CLOSE OPERATOR ALGEBRAS

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East China Normal University
A METRIC ON SUBALGEBRAS OF $\mathcal{B}(\mathcal{H})$  
Kadison-Kastler 1972

**Definition**

Let $A, B$ be $C^*$-subalgebras of $\mathcal{B}(\mathcal{H})$. The Kadison-Kastler distance $d(A, B)$ is the infimum of $\gamma > 0$ such that for all operators $x$ in the unit ball of one algebra, there exists $y$ in the unit ball of the other algebra with $\|x - y\| < \gamma$.

**Theme of the Talk**

What can be said when $d(A, B)$ is small?

- Aim: Give survey of what is known.
- See similarities and differences between $C^*$-algebra and von Neumann algebra settings.
- Establish connections to similarity and derivation problems.
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**A Metric on Subalgebras of \( \mathcal{B}(\mathcal{H}) \)**

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**Theme of the Talk**

What can be said when \( d(A, B) \) is small?

- **Aim**: Give survey of what is known.
- See similarities and differences between \( C^* \)-algebra and von Neumann algebra settings.
- Establish connections to similarity and derivation problems.
Questions about close operator algebras

Easy construction

For a unitary $u$, $d(A, uAu^*) \leq 2\|u - 1_H\|$.

Is this the only way of constructing a close pair of operator algebras?

More generally, we have a range of questions

Suppose $A, B \subseteq B(H)$ have $d(A, B)$ small.

- Must $A$ and $B$ share the same properties and invariants?
- Must $A$ and $B$ be $^*$-isomorphic?
- Must $A$ and $B$ be spatially isomorphic? Can one find a unitary implementing a spatial isomorphism in $(A \cup B)^{''}$?
- Is there a unitary $u \approx 1_H$ with $uAu^* = B$?

- Kadison-Kastler conjectured ??. Open for separable $C^*$-algebras.
- ?? is open for von Neumann algebras. Fails for separable $C^*$-algebras.
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For a unitary $u$, $d(A, uAu^*) \leq 2\|u - 1_{\mathcal{H}}\|$. Is this the only way of constructing a close pair of operator algebras?

MORE GENERALLY, WE HAVE A RANGE OF QUESTIONS

Suppose $A, B \subset \mathcal{B}(\mathcal{H})$ have $d(A, B)$ small.

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**Some Properties and Invariants**

**Type Decomposition**

**Theorem (Kadison-Kastler 1972)**

*Close von Neumann algebras have the same type decompositions.*

Precisely, suppose:

- $M, N$ are von Neumann algebras on $\mathcal{H}$ with $d(M, N)$ sufficiently small.
- $p_{I_n}, p_{II_1}, p_{II_\infty}, p_{III}$ be the central projections in $M$ onto the parts of types $I_n, II_1, II_\infty$ and III respectively.
- $q_{I_n}, q_{II_1}, q_{II_\infty}, q_{III}$ corresponding projections for $N$.

Then each $\|p_j - q_j\|$ is small.

They also show that if $d(M, N)$ is small ($< 1/10$), then $M$ is a factor if and only if $N$ is a factor.
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**Theorem (Phillips 1974)**

Suppose $A$ and $B$ are sufficiently close $C^*$-algebras. Then

- $A$ and $B$ have isomorphic and close ideal lattices.
- This takes primitive ideals to primitive ideals and is a homeomorphism for the hull-kernel topology.
- $A$ is type $I$ if and only if $B$ is type $I$.

By isomorphic and close ideal lattices, we mean that there is a lattice isomorphism $A \triangleright l \leftrightarrow \theta(l) \triangleleft B$ such that $d(l, \theta(l))$ is small for all $l$.

**Corollary**

If $d(A, B)$ is sufficiently small and $A$ is abelian, then $A \cong B$. 

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If $d(A, B)$ is sufficiently small and $A$ is abelian, then $A \cong B$. 
Near Containments
Christensen 1980

**Definition**

For $A, B \subseteq B(\mathcal{H})$ write $A \subseteq_\gamma B$ if given $x \in A$, there exists $y \in B$ such that $\|x - y\| \leq \gamma \|x\|$. In this case say $A$ is $\gamma$-contained in $B$.

Similar range of questions:

1. Must a sufficiently small near containment $A \subseteq B$ give rise to an embedding $A \hookrightarrow B$?
2. If so, can an embedding $\theta : A \hookrightarrow B$ with $\|\theta - \iota\|$ small be found?
3. Must a sufficiently small near containment arise from a small unitary conjugate of a genuine inclusion?
A CB-VERSION OF THE METRIC

- It’s natural to take matrix amplifications of operator algebras
- \( A \subset B(H) \), gives \( M_n(A) \subseteq M_n(B(H)) \cong B(H^n) \).

**DEFINITION**

Given \( A, B \subset B(H) \), define

\[
d_{cb}(A, B) = \sup_n (M_n(A), M_n(B)).
\]

Similarly \( A \subseteq_{cb, \gamma} B \) iff \( M_n(A) \subseteq_{\gamma} M_n(B) \) for all \( n \).

**THEOREM (KHOSHKAM 1984)**

Suppose \( A, B \) are C*-algebras with \( d_{cb}(A, B) < 1/3 \). Then
\( K_0(A) \cong K_0(B) \) and \( K_1(A) \cong K_1(B) \).
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**Theorem (Khoshkam 1984)**

Suppose $A, B$ are $C^*$-algebras with $d_{cb}(A, B) < 1/3$. Then $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. 
**Arveson’s Distance Formula**

Let $A \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra and $T \in \mathcal{B}(\mathcal{H})$. Then

$$d'(T, A') = \|\text{ad}(T)|_A\|_{cb}/2.$$ 

Here $\text{ad}(T)|_A$ is the spatial derivation $x \mapsto [T, x] = Tx - xT$.

**Consequence**

- $A \subseteq_{\gamma, cb} B \implies B' \subseteq_{\gamma, cb} A'$

**Two Questions**

1. Are $d$ and $d_{cb}$ locally equivalent? i.e. for each $A$ is there some $K_A$ such that $d_{cb}(A, \cdot) \leq K_A d(A, \cdot)$?

2. How does commutation behave in the metric $d'$?
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THE SIMILARITY PROPERTY

**Question (Kadison ’54)**

Let $A$ be a $C^*$-algebra. Is every bounded homomorphism $\pi : A \to B(\mathcal{H})$ similar to a *-homomorphism?

- Still open. If yes, say $A$ has the **similarity property**.
- Yes if $A$ has no bounded traces, $A$ is nuclear.
- For $\text{II}_1$ factors $M$, yes when $M$ has Murray and von Neumann’s property $\Gamma$.

**Reformulation using Christensen, Kirchberg**

Let $A$ be a $C^*$-algebra. Then $A$ has the similarity property if and only if there exists a constant $K > 0$ such that for every representation $\pi : A \to B(\mathcal{H})$, we have $\|\text{ad}(T)\pi(A)\|_{cb} \leq K \|\text{ad}(T)\pi(A)\|$, $T \in B(\mathcal{H})$.

- If $A$ has SP, then $\exists K$ such that $A \subseteq_{\gamma} B \implies B' \subseteq_{cb,K\gamma} A'$. 

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**The Similarity Property**

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- If $A$ has SP, then $\exists K$ such that $A \subseteq_\gamma B \implies B' \subseteq_{cb,K\gamma} A'$. 
When are $d_{cb}$ and $d$ equivalent?

**Theorem (Christensen, Sinclair, Smith, W))**

Suppose $A$ is a $C^*$-algebra with the similarity property. Then there exists $\gamma_0 > 0$ such that if $d(A, B) < \gamma_0$, then $B$ has the similarity property.

- $\gamma_0$ depends only on how well $A$ has the similarity property;
- Also obtain quantitative estimates on how well $B$ has similarity property.

**Corollary**

If $A$ has similarity property, then $\exists C > 0$ such that $d_{cb}(A, B) \leq Cd(A, B)$ for all $B$ and so if $d(A, B)$ small, then $K_\ast(A) \cong K_\ast(B)$.

In fact this characterises the similarity property for $A$. Further, the similarity problem has a positive answer if and only the map $A \mapsto A'$ is continuous on $C^*$-subalgebras of $B(\mathcal{H})$ (for a separable infinite dimensional Hilbert space).
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**Proposition (Christensen, Sinclair, Smith, W.)**

Suppose \( d(A, B) < 1/14 \). Then \( A \) has real rank zero iff \( B \) has real rank zero.

The definition of real rank zero (the invertible self-adjoints are dense in the self-adjoints) wasn’t very helpful. Used every hereditary subalgebra has an approximate unit of projections reformulation.

**Question**

What about higher values of the real rank, stable rank? It’s unknown whether stable rank one transfers to sufficiently close algebras.

**Theorem (Christensen, Raeburn-Taylor)**

Let \( M \) and \( N \) be sufficiently close von Neumann algebras. Then \( M \) is injective if and only if \( N \) is injective. Similarly for nuclear \( C^* \)-algebras.

What about exactness?
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What about exactness?
More on invariants

**Theorem (Perera, Toms, W, Winter)**

Suppose $d_{cb}(A, B) < 1/42$. Then $A$ and $B$ have isomorphic Cuntz semigroups.

**Theorem**

If $d_{cb}(A, B)$ is sufficiently small, and $A$ is unital, then $A$ and $B$ have the same Elliott invariant.

This uses Khoskham’s work, to get an isomorphism between $K$-theories, a method for transferring trace spaces from CSSW, then the Cuntz semigroup result (which gives a method for transferring quasi-trace spaces in a homeomorphic fashion, extending the map at the level of traces from CSSW).

**More Questions**

What about Ext, $KK$-theory, the UCT?
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Suppose $d_{cb}(A, B) < 1/42$. Then $A$ and $B$ have isomorphic Cuntz semigroups.

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If $d_{cb}(A, B)$ is sufficiently small, and $A$ is unital, then $A$ and $B$ have the same Elliott invariant.

This uses Khoskham’s work, to get an isomorphism between $K$-theories, a method for transferring trace spaces from CSSW, then the Cuntz semigroup result (which gives a method for transferring quasi-trace spaces in a homeomorphic fashion, extending the map at the level of traces from CSSW).

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What about Ext, $KK$-theory, the UCT?
Tensorial absorption a key theme in operator algebras, since Connes showed that an injective II$_1$ factor $M$ is McDuff, i.e. $M \simeq M \otimes R$, where $R$ is the hyperfinite II$_1$ factor.

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Isomorphism Results

Theorem (Christensen, Johnson, Raeburn-Taylor, 1977)

Suppose $M$ and $N$ are von Neumann algebras, with $d(M, N)$ sufficiently small and $M$ injective. Then there exists a unitary $u \in (M \cup N)^{''}$ such that $uMu^* = N$ and $\|u - 1\| \leq O(d(M, N)^{1/2})$.

This gives the strongest form of the conjecture for injective von Neumann algebras. Also:

Theorem (Christensen 1980)

Suppose $M \subseteq_\gamma N$ for $\gamma$ sufficiently small, where $M$ is an injective von Neumann algebra. Then there exists a unitary $u \in (M \cup N)^{''}$ with $uMu^* \subseteq N$ and $\|u - 1\| \leq 150\gamma$.

- Taking commutants, one gets a similar near inclusion result for $M \subseteq_\gamma N$ when $M$ has the similarity property and $N$ is injective.
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**The AMNM-strategy**

**Idea (Christensen, Johnson, Raeburn+Taylor)**

Suppose $M, N \subseteq \mathcal{B}(\mathcal{H})$ are injective (for simplicity) and $d(M, N)$ small.

- Consider a ucp map $\Phi : \mathcal{B}(\mathcal{H}) \to N$ with $\Phi|_N = \text{id}_N$.
- This is almost multiplicative on $M$, i.e. $\Phi(xy) \approx \Phi(x)\Phi(y)$.
- Find (using injectivity) a $\ast$-homomorphism $\tilde{\Phi} : M \to N$ close to $\Phi$.

One way to do this, is to do it for finite dimensional subalgebras of $M$ with constants independent of the choice of subalgebra, then take a weak$\ast$-limit point.

Subsequently, Johnson extensively studied these ideas. He called a pair of Banach algebras $(A, B)$ AMNM, if every almost multiplicative map $T : A \to B$ is near to a multiplicative map $S : A \to B$.

- $(A, B)$ AMNM, whenever $A$ an amenable Banach algebra, and $B$ a dual Banach algebra.
- $(\ell^1, C(X))$ AMNM when $X$ is compact and Hausdorff.
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Limits of the AMNM method

Two counter examples

**Counterexample (Choi, Christensen ’83)**

For $\epsilon > 0$, there exist non-isomorphic amenable C*-algebras $A, B \subset B(H)$ with $d(A, B) < \epsilon$.

- Examples are not separable.

**Example (Johnson ’82)**

For $\epsilon > 0$, there exist two faithful representations of $C([0, 1], \mathcal{K})$ on $H$ with images $A, B$ s.t. $d(A, B) < \epsilon$, yet any isomorphism $\theta : A \to B$ has $\|\theta(x) - x\| \geq \|x\|/70$ for some $x \in A$.

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The uniform topology isn’t the right topology for maps between $C^*$-algebras. Use the point norm-topology instead.

POINT NORM AMNM:
Given $C^*$-algebra $A$, a finite subset $X$ of the unit ball of $A$ and $\varepsilon > 0$, does there exist a $Y$ such that cpc maps $T_Y : A \to B$ which are almost multiplicative on $Y$ are close to a linear map $T_{X,\varepsilon} : A \to B$ with

$$\| T(x_1 x_2) - T(x_1) T(x_2) \| < \varepsilon, \quad x_1, x_2 \in X?$$

- Yes when $A$ is nuclear, using Haagerup’s approximate diagonal.
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**Theorem (Christensen, Sinclair, Smith, W, Winter)**

Let $\mathcal{H}$ be a separable Hilbert space and let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be $C^*$-algebras. Suppose $A$ is separable and nuclear and $d(A, B)$ is sufficiently small. Then $A \cong B$.

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Question

Which \( C^* \)-algebras \( A \) have the property that when \( d(A, B) \) is sufficiently small, there exists an isomorphism \( \theta : A \to B \) with \( \sup_{x \in A, \|x\| \leq 1} \|\theta(x) - x\| \) small?

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For $\epsilon > 0$, there exist two faithful representations of $C([0, 1], K)$ on $H$ with images $A, B$ s.t. $d(A, B) < \epsilon$, yet any isomorphism $\theta : A \to B$ has $\|\theta(x) - x\| \geq \|x\|/70$ for some $x \in A$.

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Near containments

Recall that if $M \subseteq_{\gamma} N$ and $M$ is injective, then there exists a unitary $u \approx 1$ with $uMu^* \subseteq N$.

**Theorem (Hirshberg, Kirchberg, W ’11)**

Let $A$ be separable and nuclear and suppose $A \subseteq_{\gamma} B$ for $\gamma < 10^{-6}$. Then $A \hookrightarrow B$.

**Key ingredients**

- A strengthening of the completely positive approximation property (due to Kirchberg) for nuclear $C^*$-algebras: the approximating maps can be taken to be convex combinations of cpc order zero maps.
- A perturbation theorem for order zero maps from Christensen, Sinclair, Smith, W, Winter.
- When $A$ is separable and nuclear and $A \subseteq_{\gamma} B$ for $\gamma$ small, can one get a spatial embedding $A \hookrightarrow B$?
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Consider a free, ergodic, probability measure preserving action $\alpha : \Gamma \curvearrowright (X, \mu)$ of a discrete group $\Gamma$ and form the crossed product

$$L^\infty(X) \rtimes_\alpha \Gamma.$$ 

This is a $\text{II}_1$ factor, generated by $A = L^\infty(X)$ and unitaries $(u_g)_{g \in \Gamma}$ satisfying

$$u_g f u_g^* = f \circ \alpha_g^{-1}, \quad u_g u_h = u_{gh}, \quad g, h \in \Gamma, \ f \in L^\infty(X).$$

Note:

- $L^\infty(X)$ injective;
- Each $(L^\infty(X) \cup \{u_g\})''$ is injective.
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- \( L^\infty(X) \) injective;
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**Theorem (Cameron, Christensen, Sinclair, Smith, W, Wiggins ’11)**

Let $M_0 = L^\infty(X) \rtimes_\alpha \Gamma$ be as above and suppose $\Gamma$ is a lattice in a semisimple Lie group of rank at least 2 with no hermitian factors (e.g. $SL_n(\mathbb{Z})$ for $n \geq 3$). Let $M = M_0 \overline{\otimes} R$. If $M, N \subset \mathcal{B}(\mathcal{H})$ has $d(M, N)$ is sufficiently small, then there exists a unitary $u \approx 1$ with $uMu^* = N$.

- First factorise $N = N_0 \overline{\otimes} R$, conjugating by a unitary so that the copies of $R$ are identical.
- As $M$ is McDuff, it has the similarity property. This enables us to transfer to and from a standard form.
- Use the embedding theorems for injective von Neumann algebras to embed each $(L^\infty(X) \cup \{u_g\})''$ into $N_0$.
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- Use a standard form trick, to get a unitary $u \approx 1$ implementing such an isomorphism.
**Theorem (Cameron, Christensen, Sinclair, Smith, W, Wiggins ’11)**

Let \( M_0 = L^\infty(X) \rtimes_\alpha \Gamma \) be as above and suppose \( \Gamma \) is a lattice in a semisimple Lie group of rank at least 2 with no hermitian factors (e.g. \( SL_n(\mathbb{Z}) \) for \( n \geq 3 \)). Let \( M = M_0 \bar{\otimes} R \). If \( M, N \subset \mathcal{B}(\mathcal{H}) \) has \( d(M, N) \) is sufficiently small, then there exists a unitary \( u \approx 1 \) with \( uMu^* = N \).

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