1(a)v^* \) for all \( a \in A \). Denote by \( \text{Ext}_w(A,B) \) [resp. \( \text{Ext}^w_u(A,B) \) when \( A \) is unital] the set of equivalence classes of extensions [resp. unital extensions] of \( A \) by \( B \) under weak unitary equivalence.

2.3. Suppose that \( A_i \) and \( B_i \) are \( C^* \)-algebras. Let \( e_i : 0 \to B_i \to E_i \to A_i \to 0 \) be two extensions with Busby invariants \( \tau_i \).

If there are tree homomorphisms \( \beta : B_1 \to B_2, \eta : E_1 \to E_2 \) and \( \alpha : A_1 \to A_2 \) such that the following diagram commutes:

\[
\begin{array}{c}
0 \to B_1 \to E_1 \to A_1 \to 0 \\
\downarrow \beta \quad \quad \quad \quad \quad \downarrow \eta \quad \quad \quad \quad \quad \downarrow \alpha \\
0 \to B_2 \to E_2 \to A_2 \to 0,
\end{array}
\]

then we call \( (\beta, \eta, \alpha) : e_1 \to e_2 \) an extension morphism from \( e_1 \) to \( e_2 \).

Two extensions \( e_1 \) and \( e_2 \) are called isomorphic (called ”weakly isomorphic” in [1]), denoted by \( e_1 \cong e_2 \), if there exist isomorphisms \( \beta, \eta, \alpha \) such that \( (\beta, \eta, \alpha) : e_1 \to e_2 \) is an extension morphism from \( e_1 \) to \( e_2 \).
2.4. Let $H$ be a separable infinite dimensional Hilbert space and $\mathcal{K}$ the ideal of compact operators in $B(H)$. If $B$ is a stable $C^*$-algebra (i.e. $B \otimes \mathcal{K} \cong B$), then the sum of two extensions $\tau_1$ and $\tau_2$ is defined to be the homomorphism $\tau_1 \oplus \tau_2$, where

$$\tau_1 \oplus \tau_2 : A \to \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq M_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B).$$

It is easy to see that the sets of equivalence classes of extensions in 2.2 are commutative semigroups with respect to the above addition when $B$ is stable. One can similarly define these semigroups replacing $B$ by $B \otimes \mathcal{K}$ if $B$ is not stable.

2.5. A trivial extension $\tau$ is called strongly unital if $\tau$ can lift to a unital homomorphism from $A$ to $M(B)$.

Denote by $\text{Ext}(A,B)$ the quotient of $\text{Ext}_s(A,B)$ by the subsemigroup of trivial extensions. If $A$ is unital, $\text{Ext}_s^u(A,B)$ [resp. $\text{Ext}_w^u(A,B)$] is the quotient of $\text{Ext}_s^u(A,B)$ [resp. $\text{Ext}_w^u(A,B)$] by the subsemigroup of strong unital trivial extensions.

Let $e_1, e_2 \in \text{Ext}(A,B)$. If $[e_1] = [e_2]$ in $\text{Ext}(A,B)$, then $e_1$ and $e_2$ are called stably unitarily equivalent, denoted by $e_1 \overset{ss}{\simeq} e_2$.

Let $e : 0 \to B \to E \to A \to 0$ be an extension of $A$ by $B$ with Busby invariant $\tau : A \to \mathcal{Q}(B)$. Then $e$ is called absorbing [unital-absorbing when $A$ is unital] if $e$ is unitarily equivalent to $e \oplus \sigma$ for any trivial [strong unital trivial] extension $\sigma$.

We call $e$ full if its Busby invariant $\tau : A \to \mathcal{Q}(B)$ is a full homomorphism.

Recall that [8] $e$ is called purely large if the $C^*$-algebra $E$ is purely large with respect to the ideal $B$.

2.6. Let $e \in \text{Ext}(A,B)$ and let $C$ and $D$ be $C^*$-algebras. Suppose that $\beta : B \to C$ is a surjective homomorphism and $\alpha \in \text{Hom}(D,A)$. Then there are two induced extensions $\beta e$ and $e \alpha$, making the following diagrams commute respectively:

$$
\begin{array}{ccc}
e \alpha : 0 & \longrightarrow & B \longrightarrow E' \longrightarrow D \longrightarrow 0 \\
& & \| \hspace{1cm} \downarrow \hspace{1cm} \| \hspace{1cm} \alpha \\
e : 0 & \longrightarrow & B \longrightarrow E \longrightarrow A \longrightarrow 0 \\
\end{array}
$$

and

$$
\begin{array}{ccc}
\beta e : 0 & \longrightarrow & C \longrightarrow E'' \longrightarrow A \longrightarrow 0 \\
\| \hspace{1cm} \downarrow \hspace{1cm} \| \\
e : 0 & \longrightarrow & B \longrightarrow E \longrightarrow A \longrightarrow 0 \\
\end{array}
$$

One can see [23] for more details.

2.7. Let $D$ be a $C^*$-algebra. Denote by $P(D)$ and $U(D)$ all projections and all unitaries in $D$, respectively. The notations of equivalences of projections and unitaries are referred to [24].
For example, for any unitary $u$ and any projection $p$, $[u]_1$ and $[p]_0$ are the images of $u$ and $p$ in $K_1(D)$ and $K_0(D)$ respectively, and $[p]$ is the Murray-von Neumann equivalence class of $p$. Denote by $p \sim q$ when two projections $p$, $q$ are Murray-von Neumann equivalent.

Recall that ([10, 17]) a $C^*$-algebra $B$ has corona factorization property if for every full projection $p \in \mathcal{Q}(B)$ there exists $y \in \mathcal{Q}(B)$ such that $ypy^* = 1_{\mathcal{Q}(B)}$.

We call a $C^*$-algebra $B$ $\sigma_p$-unital if $B$ has a countable approximate unit consisting of projections.


2.8. Let $e : 0 \to B \to E \to A \to 0$ be an extension of $A$ by $B$. Denote by $K(e)$ the six term exact sequence of $e$ in $K$-theory:

\[
\begin{array}{cccc}
K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\
\delta_1 & \uparrow & & \downarrow \delta_0 & \\
K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B)
\end{array}
\]

Let $e_i \in \text{Ext}(A_i, B_i)$ $(i = 1, 2)$. We call $(\alpha_*, \beta_*, \lambda_*) : K(e_1) \to K(e_2)$ a morphism if there are homomorphisms $\alpha_* : K_*(A_1) \to K_*(A_2)$, $\beta_* : K_*(B_1) \to K_*(B_2)$, and $\lambda_* : K_*(E_1) \to K_*(E_2)$ making the obvious diagram commutative.

If $\alpha_*$, $\beta_*$ and $\lambda_*$ are isomorphisms, then $K(e_1)$ and $K(e_2)$ are called isomorphic, written $K(e_1) \cong K(e_2)$. Furthermore, if $[p]_0 \in K_0(E_1)$ and $[q]_0 \in K_0(E_2)$ such that $\lambda_0([p]_0) = [q]_0$, then this isomorphism is denoted by $(K(e_1), [p]_0) \cong (K(e_2), [q]_0)$.

If $A_1 = A_2 = A$, $B_1 = B_2 = B$ and there is an isomorphism $(id_{K_*(A)}, id_{K_*(B)}, \lambda_*) : K(e_1) \to K(e_2)$, then they are called congruent, written $K(e_1) \equiv K(e_2)$. Similarly, $(K(e_1), [p]_0) \equiv (K(e_2), [q]_0)$.

Let $A$ be a separable $C^*$-algebra. Recall that the Elliott invariant of $A$ is the 3-tuple: $(K_0(A), K_0(A)^+, K_1(A))$. We denote it by $\text{Ell}(A)$.

Let $\alpha : A_1 \to A_2$ be a homomorphism. Denote the induced maps on the invariants by $\alpha_* : K_*(A_1) \to K_*(A_2)$, respectively. Denote by $KK(\alpha)$ the induced elements in $KK(A_1, A_2)$.

2.9. Universal Coefficient Theorem (UCT). [25] Let $A$ and $B$ be separable $C^*$-algebras with $A \in \mathcal{N}$, where $\mathcal{N}$ is the bootstrap class. Then there is a short exact sequence

\[0 \to \text{Ext}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\gamma} \text{Hom}(K_*(A), K_*(B)) \to 0.\]

The map $\gamma$ has degree 0 and $\delta$ has degree 1.

3. UCTs for unital extensions.
In the following, we assume that $B$ is a $\sigma_p$-unital stable $C^*$-algebra if there is no contrary declaration. In fact this assumption that $B$ is $\sigma_p$-unital is not always necessary.

**Lemma 3.1.** Let $A$ and $B$ be $C^*$-algebras with $A$ unital and $B$ stable. Suppose $e : 0 \to B \to E \overset{\psi}{\to} A \to 0$ is an essential unital extension. If $p$ is a projection in $M(B)$ such that $\pi(p) = 1_{Q(B)}$, where $\pi : M(B) \to Q(B)$ is the quotient map, then there is an extension $p\epsilon p : 0 \to pBp \to pEp \to A \to 0$

such that the following diagram commutes

$$
\begin{array}{cccccc}
0 & \longrightarrow & pBp & \longrightarrow & pEp & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & pBp & \longrightarrow & M(pBp) & \longrightarrow & Q(pBp) & \longrightarrow & 0.
\end{array}
$$

**Proposition 3.2.** Let $A$ and $B$ be $C^*$-algebras with $A$ unital and $B$ stable. Suppose $e_i : 0 \to B \to E_i \overset{\psi_i}{\to} A \to 0$ are essential unital extensions with Busby invariants $\tau_i$. If $e_1$ and $e_2$ are weakly unitarily equivalent, then there exist two projections $p, q \in M(B)$ such that $p\epsilon_1 p \cong q\epsilon_2 q$ and the following diagram is commutative

$$
\begin{array}{cccccc}
0 & \longrightarrow & pBp & \longrightarrow & pE_1p & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & qBq & \longrightarrow & qE_2q & \longrightarrow & A & \longrightarrow & 0.
\end{array}
$$

**Proof.** Suppose that $\tau_2 = Ad u \circ \tau_1$ for some unitary $u$ in $Q(B)$. Since $B$ is stable and has a countable approximate unit consisting of projections, by [26, Remark 4.2] there exists a partial isometry $v$ in $M(B)$ such that $\pi(v) = u$. Set $p = v^*v$ and $q = vv^*$. Then $\pi(p) = \pi(q) = 1_{Q(B)}$.

Since $1 - p$ and $1 - q$ are in $B$, we have $p, q \in E_i$ for $i = 1, 2$. Hence $pE_1p$ and $qE_2q$ are $C^*$-subalgebras. By Lemma 3.1 we have a diagram for $p\epsilon_1 p$

$$
\begin{array}{cccccc}
0 & \longrightarrow & pBp & \longrightarrow & pE_1p & \psi_p & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & pBp & \longrightarrow & M(pBp) & \pi_p & \longrightarrow & Q(pBp) & \longrightarrow & 0.
\end{array}
$$

Similarly, there is a diagram for $q\epsilon_2 q$

$$
\begin{array}{cccccc}
0 & \longrightarrow & qBq & \longrightarrow & qE_2q & \psi_q & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & qBq & \longrightarrow & M(qBq) & \pi_q & \longrightarrow & Q(qBq) & \longrightarrow & 0.
\end{array}
$$

Since $p$ is equivalent to $q$ via $v$ in $M(B)$, there is an isomorphism:

$$
Ad v : pM(B)p \longrightarrow qM(B)q.
$$
We claim that the above isomorphism maps $pE_1p$ onto $qE_2q$.

For any $x \in pE_1p$, there exists $a \in A$ such that $\pi(x) = \tau_1(a) \in \tau_1(A)$. From $\pi_q(vxv^*) = \text{Ad}(v)\tau_1(a) = \tau_2(a)$, we have $vxv^* \in qE_2q$. Hence $\text{Adj}(pE_1p) \subset qE_2q$. Conversely, for any $y \in qE_2q$, set $z = v^*yv$. Then $\pi_p(z) = \text{Ad}(v^*)(\pi(y)) \in \tau_1(A)$ and hence $z \in pE_1p$. Since $\text{Adj}(z) = qyq = y$, it follows that $\text{Adj}(pE_1p) = qE_2q$. Therefore, $\text{Adj} : pE_1p \to qE_2q$ is an isomorphism and $\text{Adj}(pBp) \subset qBq$.

Finally, for any $x \in E_1$, set $a = \psi_p(pxp)$ and $b = \psi_q \circ \text{Adj}(pxp)$. Then we have $a = \tau_1^{-1}(\pi(x))$ and $b = \tau_2^{-1} \circ \text{Adj}(\pi(x))$. Hence $\tau_2(b) = \text{Adj}(\pi(v) \circ \tau_1(x) = \tau_2(a)$. It follows that $a = b$. Therefore, there is a commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & pBp & \longrightarrow & pE_1p & \psi_p \longrightarrow A & \longrightarrow & 0 \\
& & \downarrow\text{Adj} & & \downarrow\text{Adj} & & \\
0 & \longrightarrow & qBq & \longrightarrow & qE_2q & \psi_q \longrightarrow A & \longrightarrow & 0.
\end{array}
\]

\[\square\]

**Lemma 3.3.** Suppose that $A$ and $B$ are $C^*$-algebras with $A$ unital and $B$ $\sigma$-unital. Let $0 \to B \to E \to A \to 0$ be an essential unital extension. If $B$ is stable or simple and $p$ is a projection in $M(B)$ such that $\pi(p) = 1$, then there are two isomorphisms

\[K_*(pBp) \to K_*(B), \ K_*(pEp) \to K_*(E)\]

which are induced by the inclusion maps.

**Proof.** Suppose $p$ is a projection in $M(B)$ with $\pi(p) = 1$. Then $p \in E$. If $B$ is simple, then $pBp$ is full in $B$. If $B$ is stable, by [10, Proposition 3.3] or [23, Proposition 5.1] $p$ is full in $M(B)$ and so is $pBp$. It follows that $pBp$ is full in $B$.

Let $I(pBp)$ and $I(pEp)$ be the ideals generated by $pBp$ and $pEp$ in $E$ respectively. Then we have $I(pBp) \subset I(pEp)$. Since $pBp$ is full in $B$, we have $B \subset I(pEp)$. Note that $1_E - p$ is in $B$, so $1_E$ is in $I(pEp)$. Hence $I(pEp)$ is full in $E$.

Finally, the conclusion follows from [22, Corollary 3.3]. \[\square\]

**Lemma 3.4.** Suppose $B$ is a stable $C^*$-algebra. Let $p, q$ be two projections in $M(B)$ with $1 - p \in B$ and $1 - q \in B$. If there is a partial isometry $v \in M(B)$ such that $p = v^*v$ and $q = vv^*$, then the identity map on $K_*(B)$ factors through $K_*(\text{Adj}: K_*(pBp) \to K_*(qBq)$ which is induced by $\text{Adj}$. 

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Proof. Let $v$ be a partial isometry in $M(B)$ such that $p = v^*v$ and $q = vv^*$. Then by the proof of Proposition 3.2, $\text{Ad} v : pBp \rightarrow qBq$ is an isomorphism. Note that there is a sequence for any natural number $n$

$$ M_n(B) \xleftarrow{l_p} M_n(pBp) \xrightarrow{\text{Ad} v} M_n(qBq) \xrightarrow{l_q} M_n(B), $$

where $l_p$ and $l_q$ are the inclusion maps.

For any projection $x \in M_n(pBp)$, let $y = (v \otimes 1_n)x(v^* \otimes 1_n)$. Then $y \in M_n(qBq)$ and $x$ is Murray-von Neumann equivalent to $y$ in $M_n(B)$. Lemma 3.3 implies that $K_*(l_p)$ and $K_*(l_q)$ are isomorphisms. Hence we obtain

$$ id_{K_0(B)} = K_0(l_q) \circ K_0(\text{Ad} v) \circ K_0(l_p)^{-1}. $$

Using similar discussion, one can also check the result on $K_1$ level.

Lemma 3.5. ([27]) Let $A$ be a unital separable nuclear C*-algebra. Then there is a six-term exact sequence

$$ K_0(B) \longrightarrow \text{Ext}_a^*(A, B) \longrightarrow \text{Ext}(A, B) $$

$$ \begin{array}{c}
\uparrow \\
\text{Ext}(SA, B) \longleftarrow \text{Ext}_a^*(A, SB) \longleftarrow K_1(B).
\end{array} $$

Proof. Define $i : \mathbb{C} \rightarrow A$ by $i(1) = 1_A$. Let $C_i$ be the cone of $i$, i.e. $C_i$ is defined by the following diagram

$$ \begin{array}{c}
0 \longrightarrow SA \longrightarrow C_i \longrightarrow \mathbb{C} \longrightarrow 0 \\
\| \quad \downarrow \quad \quad \| \quad \downarrow \quad \quad \|
\end{array} $$

$$ \begin{array}{c}
0 \longrightarrow SA \longrightarrow CA \longrightarrow A \longrightarrow 0.
\end{array} $$

Then there is a six-term exact sequence for the functor $KK(-, B)$

$$ \begin{array}{c}
KK(\mathbb{C}, B) \longrightarrow KK(C_i, B) \longrightarrow KK(SA, B) \\
\uparrow \\
KK^1(SA, B) \longleftarrow KK^1(C_i, B) \longleftarrow KK^1(\mathbb{C}, B).
\end{array} $$

By the identifications $KK^*(\mathbb{C}, B) \cong K_*(B)$, $KK(C_i, B) \cong \text{Ext}_a^*(A, B)([27])$ and $KK^1(SA, B) \cong \text{Ext}(SA, B)$, we have the following six-term exact sequence

$$ \begin{array}{c}
K_0(B) \longrightarrow \text{Ext}_a^*(A, B) \longrightarrow \text{Ext}(A, B) \\
\uparrow \\
\text{Ext}(SA, B) \longleftarrow \text{Ext}_a^*(A, SB) \longleftarrow K_1(B).
\end{array} $$

\[\square\]
Let $H$ and $K$ be abelian groups and let $h_0 \in H$. We denote by $\text{Ext}((H, h_0), K)$ the set of equivalence classes of all extensions of abelian groups with base points of the form

$$0 \to K \to (G, g_0) \to (H, h_0) \to 0.$$ 

Then there is a short exact sequence of groups

$$0 \to K/\{f(h_0) | f \in \text{Hom}(H, K)\} \to \text{Ext}((H, h_0), K) \to \text{Ext}(H, K) \to 0.$$ 


Let $A$ and $B$ be $C^*$-algebras. When $A$ is unital, we set

$$\text{Ext}^u(A, B) = \text{Ext}((K_0(A), K_1(B)) \oplus \text{Hom}(K_1(A), K_0(B)),$$

and

$$G = \{f([1]_0) | f \in \text{Hom}(K_0(A), K_0(B))\}.$$

**Theorem 3.6.** ([28, Theorem 3.10]) Let $A$ be a unital separable nuclear $C^*$-algebra with $A \in \mathcal{N}$. Then there is a short exact sequence of groups

$$0 \to K_1(Q(B))/G \to \text{Ext}_s^u(A, B) \to \text{Ext}^u(A, B) \to 0.$$ 

**Proof.** Identifying $K_0(B)$ with $K_1(Q(B))$, we have

$$\text{Ext}(SA, B) \xrightarrow{\beta} K_1(Q(B)) \xrightarrow{\alpha} \text{Ext}^u(A, B)$$

which is exact at the middle term by Lemma 3.5.

For any extension $\tau : SA \to Q(B)$, let $\tilde{\tau} : C(T) \otimes A \to Q(B)$ be the unitization of $\tau$. Then $\beta([\tau]) = [\tilde{\tau}(z \otimes 1_A)]_1$. Suppose that $\tau_0$ is a strong-unital trivial extension. Then for any unitary $u \in Q(B)$ we have $\alpha([u]_1) = [Au \circ \tau_0]$.

Since $\text{Ker} \alpha = \text{Im} \beta$, it follows that $x \in \text{Ker} \alpha$ if and only if there is a unital extension $\tau : A \to Q(B)$ and $u \in K_1(Q(B))$ such that $Au \circ \tau = \tau$ and $x = [u]_1$.

As in the proof of [28, Theorem 3.10], one can show $\text{Im} \beta = G$. So we obtain the following exact sequence

$$0 \to K_1(Q(B))/G \to \text{Ext}_s^u(A, B) \to \text{Ext}^u(A, B) \to 0.$$ 

$\square$
**Theorem 3.7.** Let $A$ be a unital separable nuclear $C^*$-algebra with $A \in \mathcal{N}$. Let $\tau_i : A \to \mathcal{Q}(B)$ be unital-absorbing extensions. If there is a unitary $u \in \mathcal{Q}(B)$ such that $\tau_2 = Adu \circ \tau_1$, then $\tau_1$ and $\tau_2$ are unitarily equivalent if and only if $[u]_1 \in G$.

**Proof.** If $\tau_1$ and $\tau_2$ are unitarily equivalent then there is unitary $w \in \mathcal{Q}(B)$, which can lift to a unitary in $M(B)$, such that $\tau_2 = Adw \circ \tau_1$. Hence $Adw^*u \circ \tau_1 = \tau_1$. By the definition of $\alpha$ in Theorem 3.6, it follows that $[u]_1 = [w^*u]_1 \in G$.

Conversely, since $\tau_1 \sim w \tau_2$, we have $[\tau_1] - [\tau_2] \in K_1(\mathcal{Q}(B))/G$. When $[u]_1 \in G$, it follows that $[\tau_1] - [\tau_2] = 0$ in $K_1(\mathcal{Q}(B))/G$. Hence $[\tau_1] = [\tau_2]$ in $\text{Ext}^u(A, B)$. Since $\tau_1$ are unital-absorbing, we have $\tau_1 \sim \tau_2$.

**Corollary 3.8.** ([18, Theorem 3.7]) Let $A$ be a unital separable nuclear $C^*$-algebra with $A \in \mathcal{N}$. Let $\tau_i : A \to \mathcal{Q}(B)$ be unital full extensions. If $B$ has corona factorization property and there is a unitary $u \in \mathcal{Q}(B)$ such that $\tau_2 = Adu \circ \tau_1$, then $\tau_1$ and $\tau_2$ are unitarily equivalent if and only if $[u]_1 \in G$.

**Proof.** Since $B$ has corona factorization property and $\tau_i : A \to \mathcal{Q}(B)$ are unital full extensions, $\tau_i$ are unital-absorbing by [10]. Therefore, the conclusion is immediate from Theorem 3.7. □

**Remark 3.9.** Lemma 3.5 first appeared in [27], but its proof is based on Kasparov module. We have completed [28] without knowing [27]. When the author was visiting Purdue University during 2011, he began to notice it from there. Now with the help of Lemma 3.5 and some technique, we can remove most restrictive conditions and simplify the proofs in [28, 3.10-3.12, 3.15].

**Theorem 3.10.** Suppose that $A$ and $B$ are $C^*$-algebras with $A$ unital and $B$ stable. Let $e_i : 0 \to B \to E_i \to A \to 0$ be essential unital extensions. Suppose $\tau_2 = Adu \circ \tau_1$ for some unitary $u$ in $\mathcal{Q}(B)$. Let $v$ be a partial isometry in $M(B)$ such that $\pi(v) = u$, and let $p = v^*v$ and $q = vv^*$. Then

$$\left(K(e_1), [1]_0\right) \equiv \left(K(e_2), [q]_0 + [1 - p]_0\right).$$

**Proof.** Let $\tau_i$ be the Busby invariants of $e_i$. Then there is a unitary $u \in \mathcal{Q}(B)$ such that $\tau_2 = Adu \circ \tau_1$. Choose a partial isometry $v \in M(B)$ such that $\pi(v) = u$. Let $p = v^*v$ and $q = vv^*$. Then $1 - p, 1 - q \in B$. Since $e_i$ are unital extensions, we have $p, q \in E_i$. 

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By Proposition 3.2 there is a commutative diagram:

\[
\begin{array}{ccccccccc}
  e_1 : 0 & \longrightarrow & B & \xrightarrow{t_1} & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
  \uparrow & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
  p e_1 p : 0 & \longrightarrow & p B p & \xrightarrow{\operatorname{Adv}} & p E_1 p & \longrightarrow & A & \longrightarrow & 0 \\
  \downarrow & & \downarrow & & \downarrow & & \uparrow & & \uparrow \\
  q e_2 q : 0 & \longrightarrow & q B q & \xrightarrow{\operatorname{Adv}} & q E_2 q & \longrightarrow & A & \longrightarrow & 0 \\
  \downarrow & & \downarrow & & \downarrow & & \uparrow & & \uparrow \\
  e_2 : 0 & \longrightarrow & B & \xrightarrow{t_2} & E_2 & \longrightarrow & A & \longrightarrow & 0
\end{array}
\]

Hence \( e_1 \leftarrow p e_1 p \cong q e_2 q \rightarrow e_2 \). By Lemma 3.3 we have

\[
(K(pe_1 p), [p]_0) \cong (K(e_1), [p]_0), \quad (K(qe_2 q), [q]_0) \cong (K(e_2), [q]_0).
\]

Note that \((\operatorname{Adv}, \operatorname{Adv}, \text{id}) : pe_1 p \rightarrow qe_2 q\) is an isomorphism and \(\operatorname{Adv}(p) = q\), so there is an isomorphism

\[
(K(pe_1 p), [p]_0) \rightarrow (K(qe_2 q), [q]_0).
\]

By Lemma 3.4 and the six-term exact sequence in K-theory, we have the following commutative diagram

\[
\begin{array}{ccccccc}
  & \longrightarrow & K_0(B) & \xrightarrow{K_0(t_1)} & K_0(E_1) & \rightarrow & K_0(A) & \rightarrow \\
  & \uparrow & \uparrow & \uparrow & \uparrow & \downarrow & \downarrow & \downarrow \\
  & \longrightarrow & K_0(B) & \xrightarrow{K_0(t_2)} & K_0(E_2) & \rightarrow & K_0(A) & \rightarrow
\end{array}
\]

Hence there is an isomorphism \((\text{id}, \eta_0, \text{id}) : (K(e_1), [p]_0) \rightarrow (K(e_2), [q]_0)\).

Note that \(\eta_0([p]_0) = [q]_0\) and \(\eta_0([1-p]_0) = [1-p]_0\). Hence we have \(\eta_0([1]_0) = [q]_0 + [1-p]_0\).

Therefore,

\[
(K(e_1), [1]_0) \equiv (K(e_2), [q]_0 + [1-p]_0).
\]

\[\square\]

Let \(0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0\) be an extension with index maps \(\delta_0\) and \(\delta_1\) in its K-theory. We set \(G' = \{f([1]_0)\mid f \in \operatorname{Hom}(\operatorname{Ker}\delta_0, \operatorname{Coker}\delta_1)\}\) and let \(\pi : K_0(B) \rightarrow \operatorname{Coker}\delta_1\) be the quotient map.

**Theorem 3.11.** Let \(e_i\) be essential unital extensions with Busby invariant \(\tau_i\). If \(e_1\) is weakly unitarily equivalent to \(e_2\) by a unitary \(u \in Q(B)\). Then

\[
(K(e_1), [1]_0) \equiv (K(e_2), [1]_0)
\]

if and only if \(\pi([u]_1)\) is in \(G'\).
Proof. Let $v$ be a partial isometry in $M(B)$ such that $\pi(v) = u$, and let $p = v^*v$ and $q = vv^*$. Then by Theorem 3.10 we have

$$(K(e_1), [1]_0) \equiv (K(e_2), [q]_0 + [1 - p]_0).$$

Hence, $(K(e_1), [1]_0) \equiv (K(e_2), [1]_0)$ if and only if

$$(K(e_2), [1]_0) \equiv (K(e_2), [q]_0 + [1 - p]_0).$$

From [23, Proposition 2.3], this is equivalent to

$$(\zeta(e_2), [1]_0) \equiv (\zeta(e_2), [q]_0 + [1 - p]_0).$$

For abelian group extensions with base point $[1]_0$ of Ker$\delta_0$ by Coker$\delta_1$, there is a short exact sequence of groups

$$0 \to \text{Coker}\delta_1 / G' \to \text{Ext}((\text{Ker}\delta_0, [1]_0), \text{Coker}\delta_1) \to \text{Ext}((\text{Ker}\delta_0, \text{Coker}\delta_1)) \to 0,$$

where $G' = \{ f([1]_0) | f \in \text{Hom}(\text{Ker}\delta_0, \text{Coker}\delta_1) \}$. Hence, it follows that

$$(\zeta(e_2), [1]_0) \equiv (\zeta(e_2), [q]_0 + [1 - p]_0)$$

if and only if $[q]_0 - [p]_0$ is in $G'$.

We identify $K_1(\mathbb{Q}(B))$ with $K_0(B)$. By the following commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Coker}\delta_1 & \longrightarrow & K_0(E_2) & \longrightarrow & \text{Ker}\delta_0 & \longrightarrow & 0 \\
& & \uparrow\pi & & \| & & \downarrow & & \\
& & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A), & & 
\end{array}
$$

we have

$$\pi([u]_1) = \pi([1 - p]_0 - [1 - q]_0) = [q]_0 - [p]_0$$

in $K_0(E_2)$. Therefore, $(K(e_1), [1]_0) \equiv (K(e_2), [1]_0)$ is equivalent to $\pi([u]_1) \in G'$.

Theorem 3.12. Suppose $e_i$ are essential unital extensions with Busby invariant $\tau_i$ and $e_1$ is weakly unitarily equivalent to $e_2$. If the index maps of $e_i$ are trivial and

$$(K(e_1), [1]_0) \equiv (K(e_2), [1]_0),$$

then $[e_1] = [e_2]$ in $\text{Ext}_u^*(A, B)$. 

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**Proof.** Set \( \sigma_1 = (K(e_2), [1]_0) \) and \( \sigma_2 = (K(e_2), [q]_0 + [1 - p]_0) \) in \( \text{Ext}_A(K_+(A), K_+(B)) \). Since \( \tau_1 \) and \( \tau_2 \) are weakly unitarily equivalent, by Theorem 3.10 we have

\[
(K(e_1), [1]_0) \equiv (K(e_2), [q]_0 + [1 - p]_0).
\]

Hence, by the assumption we obtain \( \sigma_1 = \sigma_2 \) in \( \text{Ext}((K_0(A), [1_A]_0), K_0(B)) \). Then from the short exact sequence

\[
0 \to K_0(B)/G \to \text{Ext}((K_0(A), [1_A]_0), K_0(B)) \to \text{Ext}(K_0(A), K_0(B)) \to 0
\]
or from Theorem 3.11, it follows that

\[
[1 - p]_0 - [1 - q]_0 = [q]_0 + [1 - p]_0 - [1]_0 \in G.
\]

Since \( v \) is a lifting of \( u \), then \([u]_1 = [1 - p]_0 - [1 - q]_0\) under the isomorphism \( K_1(Q(B)) \cong K_0(B) \). This implies that \([u]_1 \in G\). By Theorem 3.6, we have \([\tau_1] = [\tau_2]\) in \( \text{Ext}_A^e(A, B) \). \(\square\)

**Lemma 3.13.** [28, Lemma 3.11] Let \( A \) be a separable, nuclear \( C^* \)-algebra with unit. Then the natural homomorphism \( \text{Ext}_A^e(A, B) \to \text{Ext}(A, B) \) is injective. Moreover, this map is an isomorphism if and only if the map \( \delta : \text{Ext}(A, B) \to K_0(Q(B)) \cong K_1(B) \) is trivial, where \( \delta([\tau]) = [\tau(1)]_0 \) for any \([\tau] \in \text{Ext}(A, B)\).

**Proof.** Suppose that \( \tau_i : A \to Q \) are unital essential extensions such that \([\tau_1] = [\tau_2]\) in \( \text{Ext}(A, B) \). Then there is a trivial absorbing extension \( \sigma_0 \) such that

\[
\tau_1 \oplus \sigma_0 \sim \tau_2 \oplus \sigma_0.
\]

Let \( \sigma_1 \) be a trivial unital extension and set \( p = \sigma_0(1) \). Note that \( p \oplus 1 \) and \( 1 \oplus 1 \) are infinite projections in \( M_2(Q(B)) \). Since \([p \oplus 1]_0 = [1 \oplus 1]_0 \) in \( K_0(Q(B)) \), there is a partial isometry \( v \in M_2(Q(B)) \) such that \( v^*v = p \oplus 1 \) and \( vv^* = 1 \oplus 1 \). By \( M_2(Q(B)) \cong Q(B) \) there exists \( w \in Q(B) \) such that \( ww^* = (\sigma_0 \oplus \sigma_1)(1) \) and \( w^*w = 1 \).

Set \( \sigma_2 = \sigma_0 \oplus \sigma_1 \). Then \( \sigma_2 \) is trivial and absorbing and \( \sigma_2(p) \sim 1 \). Replacing \( \sigma_0 \) by \( \sigma_2 \) in the proof of Lemma 3.11 in [28] and applying a similar argument, we obtain \([\tau_1] = [\tau_2]\) in \( \text{Ext}_A^e(A, B) \).

Finally, the last part of the theorem follows from Lemma 3.5. \(\square\)

**Lemma 3.14.** Suppose that \( \tau : A \to Q(B) \) is an extension with \( p = \tau(1) \). Then there is a unital extension \( \sigma : A \to Q(B) \) such that \( \tau \sim \sigma \) if and only if \([p]_0 = 0\) in \( K_0(Q(B)) \).
Since $\tau \sim \sigma$, there is a trivial absorbing extension $\delta$ such that $\tau \oplus \delta \sim \sigma \oplus \delta$. Set $q = \delta(1)$. Then $\begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}$ is unitarily equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$. This implies $[p]_0 = [1]_0 = 0$. By $[p]_0 = 0$ we have $[1 \oplus p]_0 = [1 \oplus 1]_0$ in $K_0(Q(B))$. By [1, 12.2] we note that

\[ K_0(Q(B)) = \{ [p] : p \text{ is infinite in } Q(B) \} \]

Hence, there is a partial isometry $v \in M_2(Q(B))$ such that $1 \oplus p \sim v \oplus 1$.

Choose a unital-absorbing extension $\tau_0$. Then we have

\[ (\tau_0 \oplus \tau) : A \to M_2(Q(B)), \; 1 \mapsto \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \]

and

\[ \text{Ad}v \circ (\tau_0 \oplus \tau) : A \to M_2(Q(B)), \; 1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

Let $\sigma = \theta \circ \text{Ad}v \circ (\tau_0 \oplus \tau)$. Then $\sigma$ is unital and $\sigma = \text{Ad}(\theta(v))(\tau_0 \oplus \tau)$.

Set $T = \theta(v)$. Then $(\tau_0 \oplus \tau)(1) = \theta(1 \oplus p) \sim 1$. Therefore, $\sigma \sim \tau_0 \oplus \tau \sim \tau$.

**Theorem 3.15.** [28, Theorem 3.12] Let $A$ be a unital separable nuclear $C^*$-algebra with $A \in \mathcal{N}$. Then there is a short exact sequence

\[ 0 \to \text{Ext}(K_*(A), K_*(B)) \to \text{Ext}_u(A, B) \to \text{Hom}_{[1]}(K_*(A), K_*(B)) \to 0. \]

**Proof.** Let $\gamma_0$ be the quotient map in the UCT. Note that $\gamma_0([\tau]) = (K_0(\tau), K_1(\tau))$ for any extension $\tau$. By Lemma 3.13 we can set

\[ \gamma_1 = \gamma_0 \vert_{\text{Ext}_u(A, B)} : \text{Ext}_u(A, B) \to \text{Hom}_{[1]}(K_*(A), K_*(B)). \]

If $\gamma_0([\tau]) \in \text{Hom}_{[1]}(K_*(A), K_*(B))$, then

\[ [\tau(1_A)]_0 = K_0(\tau)([1_A]_0) = 0 \]

in $K_0(Q(B))$.

By Lemma 3.14, there is a unital extension $\tau' : A \to Q(B)$ such that $[\tau] = [\tau']$ in $\text{Ext}(A, B)$. Then $\gamma$ is surjective. By a similar argument as above, one can show

\[ \text{Ker}\gamma_1 = \text{Ker}\gamma_0 = \text{Ext}(K_*(A), K_*(B)). \]

Therefore, we obtain the required exact sequence. \qed
Theorem 3.16. [28, 27] Let $A$ be a unital separable nuclear $C^*$-algebra with $A \in \mathcal{N}$. Then there is a short exact sequence of groups

$$0 \rightarrow \text{Ext}_1(K_*(A), K_*(B)) \rightarrow \text{Ext}_u^u(A, B) \rightarrow \text{Hom}_{[1]}(K_*(A), K_*(B)) \rightarrow 0,$$

where $\text{Ext}_{[1]}(K_*(A), K_*(B)) = \text{Ext}((K_0(A), [1]_A), K_0(B)) \oplus \text{Ext}(K_1(A), K_1(B))$.

**Proof.** Define

$$\gamma_2 : \text{Ext}_u^u(A, B) \rightarrow \text{Hom}_{[1]}(K_*(A), K_*(B))$$

by $\gamma_2([\tau]) = (K_0(\tau), K_1(\tau))$ for any extension $\tau$. As in the proof of Theorem 3.15, one can check the above sequence is exact at $\text{Ext}_u^u(A, B)$ and $\text{Hom}_{[1]}(K_*(A), K_*(B))$.

Define

$$\varepsilon : \text{Ker}\gamma_2 \rightarrow \text{Ext}_{[1]}(K_*(A), K_*(B))$$

by $\varepsilon([\tau]) = (K(\tau), [1]_0)$ for any extension $\tau$, where $e$ is the exact sequence of $\tau$. By Theorem 3.12, $\varepsilon$ is well-defined and injective.

Let $\gamma_1$ be as in Theorem 3.15. By Theorem 3.6, we have the following commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & K_1(Q(B))/G & \rightarrow & \text{Ker}\gamma_2 & \rightarrow & \text{Ker}\gamma_1 & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\varepsilon} & & & & \downarrow & \\
0 & \rightarrow & K_1(Q(B))/G & \rightarrow & \text{Ext}_{[1]}(K_*(A), K_*(B)) & \rightarrow & \text{Ext}(K_*(A), K_*(B)) & \rightarrow & 0.
\end{array}
$$

Since the map $\text{Ker}\gamma_1 \rightarrow \text{Ext}(K_*(A), K_*(B))$ is an isomorphism, it follows from the Five Lemma that $\varepsilon$ is an isomorphism. $\square$

4. Applications to BDF-theory

In the 1970’s, L. Brown, R. Douglas and P. Fillmore established the famous BDF-theory in order to study essentially normal operators on an infinite dimensional Hilbert space and extensions of the $C^*$-algebra $C(X)$ by the compact operators. They proved the following UCT

$$\text{Ext}(X) \cong \text{Hom}(K^1(X), \mathbb{Z})$$

for $A = C(X)$ with $X$ is a compact subset of the plane. The theory of classification of $C^*$-algebra extensions has developed rapidly since then.

Subsequently, G. Kasparov introduced $KK$-theory which generalizes both $K$-theory and extension theory. J. Rosenberg and C. Schochet proved the UCT for $KK$-groups in a general form. Hence many classes of $C^*$-algebra extensions have been classified. But the main part of these classifications of extensions is not in the original sense of the BDF-theory, which

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considered unital extensions and classified extensions up to unitary equivalence but not stable unitary equivalence. As we know, they are very different.

Hence, along the original intention of the BDF-theory, we meet a question: for which classes of $C^*$-algebras the classic BDF-theory still holds? In the following, we will use the results in Section 3 to classify certain classes of unital extensions in the sense of the BDF-theory.

Suppose $\mathcal{C}$ is a class of unital separable $C^*$-algebras and let $\mathcal{KC} = \{B \otimes K : B \text{ is in } \mathcal{C}\}$. We consider the following conditions:

(i) For any $A$ in $\mathcal{C}$, $A \otimes K$ has the corona factorization property.

(ii) For any $A, B$ in $\mathcal{C}$ and $a \in KK(A,B)$ such that $K_*(a) : (Ell(A), [1]_0) \to (Ell(B), [1]_0)$ is an isomorphism, there is an isomorphism $\varphi : A \to B$ such that $KK(\varphi) = a$.

(iii) For any $A, B$ in $\mathcal{C}$ and $a \in KK(A\otimes K, B\otimes K)$ such that $K_*(a) : Ell(A \otimes K) \to Ell(B \otimes K)$ is an isomorphism, there is an isomorphism $\varphi : A \otimes K \to B \otimes K$ such that $KK(\varphi) = a$.

Note that with the help of certain other assumptions one can obtain (iii) from (ii). In order to simplify the proof, we use directly the condition (iii) here.

Till now we have known that there are many classes of $C^*$-algebras with these properties. One can check that among the following classes (they are not independent) $CI-IV$ and $CVII$ satisfy (i) and $CI-CVI$ satisfy (ii) and (iii).

(CI) unital $AF$-algebras (see [17, Corollary 3.6]);

(CII) unital simple purely infinite $C^*$-algebras satisfying the UCT (see [21]);

(CIII) unital simple $C^*$-algebras with real rank zero, stable rank one and weakly unperforated $K_0$-group (see [17]);

(CIV) $C(X)$, where $X$ is a compact subset of the plane (see [17]);

(CV) unital simple $AH$-algebras with slow dimension growth and real rank zero;

(CVI) unital simple nuclear $C^*$-algebras satisfying the UCT with tracial rank zero (see [19]).

(CVII) unital separable $C^*$-algebras with finite decomposition ranks (see [20], Corollary 5.12).

For the convenience of our referring, we still need the following condition for a unital $C^*$-algebra $A$.

(iv) For any $C^*$-algebra $B$, $K_0(B) = \{\rho([1]_0) | \rho \in \text{Hom}(K_0(A), K_0(B))\}$.

It should be noted that there are many $C^*$-algebras satisfying (iv), for example, all unital separable commutative $C^*$-algebras, Cuntz algebra $O_\infty$, Jiang-Su algebra $Z$, irrational rotation $C^*$-algebras, and all unital $C^*$-algebras with $K_0(A) = \mathbb{Z} \oplus H$ and $[1_A]_0 = (1, 0)$.

The following theorem can be viewed as a generalization for the classic BDF-theory, which
is the closest to the original one with $\text{Ext}^u(C(X), K) \cong \text{Hom}(K^1(X), \mathbb{Z})$.

**Theorem 4.1.** Let $A$ be a unital separable nuclear $C^*$-algebra with $A \in \mathcal{N}$ and $K_*(A)$ free.

1. If $B$ is a stable $\sigma_p$-unital $C^*$-algebra, then

$$\text{Ext}^u(A, B) \cong \text{Hom}([1], (K_0(A), K_1(B)) \oplus \text{Hom}(K_1(A), K_0(B))].$$

2. Suppose $C$ is a class of unital separable $C^*$-algebras satisfying the condition (i) and $B$ is in the class $KC$. If $A$ satisfies (iv), then

$$\text{Ext}^u(A, B) \cong \text{Hom}([1], (K_0(A), K_1(B)) \oplus \text{Hom}(K_1(A), K_0(B)).$$

**Proof.** It follows from Theorem 3.6 and Theorem 3.16.

The following corollaries (4.2-4.5) are the main results which were proved in the case of nonstable by H. Lin in his several papers. Now using Theorem 4.1 we can proved the stable case immediately.

**Corollary 4.2.** [11, Theorem 7.1] Let $X$ be a compact subset of the plane and suppose that $B$ is in $KC_I$. Then

$$\text{Ext}^u(C(X), B) \cong \text{Hom}((K_*(C(X)), [1]), (K_*(Q(B)), [1])).$$

**Corollary 4.3.** [12, Theorem 4.6] Let $X$ be a compact subset of the plane and suppose that $B$ is in $KC_{III}$. Then

$$\text{Ext}^u(C(X), B) \cong \text{Hom}((K_*(C(X)), [1]), (K_*(Q(B)), [1])).$$

**Corollary 4.4.** [13, Theorem 2.10] Let $X$ be a finite CW complex and suppose that $B$ is in $KC_{III}$. Then

$$\text{Ext}(C(X), B) \cong KK(C(X), Q(B)).$$

**Proof.** By the Dadarlat isomorphism [5], we have $\text{Ext}(C(X), B) \cong KK(C(X), Q(B))$. Since $B$ has the corona factorization property, there is an isomorphism $\text{Ext}(C(X), B) \cong \text{Ext}(C(X), B)$. Therefore $\text{Ext}(C(X), B) \cong KK(C(X), Q(B))$. 

**Corollary 4.5.** [13, Theorem 3.8]. Let $X$ and $B$ be as in Corollary 4.4. Then

$$\text{Ext}^u(C(X), B) \cong KK^u(C(X), Q(B)).$$
Finally, we give a classification theorem for certain unital full extensions of \( C(X) \) up to isomorphism.

**Lemma 4.6.** ([6, 23]) Let \( A \) and \( B \) be separable nuclear \( C^* \)-algebras in \( \mathcal{N} \) with \( B \) stable, and let \( x_1, x_2 \in \text{Ext}(A, B) \). Then \( K(x_1) = K(x_2) \) in \( \text{HExt}(A, B) \) if and only if there exist elements \( a \) in \( KK(A, A) \) and \( b \) in \( KK(B, B) \) with \( K_*(a) = K_*(id_A) \) and \( K_*(b) = K_*(id_B) \) such that \( x_1b = ax_2 \).

**Lemma 4.7.** [10, Theorem 3.1] Suppose \( A \), \( B \) are separable \( C^* \)-algebras with \( A \) unital and \( B \) stable. If \( e : 0 \to B \to E \to A \to 0 \) is a unital essential extension, then \( e \) is unital-absorbing if and only if \( e \) is a purely large extension.

**Theorem 4.8.** Let \( C \) be a class of unital separable \( C^* \)-algebras satisfying the above conditions (i)-(iii). Suppose that \( B_i \) is in the class \( K\mathcal{C} \) and \( A = C(X) \) with \( X \) a compact subset of the plane. If \( e_i : 0 \to B_i \to E_i \to A \to 0 \) are two unital full extensions, then the following statements are equivalent:

1. \( E_1 \cong E_2 \).
2. There is an extension isomorphism \((\beta, \eta, \alpha) : e_1 \to e_2\).
3. There are isomorphisms \( \beta_2 : \text{Ell}(B_1) \to \text{Ell}(B_2) \), \( \eta_2 : (K_*(E_1), [1]_0) \to (K_*(E_2), [1]_0) \) and \( \alpha_2 : \text{Ell}(A) \to \text{Ell}(A) \) such that \((\beta_2, \eta_2, \alpha_2) : (K(e_1), [1]_0) \to (K(e_2), [1]_0) \) is an isomorphism.

**Proof.** We only need to show (3) \(\implies\) (2).

From the isomorphism \((\beta_2, \eta_2, \alpha_2) : (K(e_1), [1]_0) \to (K(e_2), [1]_0)\), it follows that \( \alpha_2 \) keeps the equivalence classes of units. Hence there is an isomorphism \( \varphi : A \to A \) such that \( \varphi_2 = \alpha_2 \).

By (iii), there are isomorphism \( \psi : B_1 \to B_2 \) such that \( \psi_2 = \beta_2 \). As in the proof of Theorem 3.11 in [29], we have the following extension isomorphism:

\[
(id_{K_*(B_2)}, \eta_*, id_{K_*(A_1)}) : (K(e_1\psi), [1]_0) \longrightarrow (K(\varphi e_2), [1]_0),
\]

that is,

\[
(K(e_1\psi), [1]_0) \cong (K(\varphi e_2), [1]_0).
\]

By Lemma 4.6, there exist elements \( a \) in \( KK(A, A) \) and \( b \) in \( KK(B_2, B_2) \) with \( K_*(a) = K_*(id_A) \) and \( K_*(b) = K_*(id_{B_2}) \) such that \([e_1\psi]b = a[\varphi e_2]\). By (iii) there is isomorphism \( \lambda : B_2 \to B_2 \) such that \( KK(\lambda) = b \). Since \( K_*(A) \) is free, by the UCT, \( KK(A, A) \cong \text{Hom}(K_*(A), K_*(A)) \). Thus there is an automorphism \( h : A \to A \) such that \( KK(h) = a \).

It follows that

\[
[e_1]KK(\psi)KK(\lambda) = KK(h)KK(\varphi)[e_2].
\]
Hence \([e_1 \psi \lambda] = [h \varphi e_2]\) in \(\text{Ext}(A, B_2)\).

Note that \(e_1 \psi \lambda\) and \(h \varphi e_2\) are unital extensions. By Lemma 3.13 we have \([e_1 \psi \lambda] = [h \varphi e_2]\) in \(\text{Ext}^u(A, B_2)\). Therefore, there are strong unital trivial extensions \(\sigma_1, \sigma_2 \in \text{Ext}(A, B_2)\) such that

\[
e_1 \psi \lambda \oplus \sigma_1 \overset{w}{\sim} h \varphi e_2 \oplus \sigma_2.
\]

Since \(\lambda \circ \psi : B_1 \to B_2\) is an isomorphism, then there are isomorphisms \(\kappa : M(B_1) \to M(B_2)\) and \(\tilde{\kappa} : Q(B_1) \to Q(B_2)\) such that the following diagram is commutative

\[
\begin{array}{cccccc}
0 & \to & B_1 & \to & M(B_1) & \to & Q(B_1) & \to & 0 \\
& & \downarrow \lambda \circ \psi & & \downarrow \kappa & & \downarrow \tilde{\kappa} & \\
0 & \to & B_2 & \to & M(B_2) & \to & Q(B_2) & \to & 0.
\end{array}
\]

Let \(\tau_1\) be the Busby invariant of \(e_1\). It follows that \(\tilde{\kappa} \circ \tau_1\) is the Busby invariant of \(e_1 \psi \lambda\).

Similarly, we have the following commutative diagram

\[
\begin{array}{cccccc}
h \varphi e_2 : 0 & \to & B_2 & \to & E_2 & \to & A & \to & 0 \\
& & \| & & \| & & \| & \\
e_2 : 0 & \to & B_2 & \to & E_2 & \to & A & \to & 0 \\
& & \| & & \| & & \| & \\
0 & \to & B_2 & \to & M(B_2) & \to & Q(B_2) & \to & 0.
\end{array}
\]

Therefore, \(\tau_2 \circ \varphi \circ h : A \to Q(B_2)\) is the Busby invariant of \(h \varphi e_2\).

Since \(\tau_1, \tau_2\) are full homomorphisms and \(\psi \lambda, \tilde{\kappa}\) are isomorphisms, it follows that \(\tilde{\kappa} \circ \tau_1\) and \(\tau_2 \circ \varphi \circ h\) are full. Since \(B_2\) has the corona factorization property, by Lemma 4.7 unital full extensions of \(A\) by \(B_2\) are unital-absorbing. Hence

\[
e_1 \psi \lambda \overset{\sim}{\sim} e_1 \psi \lambda \oplus \sigma_1 \overset{w}{\sim} h \varphi e_2 \oplus \sigma_2 \overset{\sim}{\sim} h \varphi e_2.
\]

By the assumption and Theorem 3.6, \(e_1 \psi \lambda\) and \(h \varphi e_2\) are unitarily equivalent.

Finally,

\[
e_1 \cong e_1 \psi \lambda \overset{\sim}{\sim} h \varphi e_2 \cong e_2.
\]

\[\square\]

**References**


[31] C. Wei, Classification of unital extensions of AT-algebras, (submitted)