Uncertainty Principles for Abelian Groups

Liming Ge
(jointly with Jingsong Wu, Wenming Wu & Wei Yuan)

Chinese Academy of Sciences

June 18, 2013
(ECNU, Shanghai, China)
Heisenberg uncertainty principle

Heisenberg’s canonical commutation relation:

\[ [P, Q] = PQ - QP = i\hbar \]

A mathematical representation: \[ [D, M] = DM - MD = il \]

where \( D = i\frac{d}{dx} \) and \( M = M_x \). \( D = i\mathcal{F}^*M\mathcal{F}. \)

\[ ||f||^2 = \langle f, f \rangle = |\langle [D - d, M - c]f, f \rangle| \]
\[ = |\langle (M - c)f, (D - d)^*f \rangle - \langle (D - d)f, (M - c)^*f \rangle| \]
\[ \leq 2\| (D - d)f \| \cdot \| (M - c)f \|. \]
Heisenberg uncertainty principle in term of Fourier analysis:

If

\[ \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = 1, \]

then we must have

\[ \int_{\mathbb{R}} |xf(x)|^2 dx + \int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2 d\xi \geq \frac{1}{(4\pi)^2}. \]
Hardy uncertainty principle:

If $|f(x)| \leq C_1 e^{-\pi ax^2}$ and $|\hat{f}(\xi)| \leq C_2 e^{-\pi \xi^2 / a}$, then $f = Ce^{-\pi ax^2}$.

This implies immediately that $f$ and $\hat{f}$ cannot be both compactly supported.

Questions:

1. What could be the relation between the support of $f$ and that of $\hat{f}$? E.g. If $f$ has a compact support, can the support of $\hat{f}$ lie in $[0, \infty)$?

   (It is known that $f$ and $\hat{f}$ can be supported in $[0, \infty)$.)

2. What if $\mathbb{R}$ is replaced by another abelian group $G$?
We shall study Question 2 for finite abelian groups and the group of integers.

\( G \): a finite abelian group
\( f : G \to \mathbb{C} \), a complex valued function
\( \text{supp}(f) := \{ x \in G : x \in G, f(x) \neq 0 \} \)
\( \hat{f} \): the Fourier transform of \( f \)

**Theorem 1 (An uncertainty principle)**

(for a nonzero function \( f \)):

1. \( |\text{supp}(f)| \cdot |\text{supp}(\hat{f})| \geq |G| \).

For a cyclic group of prime order \( p \) (T. Tao, 2005):

2. \( |\text{supp}(f)| + |\text{supp}(\hat{f})| \geq p + 1 \).
Uncertainty principle in term of spatial properties:

Suppose $X \subset G$ and $S \subset \hat{G}$. Define

$$P_X = \{ f : \text{supp}(f) \subset X \}; \quad Q_S = \{ f : \text{supp}(\hat{f}) \subset S \}.$$ 

Let $f$ be a nonzero function on $G$, $X = \text{supp}(f)$ and $S = \text{supp}(\hat{f})$. Then $f \in P_X \cap Q_S$.

Tao’s result can be restated as follows:
For $G = \mathbb{Z}_p$ and any $X, S$ given as above, if $P_X \cap Q_S \neq 0$, then $|X| + |S| \geq p + 1$.

We shall see $\dim(P_X \cap Q_S) = 1$ when $|X| + |S| = p + 1$. 

Notation:

$G$: a finite additive abelian group, then $G$ is self-dual.

$l^2(G)$: the Hilbert space of all complex-valued functions on $G$.

Inner product: $\langle f, g \rangle := \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$.

Let $f_x$ be the characteristic function on $\{x\}$. Then $\{f_x : x \in G\}$ is an orthogonal basis for $l^2(G)$.

Let $e : G \times G \to \mathbb{T}$ be any non-degenerate bi-character of $G$.

Let $e_x$ denote the function $e(x, \cdot)$.

Then $\{e_x\}_{x \in G}$ is an orthonormal basis of $l^2(G)$.

If $f$ is a complex function on $G$, the Fourier transform $\hat{f}$ of $f$ is

$$\hat{f} := \frac{1}{|G|} \sum_{x \in G} f(x)e_{\overline{x}}.$$
More notation:

Let $X, S \subset G (= \hat{G})$. Denote also by $P_X$ the orthogonal projection from $l^2(G)$ onto the subspace $l^2(X)$ and $Q_S$ the projection from $l^2(G)$ onto the subspace $\text{span}\{e_x : x \in S\}$.

Then the uncertainty principle on $G$ given by Theorem 1, part 1) can be reformulated by:

$$|\text{supp}(f)| |\text{supp}(\hat{f})| \geq |G| (f \neq 0) \text{ is equivalent to }$$

$$|X| \cdot |S| < |G| \Rightarrow P_X \wedge Q_S = 0.$$
The proof follows from a straightforward computation: for any $f \in l^2(G)$, if $f(x) = \sum_{y \in S} \lambda_y e_y(x)$, then

$$\hat{f}(\xi) = \frac{1}{|G|} \sum_{y \in S} \lambda_y e_y(\xi) = \frac{1}{|G|} \lambda_\xi.$$ 

In fact,

$$\hat{f}(\xi) = \frac{1}{|G|} \sum_{x \in G} f(x) e(x, \xi)$$
$$= \frac{1}{|G|} \sum_{x \in G} (\sum_{y \in S} \lambda_y e(y, x)) e(x, \xi)$$
$$= \frac{1}{|G|} \sum_{y \in S} \lambda_y (\sum_{x \in G} e(y, x) e(x, \xi)) = \frac{1}{|G|} \lambda_\xi.$$

Thus $f \in (P_X \wedge Q_S)(l^2(G))) \Rightarrow \text{supp}(f) \subset X, \text{supp}(\hat{f}) \subset S$. □
Theorem 2: Let $G = \mathbb{Z}_p$ with $p$ prime. Then the FAQ

1) Chebotarev’s theorem (Resetnyak, Dieudonne, T.Tao, etc): Let 
   \{x_1, \cdots, x_n\}, \{y_1, \cdots, y_n\} \subset \mathbb{Z}_p, (n \leq p)$. Then
   \[
   \det(e^{\frac{2\pi i x_j y_k}{p}})_{1 \leq j, k \leq n} \neq 0.
   \]

2) (Tao’s uncertainty principle) $|\text{supp}(f)| + |\text{supp}(\hat{f})| \geq p + 1 (f \neq 0)$.

3) If $|X| + |S| \leq p$, then $P_X \wedge Q_S = 0$. 
**Proof.** 1) $\implies$ 2) Theorem 1.1. in [9, T.Tao].

2) $\implies$ 3) If there is a nonzero function $f \in P_X \land Q_S$, then $\text{supp}(f) \subset X$ and $\text{supp}(\hat{f}) \subset S$. Thus $|X| + |S| \geq |\text{supp}(f)| + |\text{supp}(\hat{f})| \geq p + 1$.

3) $\implies$ 2) If $|\text{supp}(f)| + |\text{supp}(\hat{f})| \leq p$, then let $X = \text{supp}(f)$ and $S = \text{supp}(\hat{f})$. We get a contradiction.

3) $\implies$ 1) If there is \{x_1, \cdots , x_n\}, \{y_1, \cdots , y_n\} $\subset \mathbb{Z}/p\mathbb{Z}$($n \leq p$) such that

$$\det(e^{\frac{2\pi i x_j y_k}{p}})_{1 \leq j, k \leq n} = 0.$$ 

Then vectors \{e_{x_1}, \cdots , e_{x_n}, f_y : y \in \{x_1, \cdots , x_n\}^c\} is linearly dependent. Thus there is a non-zero vector $(\lambda_0, \cdots , \lambda_{p-1})$ such that

$$\sum_{i=1}^{n} \lambda_{x_i} e_{x_i} + \sum_{y \in \{x_1, \cdots , x_n\}^c} \lambda_y f_y = 0.$$ 

Let $X = \{x_1, \cdots , x_n\}^c$, $S = \{x_1, \cdots , x_n\}$ and $f(x) = \lambda_x$, $x \in G$. Then $|X| + |S| = p$ but $f \in P_X \land Q_S$. $\square$
Proposition 1.

Let \( w = e^{\frac{2\pi i}{n}} \) and \( G \) be a cyclic group of order \( n \) and \( |X| + |S| = n \). Then

\[
\det(w^{jk})_{j \in X, k \in S^c} = 0 \iff \det(w^{jk})_{j \in X^c, k \in S} = 0.
\]

In particular \( P_X \wedge Q_S = 0 \iff \det(w^{jk})_{j \in X, k \in S^c} \neq 0 \).

Proof. Suppose \( |X| = l, X^c = \{j'_1, \cdots, j'_{n-l}\}, S = \{k_1, \cdots, k_{n-l}\} \). Define
\[
Tf_x = f_x, x \in X \quad \text{and} \quad Tf_{j't} = e^{kt}, t = 1, \cdots, n - l.
\]
Then \( P_X \vee Q_S = l \iff T \text{ is invertible} \iff T|_{l^2(X^c)} \text{ is invertible} \).

The matrix of \( T|_{l^2(X^c)} = (w^{jk})_{j \in X^c, k \in S} \).

\( \square \)
Proposition 2

Let $G$ be a finite abelian group and $X, S \subset G$. Then we have the following:

1) If $|X| + |S| > |G|$, then $P_X \land Q_S \neq 0$.

2) If $|X| + |S| = |G|$, then $P_X \land Q_S = 0$ if and only if $P_{X^c} \land Q_{S^c} = 0$.

3) If $|X| \cdot |S| < 2\sqrt{|G|}$, then $P_X \land Q_S = 0$.

**Proof:** $\tau(T) = \frac{1}{|G|} \sum_{x \in G} \langle Te_x, e_x \rangle$ (the trace on $B(l^2(G))$).

By Kaplansky-formula, $\tau(P_X \lor Q_S - P_X) = \tau(Q_S - P_X \land Q_S)$

$$\tau(P_X \land Q_S) = \tau(P_X) + \tau(Q_S) - \tau(P_X \lor Q_S) > 0.$$
**Proposition 3**

Suppose $G$ is a finite abelian group. Assume that there are $\alpha, \beta, \gamma \in \mathbb{N}$ such that, for any function $f \neq 0$ on $G$, we have $\alpha |\text{supp}(f)| + \beta |\hat{\text{supp}}(f)| \geq \gamma$. Then for any nonzero function $g$ on $G \times \mathbb{Z}_p$ with $p$ prime, we have

$$p\alpha |\text{supp}(g)| + \beta |\hat{\text{supp}}(g)| \geq p\gamma, \quad \alpha |\text{supp}(g)| + p\beta |\hat{\text{supp}}(g)| \geq p\gamma.$$ 

**Corollary 1**

Let $G = \mathbb{Z}_p \times \mathbb{Z}_q$ and $f$ be a non zero function on $G$, where $p$ and $q$ are prime numbers. Then we have

$$q|\text{supp}(f)| + |\hat{\text{supp}}(f)| \geq q(p + 1), \quad |\text{supp}(f)| + q|\hat{\text{supp}}(f)| \geq q(p + 1),$$

$$p|\text{supp}(f)| + |\hat{\text{supp}}(f)| \geq p(q + 1), \quad |\text{supp}(f)| + p|\hat{\text{supp}}(f)| \geq p(q + 1).$$
Corollary 2

Let $G = (\mathbb{Z}_p)^n$ for a prime number $p$ and a natural number $n$, and $f$ be a non zero function on $G$. Then we have

$$p^j |\text{supp}(f)| + p^{n-j-1} |\text{supp}(\hat{f})| \geq p^n + p^{n-1} (j = 0, \ldots, n - 1).$$

Corollary 3

Let $G = (\mathbb{Z}_p)^n$ for a prime number $p$ and a natural number $n$. For any subsets $X, S \subset G$, if there exist $0 \geq j \geq n - 1$ such that $p^j |X| + p^{n-j-1} |S| < p^n + p^{n-1}$ holds, then $P_X \wedge Q_S = 0$. 
Uncertainty Principles for $\mathbb{Z}$

Recall that an uncertainty principle for $\mathbb{R}$ states that, when $X \subset \mathbb{R}$ and $S \subset \hat{\mathbb{R}}$ are both compact, then $P_X \cap Q_S = 0$. We hope to describe the largest possible such pairs $(X, S)$. Or symmetrically the smallest pairs $(X, S)$ so that $P_X \lor Q_S = I$.

Since $\mathbb{Z}$ has no invariant finite measure, we may consider its dual group $G = \mathbb{T}$, the unit circle on the complex plane.

Now $G = \hat{\mathbb{Z}} = \mathbb{T}$, $\hat{G} = \hat{\mathbb{T}} = \mathbb{Z}$. $G$ is not self-dual.

**Goal:** To investigate the respective subsets $X$ of $\mathbb{T}$ and $S$ of $\mathbb{Z}$ such that $P_X \land Q_S = 0$ and $P_X \lor Q_S = I$. 
Notation:

\[ dm(z) = \frac{1}{2\pi i} \frac{dz}{z} = \frac{1}{2\pi} d\theta \]: the normalized Lebesgue measure on \( \mathbb{T} \), where \( z = e^{i\theta}, \theta \in [0, 2\pi) \). Also denote \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \), or simply \( [0, 2\pi) \).

\( \{ e^{im\theta} : \theta \in [0, 2\pi), m \in \mathbb{Z} \} \): an orthonormal basis of \( L^2(\mathbb{T}) \).

\( \{ e_n : n \in \mathbb{Z} \} \): the standard orthonormal basis in \( l^2(\mathbb{Z}) \), where \( e_n(m) = \delta_{n,m} \).

The Fourier transformation: \( e^{im\theta} \mapsto e_m \) is a unitary operator from \( L^2(\mathbb{T}) \) to \( l^2(\mathbb{Z}) \).
Recall:

$X \subset [0, 2\pi]$: a measurable subset, $m(X)$: the measure of $X$

$P_X$: the orthogonal projection from $L^2(\mathbb{T})$ onto $L^2(X)$

$S \subset \mathbb{Z}$

$Q_S$: the projection from $L^2(\mathbb{T})$ onto $\text{span}\{e^{im\theta} : m \in S\}$

$P_t$: the projection from $L^2(\mathbb{T})$ onto $L^2([2(1-t)\pi, 2\pi])$ for any $0 < t < 1$

$Q_{\geq j}$: the projection from $L^2(\mathbb{T})$ onto $\text{span}\{e^{im\theta} : m \geq j, m \in \mathbb{Z}\}$

When $j = 0$, the range of projection $Q_{\geq 0}$ is the Hardy space $H^2(\mathbb{T})$

For a mean $\mu_\omega$ on $\mathbb{Z}$ given by a free ultrafilter $\omega$,
we define $\mu_\omega(S) = \mu_\omega(\chi_S)$

If the above is independent of $\omega$, then we denote it by $\mu_\infty(S)$ and it is given by

$$\mu_\infty(S) = \lim_{n \to \infty} \frac{|S \cap \{-n, -(n-1), \cdots, n-1, n\}|}{2n+1}.$$
Definition

A pair \((X, S)\) is called balanced if \(P_X \land Q_S = 0\) and \(P_X \lor Q_S = I\).

When \(G\) is a finite abelian group, if \((X, S)\) is balance, then \(\tau(P_X) + \tau(Q_S) = 1\).

Examples and Questions:

**Examples** \(X = [0, \pi]\), \(S_0 = 2\mathbb{Z}\), all even integers, \(S_1 \subset \mathbb{Z}\) all odd integers. \((X, S_0)\) and \((X, S_1)\) are balanced pairs.

\(m(X) + \mu_\infty(S_0) = m(X) + \mu_\infty(S_1) = 1\).

**Questions** Is \(m(X) + \mu_\infty(S) = 1\) a necessary condition for balanced pairs? If "no", for any \(\epsilon > 0\), can one find a balanced pair \((X, S)\) so that \(m(X) + \mu_\infty(S) < \epsilon\) or \(m(X) + \mu_\omega(S) < \epsilon\)?
Some basic facts:

1) \( P_X \lor Q_S = I \iff P_X \lor Q_{-S} = I \), where \(-S = \{-s : s \in S\}\);

2) \( P_X \land Q_S = 0 \iff P_X \land Q_{-S} = 0 \);

3) \( P_X \lor Q_S = I \iff P_X \lor Q_{S+j} = I \), where \( S + j = \{s + j : s \in S\}\);

4) \( P_X \land Q_S = 0 \iff P_X \land Q_{S+j} = 0 \);

5) If \( X \subset \mathbb{T} \) with \( 0 < m(X) < 1 \), then \( P_X \land Q_{\geq j} = 0 \) and \( P_X \lor Q_{\geq j} = I(\forall j \in \mathbb{Z}) \).

From 5), we see that \( \frac{1}{2} < m(X) + \mu_\infty(Q_{\geq 0}) < \frac{3}{2} \).
Proof. Let \((Uf)(z) = \overline{f(z)}\). Then \(U\) is a conjugate linear operator such that \(U^2 = I\) and \(UP_X U = P_X, UQ_S U = Q_{-S}\). Thus 1) and 2) are true.

Let \((U_j f)(z) = z^j f(z)\). Then \(U_j\) is a unitary operator such that \(UP_X U^* = P_X, UQ_S U^* = Q_{S+j}\). Hence 3) and 4) are true.

For 5), let \((Vf)(z) = zf(z)\). Then \(V\) is a unitary operator such that \((I - P_X \wedge Q_{\geq 0}) VP_X \wedge Q_{\geq 0} = 0\). As \(P_X \wedge Q_{\geq 0} \leq Q_{\geq 0}\) and by Beurling theorem, there exists an inner function \(\phi\) such that \(P_X \wedge Q_{\geq 0}(H^2(T)) = \phi H^2(T)\). Thus \(\phi = 0\) and \(P_X \wedge Q_{\geq 0} = 0\). From 2), \(P_X^c \wedge Q_{\leq 0} = 0\). This implies that \(P_X \vee Q_{\geq 0} = I\).
For any $\varepsilon > 0$, there exists a measurable subset $X$ of $[0, 2\pi]$ with $0 < m(X) < \varepsilon$ and a subset $S$ of $\mathbb{Z}$ with $\mu_\omega(S) = 0$ for some free ultrafilter $\omega$ such that $P_X \land Q_S = 0$ and $P_X \lor Q_S = I$. 

**Theorem 3**
Proof. For any \( \epsilon > 0 \), there exist \( n \) in \( \mathbb{N} \) such that \( \frac{1}{n} < \epsilon \). Let
\[
X = [2(1 - \frac{1}{n})\pi, 2\pi].
\]
Then \( m(X) = \frac{1}{n} < \epsilon \). From Basic Fact 5), we have \( P_X \land Q_{\geq 0} = 0 \) and \( P_X \lor Q_{\geq 0} = P_X \lor Q_{\geq j} = I \) for any \( j \in \mathbb{Z} \). Then
\[
\operatorname{span}\{e^{i \frac{n-1}{n} m\theta}, m \geq j\} = L^2[0, 2\pi] \text{ for } j \text{ in } \mathbb{Z}.
\]
In fact if there is a non zero vector \( f \) in \( L^2[0, 2\pi] \) orthogonal to
\[
\operatorname{span}\{e^{i \frac{n-1}{n} m\theta}, m \geq j\},
\]
we define a function \( g(\theta) = f(\frac{n}{n-1} \theta) \) when \( 0 \leq \theta \leq 2\pi \frac{n-1}{n} \), 0 elsewhere, then we have
\[
\int_{0}^{2\pi} f(\theta) e^{-i \frac{n-1}{n} m\theta} d\theta = \int_{0}^{2\pi} g(\frac{n}{n-1} \theta) e^{-i \frac{n-1}{n} m\theta} d\theta = \int_{0}^{2\frac{n-1}{n} \pi} g(\theta) e^{-im\theta} = 0,
\]
and hence \( g \) is a non zero vector in the range of \( I - (P_{1/n} \lor Q_{\geq j}) \) which leads a contradiction.
For any $n$ in $\mathbb{N}$, since $\text{span}\{e^{i\frac{n-1}{n}m\theta}, m \geq j\} = L^2[0, 2\pi]$ for $j$ in $\mathbb{Z}$, there exists $m(n, j)$ in $\mathbb{N}$ such that the distance between $e^{ik\theta}$ and $\text{span}\{e^{i\frac{n-1}{n}1\theta}, \ldots, e^{i\frac{n-1}{n}m(n,j)\theta}\}(= \mathcal{H}_{n,j})$ is less than $\frac{1}{n}$ for any $-n \leq k \leq n$. Obviously, $m(n, j) > j$ for any $j$ in $\mathbb{Z}$. Let $S_{n,j}$ be the set $\{j, \ldots, m(n,j)\}$. We define $m_k$ in $\mathbb{N}$ by induction. Let $m_1 = m(1, 0)$. Suppose that $m_k$ is defined. Then $m_{k+1} = m(k + 1, m_k^2)$ for $k \geq 1$ and $m_{k+1} > m_k^2$. It is clear that the closure of the union of $\mathcal{H}_{k, m_k}$, $k \geq 1$ is $L^2[0, 2\pi]$ and its corresponding set $S$ is $\bigcup_{k \geq 1} S_{k, m_k}$. For the sequence $\frac{\#S \cap \{-n, \ldots, 0, \ldots, n\}}{2n+1}$, there is a subsequence $\left\{ \frac{\sum_{j=1}^{k} (m_j - m_{j-1}^2)}{2m_k^2 + 1} \right\}_{k \geq 1}$ with limit zero, since $\sum_{j=1}^{k} (m_j - m_{j-1}^2) < m_k$. Hence there is a free ultrafilter $\omega$ such that $\lim_{n \to \omega} \frac{\#S \cap \{-n, \ldots, 0, \ldots, n\}}{2n+1} = 0$. \(\square\)
Uncertainty Principles on \( \mathbb{Z} \)

**Corollary**

Let \( X_n = [0, \frac{1}{n}] \subset \mathbb{T} \). For any free ultrafilter \( \omega \), there is a subset \( S \) of \( \mathbb{Z} \) with \( \mu_\omega(S) = 0 \) such that \( P_{X_n} \land Q_S = 0 \) and \( P_{X_n} \lor Q_S = I \), for any \( n \geq 1 \). Thus, for any \( f, g \in L^2(\mathbb{T}) \), if there is an \( n \) such that \( f|_{X_n} = g|_{X_n} \) and \( \hat{f}|_S = \hat{g}|_S \), then \( f = g \).

**Conjecture:** \( S = \{0, \pm 1, \pm p, \pm 2p : p \text{ a prime number}\} \) is such a set satisfies our Theorem 4, i.e., \( ([0, \varepsilon], S) \) is balanced for any \( \varepsilon > 0 \).

In other words, two functions on \( \mathbb{T} \) agree on \( [0, \varepsilon] \) and their Fourier expansions agree on \( S \). Then they must be the same function.
One possible application:

If \((X, S)\) is a balanced pair for \(\mathbb{T}\) and \(f \in L^2(\mathbb{T})\), then how can we recover \(f\) from \(f|_X\) and \(\hat{f}|_S\)?

It is not an easy question. In the following we shall workout a concrete example.
Theorem 4

Let \( \{a_n\}_{n=1}^{\infty} \) be an increasing sequence of odd natural numbers such that

\[
\sum_{n=1}^{\infty} \frac{1}{a_n} = +\infty.
\]

Suppose \( S = \{2k : k \in \mathbb{N}\} \cup \{a_n\} \). Then \( P_{1/2} \lor Q_S = 1 \) and \( P_{1/2} \land Q_S = 0 \).

In this case, \( X = [\pi, 2\pi] \). We may choose \( \{a_n\} \) so that \( m(X) + \mu_\infty(S) = \frac{3}{4} \).
Lemma Suppose $p$ is a prime number and $l_j = \{ w^j e^{i\theta} \in \mathbb{T} : \theta \in [0, \frac{2\pi}{p}) \}$ for $j = 0, 1, \cdots, p - 1$. Let $X(i_1, \cdots, i_m) = l_{i_1} \cup \cdots \cup l_{i_m}$ where $0 \leq i_1 < \cdots < i_m \leq p - 1$. Let $S_0 \subset \{0, 1, \cdots, p - 1\}$ and $\emptyset \neq S_1 \subset S_0^c$. Let $S = \{ kp + s_0 : k \in \mathbb{Z}, s_0 \in S_0 \} \cup \{ kp + s_1 : k \geq 0, s_1 \in S_1 \}$. Then we have

$$P_{X(i_1, \cdots, i_m)} \wedge Q_S = 0 \iff |S_0^c| \geq m + 1.$$
Proof of Theorem 4. $Q_S < Q \geq 0$, $P_{1/2} \land Q \geq 0 = 0 \Rightarrow P_{1/2} \land Q_S = 0$. Assume that $P_{1/2} \lor Q_S \neq I$. Then there exists a non-zero function $f$ in $L^2([0, 2\pi])$ such that $f$ is orthogonal to the ranges of $P_{1/2}$ and $Q_S$. Thus $\text{supp}(f) \subset [0, \pi]$ and for any $s \in S$, we have

$$\frac{1}{2\pi} \int_0^\pi f(\theta) e^{-is\theta} d\theta = \frac{1}{4\pi} \int_0^{2\pi} f\left(\frac{\theta}{2}\right) e^{-is\theta/2} d\theta = 0.$$ 

Claim. $\mathcal{H}_S := \text{span}\{e^{is\theta/2} : s \in S\} = L^2([0, 2\pi])$. 


Firstly when $s = 2k (k \in \mathbb{N})$, we have $e^{ik\theta} \in \mathcal{F}_S$. When $s = a_n$ for $n \geq 1$, for any $m \in \mathbb{Z}$, we have

$$\langle e^{ian\theta/2}, e^{im\theta} \rangle = \frac{2i}{\pi(a_n - 2m)}.$$

Then $e^{ian\theta/2} = \sum_{m \in \mathbb{Z}} \frac{2i}{\pi} \frac{e^{im\theta}}{a_n - 2m}$. Let $\xi_n = \sum_{m=-\infty}^{-1} \frac{e^{im\theta}}{a_n - 2m} = \sum_{m=1}^{\infty} \frac{e^{-im\theta}}{a_n + 2m}$. To show that the claim holds, we just need to show that $\overline{\text{span}}\{\xi_n : n \geq 1\} = \overline{\text{span}}\{e^{-im\theta} : m \geq 1\}$ which is equivalent to $\{\xi_n : n \geq 1\}^\perp \cap \overline{\text{span}}\{e^{-im\theta} : m \geq 1\} = 0$. 
Suppose that $\alpha^{(0)} = \sum_{m \geq 1} \alpha^{(0)}_m e^{-im\theta}$ such that $\alpha^{(0)} \perp \{\xi_n : n \geq 1\}$ and $\sum_{m \geq 1} |\alpha^{(0)}_m|^2 < \infty$. Thus for any $n \geq 1$, we have

$$\sum_{m \geq 1} \frac{\alpha^{(0)}_m}{a_n + 2m} = 0.$$  

This implies that for any $n \geq 2$, we have

$$0 = \frac{1}{a_n - a_1} \sum_{m=1}^{\infty} \left( \frac{\alpha^{(0)}_m}{a_1 + 2m} - \frac{\alpha^{(0)}_m}{a_n + 2m} \right) = \sum_{m=1}^{\infty} \frac{\alpha^{(0)}_m}{a_1 + 2m} - \frac{1}{a_n + 2m}.$$  

Let $\alpha^{(1)}_m := \frac{\alpha^{(0)}_m}{a_1 + 2m}$ and $\alpha^{(1)} := \sum_{m \geq 1} \alpha^{(1)}_m e^{-im\theta}$. Then $\alpha^{(1)} \perp \{\xi_n : n \geq 2\}$ and

$$\sum_{m=1}^{\infty} |\alpha^{(1)}_m| \leq \|\alpha^{(0)}\| \cdot \left( \sum_{m=1}^{\infty} \frac{1}{(a_n + 2m)^2} \right)^{1/2} < \infty.$$
Iterating the process, for any $N > 0$, we can define $\alpha^{(N)}_m = \frac{\alpha^{(N-1)}_m}{a_N + 2m}$ and $\alpha^{(N)} = \sum_{m \geq 1} \alpha^{(N)}_m e^{-im\theta}$ with $\alpha^{(N)} \perp \{\xi_n : n \geq N + 1\}$.

Without loss of generality, we can assume that $\alpha^{(0)}_1 = 1$. Then $\alpha^{(N)}_1 = \prod_{n=1}^N \frac{1}{a_n + 2}$. We define

$$\beta^{(N)}_m = \frac{\alpha^{(N)}_m}{\alpha^{(N)}_1} = (a_1 + 2) \prod_{n=2}^N \frac{a_n + 2}{a_n + 2m} \alpha^{(1)}_m, \ m \geq 1.$$
Then we have $\beta^{(N)} = \frac{\alpha^{(N)}}{\alpha_1}$ and $\beta^{(N)} \perp \{\xi_n : n \geq N + 1\}$ and

$$\sum_{m \geq 2} |\beta_m^{(N)}| = \sum_{m \geq 2} (a_1 + 2)(\prod_{n=2}^{N} \frac{a_n + 2}{a_n + 2m})|\alpha_m^{(1)}|$$

$$\leq (a_1 + 2)(\prod_{n=2}^{N} \frac{a_n + 2}{a_n + 4}) \sum_{m \geq 2} |\alpha_m^{(1)}|.$$ 

Then as $\sum \frac{1}{a_n} = +\infty$, thus $\prod_{n=2}^{N} \frac{a_n + 2}{a_n + 4} = \prod (1 - \frac{2}{a_n + 4}) \to 0$ as $N \to \infty$. Then $\exists$ sufficient large $N_0$ such that for any $N \geq N_0$ we have

$$(a_1 + 2)(\prod_{n=2}^{N} \frac{a_n + 2}{a_n + 4}) \sum_{m \geq 2} |\alpha_m^{(1)}| < 1. \quad (1)$$

Thus $\sum_{m \geq 2} |\beta_m^{(N)}| < 1$ for any $N \geq N_0$. 

33 / 38
On the other hand for any vector $\beta = e^{-i\theta} + \sum_{m \geq 2} \beta_m e^{-im\theta}$ which is orthogonal some $\xi_k$, $k \geq 1$, then we have

$$1 = -\sum_{m \geq 2} \beta_m \frac{a_k + 2}{a_k + 2m} \leq \sum_{m \geq 2} |\beta_m|.$$ 

Thus by (1) and (2), we get a contradiction. Thus $\alpha^{(0)} = 0$. $\square$
Corollary Let $S = \{nk : k \geq 0\} \cup \{a_m\}$ where $\{a_m\}$ is an increasing sequence of positive integers in $(n\mathbb{Z})^c$ and $\sum_m \frac{1}{a_m} = \infty$, then $P_{(n-1)/n} \lor Q_S = I$ and $P_{(n-1)/n} \land Q_S = 0$.

Finding $f$ from the restrictions to $(X, S)$ is related to finding the inverse of certain Hankel operators. A special one is the following:

Let $H(s)(0 < s < 1)$ be the Hankel operator with the following matrix form

$$
\begin{pmatrix}
\frac{1}{1+s} & \frac{1}{2+s} & \frac{1}{3+s} & \cdots \\
\frac{1}{2+s} & \frac{1}{3+s} & \frac{1}{4+s} & \cdots \\
\frac{1}{3+s} & \frac{1}{4+s} & \frac{1}{5+s} & \cdots \\
\cdots & \cdots & \cdots & \ddots
\end{pmatrix}.
$$
Some References


Thanks