On generalized universal irrational rotation algebras

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- Examples of strongly irreducible operators on infinite dimensional Hilbert spaces are from analytic Toeplitz operators, Cowen-Douglas operators, shift operators and so on.

- The main purpose of studying strongly irreducible operators is to extend the Jordan Canonical Form Theorem to infinite dimensional spaces.
**Strongly irreducible operators relative to type II\(_1\) factors**

Let \(M\) be a type II\(_1\) factor, an operator \(T \in M\) is said to be a strongly irreducible operator relative to \(M\), if there exists no non-trivial idempotents in \(\{T\}' \cap M\).
Strongly irreducible operators relative to type $\text{II}_1$ factors

Relative strongly irreducible operator
Let $M$ be a type $\text{II}_1$ factor, an operator $T \in M$ is said to be a strongly irreducible operator relative to $M$, if there exists no non-trivial idempotents in $\{T\}' \cap M$.

Question
Are there any relative strongly irreducible operators in type $\text{II}_1$ factors?
Let $R$ be the hyperfinite type II$_1$ factor. For an irrational number $\theta \in (0, 1)$, there are two unitary operators $u, v \in R$ satisfying the following properties:

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- $R = \{u, v\}''$;
- $vu = e^{2\pi i \theta} uv$. 

Theorem: $u + v$ is a relative strongly irreducible operator in $R$, i.e., there exists no nontrivial idempotents in $\{u + v\} \cap R$.

Corollary: The spectrum $\sigma(u + v)$ is connected.
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**Corollary**

The spectrum $\sigma(u + v)$ is connected.
The almost Mathieu operator

In mathematical physics, the almost Mathieu operator is given by

$$(H_{\lambda, \theta, \beta} u)(n) = u(n + 1) + u(n - 1) + 2\lambda \cos(2\pi(n\theta + \beta))u(n)$$

acting as a self-adjoint operator on the Hilbert space $l^2(\mathbb{Z})$. Here $\theta, \beta, \lambda \in \mathbb{R}$ are parameters.

Barry Simon raised several problems on the almost Mathieu operator. One problem (known as the "Ten martini problem" after Kac and Simon) conjectures that the spectrum of the almost Mathieu operator is a cantor set for all $\lambda \neq 0$ and irrational number $\theta$.

The almost Mathieu operator $H_{\lambda, \theta, \beta}$ can be viewed as the operator

$$(u + \lambda e^{2\pi i \beta} v) + (u + \lambda e^{2\pi i \beta} v)^*$$

in the hyperfinite type $\text{II}_1$ factor $R$. 
If $a = (u + \lambda e^{2\pi i \beta} v) + (u + \lambda e^{2\pi i \beta} v)^*$, then $\sigma(a) = \text{Cantor set}$,

where $\lambda \neq 0$, $vu = e^{2\pi i \theta} uv$, $\theta$-irrational number.

Conjecture

If $a = (u + \lambda e^{2\pi i \beta} v) + (u + \lambda e^{2\pi i \beta} v)^*$, then $\sigma(a) = \text{Cantor set}$, where $\lambda \neq 0$, $\nu u = e^{2\pi i \theta} uv$, $\theta$-irrational number.


Question

A natural question: What is the spectrum of $u + \lambda e^{2\pi i \beta} v$?
Spectrum of $u + \lambda v$

Theorem

The spectrum of $u + \lambda v$ is given by:

1. $\sigma(u + v) = B(0, 1)$;
2. $\sigma(u + \lambda v) = S^1, \lambda \in (0, 1)$;
3. $\sigma(u + \lambda v) = \lambda S^1, \lambda > 1$. 

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Spectral radius of $u + \lambda v$

Let $r(u + \lambda v)$ be the spectral radius of $u + \lambda v$. Then

$$r(u + \lambda v) = \lim_{n \to +\infty} \|(u + \lambda v)^n\|^\frac{1}{n}$$

$$= \lim_{n \to +\infty} \|u^n (1 + \lambda \omega)(1 + \alpha \lambda \omega) \cdots (1 + \alpha^{(n-1)} \lambda \omega)\|^\frac{1}{n},$$

where $\omega = u^* v$ is a Haar unitary operator and $\alpha = e^{2\pi i \theta}$. Hence,

$$\|(u + \lambda v)^n\|^\frac{1}{n} = \|(1 + \lambda \omega)(1 + \alpha \lambda \omega) \cdots (1 + \alpha^{(n-1)} \lambda \omega)\|^\frac{1}{n}$$

$$= \|(1 + \lambda M_z)(1 + \alpha \lambda M_z) \cdots (1 + \alpha^{(n-1)} \lambda M_z)\|^\frac{1}{n}$$

$$= \left( \max_{z \in S^1} |(1 + \lambda z)(1 + \alpha \lambda z) \cdots (1 + \alpha^{(n-1)} \lambda z)| \right)^\frac{1}{n}$$

$$= \left( \max_{x \in [0,1]} \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^\frac{1}{2n}$$
Lemma

Suppose $0 < \lambda \leq 1$. For any $\varepsilon > 0$, there exists $x \in [0, 1]$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$
\left( \prod_{k=0}^{n-1} \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right) \right)^{\frac{1}{2n}} \geq 1 - \varepsilon.
$$
**Lemma**

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$$\left(\prod_{k=0}^{n-1} \left(1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))\right)\right)^{\frac{1}{2n}} \geq 1 - \varepsilon.$$ 

**Proof**

Key point: Let $T : x \to x + \theta \,(\text{mod}1)$. Then $T$ is a measure preserving ergodic transformation of $[0, 1]$. By Birkhoff’s Ergodic theorem, for almost all $x \in [0, 1]$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) = \int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) \, dx = 0.$$
**Spectral radius of** $u + \lambda v$

**Corollary**

$r(u + \lambda v) \geq 1, \forall 0 < \lambda \leq 1.$
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**Corollary**

$r(u + \lambda v) \geq 1$, $\forall 0 < \lambda \leq 1$.

**Lemma**

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- For all $x \in [0, 1]$, the points \(\{e^{2\pi i(x+k\theta)} : 0 \leq k \leq n - 1\}\) is almost uniformly distributed on $\mathbb{T}$. 

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Spectral radius of $u + \lambda v$

**Corollary**

$$r(u + \lambda v) \geq 1, \forall 0 < \lambda \leq 1.$$  

**Lemma**

Suppose $0 < \lambda \leq 1$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0, 1]$

$$\left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^{\frac{1}{2n}} \leq e^{2\varepsilon}, \forall n \geq N.$$  

- For all $x \in [0, 1]$, the points $\{ e^{2\pi i(x+k\theta)} : 0 \leq k \leq n - 1 \}$ is almost uniformly distributed on $\mathbb{T}$.

**Corollary**

$$r(u + \lambda v) = 1, \forall 0 < \lambda \leq 1.$$
Notice that, \( u + \nu = u(1 + \nu^* \nu) \). Since \( \nu^* \nu \) is a Haar unitary operator, \(-1 \in \sigma(\nu^* \nu)\). This implies that \( u + \nu \) is not invertible and therefore \( 0 \in \sigma(u + \nu) \). Observe that \( \sigma(u + \nu) \) is rotation symmetric. Since \( \sigma(u + \nu) \) is connected and \( r(u + \nu) = 1 \), \( \sigma(u + \nu) = B(0, 1) \).

For \( 0 < \lambda < 1 \), \( u + \lambda \nu = u(1 + \lambda \nu^* \nu) \), so \( 0 \not\in \sigma(u + \lambda \nu) \). We can also show that \( r((u + \lambda \nu)^{-1}) = 1 \). Then

\[
\sigma(u + \lambda \nu) = S^1.
\]

For \( \lambda > 1 \), consider \( \lambda(\frac{1}{\lambda} u + \nu) \), we have \( \sigma(u + \lambda \nu) = \lambda S^1 \).
The von Neumann algebra and $C^*$-algebra generated by $u + v$

**Theorem**

For $\lambda > 0$, the von Neumann algebra generated by $u + \lambda v$ is $R$. However, $C^*(u + v)$ is a proper $C^*$-subalgebra of $C^*(u,v)$. Indeed, we have $	ext{dist}(u, C^*(u + v)) = 1$.

**Question**

Is $C^*(u + v)$ $*$-isomorphic to $C^*(u,v)$?
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On generalized universal irrational rotation algebras
The von Neumann algebra and $C^*$-algebra generated by $u + \nu$

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For $\lambda > 0$ and $\lambda \neq 1$, $C^*(u + \lambda\nu) = C^*(u, \nu)$. However, $C^*(u + \nu)$ is a proper $C^*$-subalgebra of $C^*(u, \nu)$. Indeed, we have $\text{dist}(u, C^*(u + \nu)) = 1$.

**Question**
Is $C^*(u + \nu) \ast$-isomorphic to $C^*(u, \nu)$?
It is well known that the universal irrational rotation $C^*$-algebra $A_\theta = C^*(u, v)$ has the following properties (Rieffel [1981], Pimsner-Voiculescu [1980], Elliott-Evans [1993])

(a) $A_\theta$ is simple.
(b) There is a unique trace on $A_\theta$.
(c) For every $\alpha$ in $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, there exists a projection $p$ in $A_\theta$ such that $\tau(p) = \alpha$.
(d) $K_0(A_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta$ and $K_1(A_\theta) \cong \mathbb{Z}^2$.
(f) $A_\theta \cong A_\eta$ $\iff$ $\theta = \pm \eta$ (mod $\mathbb{Z}$).
(g) $A_\theta$ is an $AT$-algebra and hence with real rank 0 and stable rank 1.
A generalized universal irrational rotation algebra $A_{\theta, \gamma} = C^* (x, w)$ is the universal $C^*$-algebra satisfying the following properties:

\begin{align*}
  w^* w &= w w^* = 1, \\
  x^* x &= \gamma (w), \\
  x x^* &= \gamma (e^{-2\pi i \theta} w), \\
  x w &= e^{-2\pi i \theta} w x,
\end{align*}

where $\gamma (z) \in C(S^1)$ is a positive function.

- If $\gamma (z) \equiv 1$, then $A_{\theta, \gamma}$ is the universal irrational $C^*$-algebra,
A generalized universal irrational rotation algebra \( A_{\theta, \gamma} = C^*(x, w) \) is the universal \( C^* \)-algebra satisfying the following properties:

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\begin{align*}
    w^*w &= ww^* = 1, \quad (1) \\
    x^*x &= \gamma(w), \quad (2) \\
    xx^* &= \gamma(e^{-2\pi i \theta} w), \quad (3) \\
    xw &= e^{-2\pi i \theta} wx, \quad (4)
\end{align*}
\]

where \( \gamma(z) \in C(S^1) \) is a positive function.

- If \( \gamma(z) \equiv 1 \), then \( A_{\theta, \gamma} \) is the universal irrational \( C^* \)-algebra,
- If \( \gamma(z) = |1 + z|^2 \), then \( A_{\theta, \gamma} \) is \(*\)-isomorphic to \( C^* (u + v) \).
For $t \in [0, 1]$, the pair $(e^{2\pi it}x, w)$ also satisfy (1)-(4). Thus there is an automorphism $\rho_t$ of $A_{\theta,\gamma} = C^*(x, w)$ such that $\rho_t(x) = e^{2\pi it}x$ and $\rho_t(w) = w$. Define a map of $A_{\theta,\gamma} = C^*(x, w)$ into itself by

$$
\Phi(a) = \int_0^1 \rho_t(a) dt.
$$

**Proposition**

The map $\Phi$ is a faithful conditional expectation of $A_{\theta,\gamma} = C^*(x, w)$ onto $C^*(w)$. Furthermore, if $\rho$ is the state on $C^*(w)$ induced by the Haar measure on $C(\mathbb{T})$, then $\tau = \rho \cdot \Phi$ is a classical faithful trace on $A_{\theta,\gamma} = C^*(x, w)$.
Applying the GNS-construction to $\tau$, $A_{\theta,\gamma} = C^*(x, w)$ is isomorphic to $C^*(u\gamma(v)^{1/2}, v) \subseteq A_{\theta}$ and the isomorphism takes $x$ to $u\gamma(v)^{1/2}$ and $w$ to $v$. 
Tracial linear functionals on generalized universal irrational rotation $C^*$-algebras

**Proposition**

If $\mu$ is a complex regular Borel measure on $\mathbb{T}$ which satisfies that

$$\int_{\mathbb{T}} f(e^{-2\pi i \theta} z) d\mu(z) = \int_{\mathbb{T}} f(z) d\mu(z)$$

for all $f(z)$ in $\gamma(z)C(\mathbb{T}) \oplus \mathbb{C}1$ and let $\sigma(f) = \int_{\mathbb{T}} f(z) d\mu(z)$ for $f(z) \in C(\mathbb{T})$, then $\sigma \cdot \Phi$ is a bounded tracial linear functional on $A_{\theta,\gamma}$. Conversely, every bounded tracial linear functional on $A_{\theta,\gamma}$ is given in this way.
Theorem

Suppose $\gamma(z)$ has finite zero points which can be divided into nonempty disjoint classes $A_1, \ldots, A_r$ in the following sense: $z_1$ and $z_2$ in $A_j$ if and only if $z_2 = e^{2\pi i k \theta} z_1$ for some $k \in \mathbb{Z}$. Then the dimension of the space of tracial linear functionals on $A_{\theta, \gamma} = C^*(x, w)$ is $1 + \sum_{j=1}^{r} (|A_j| - 1)$, where $|A_j|$ is the number of elements in $A_j$. 
Theorem

Let $Y$ be the set of zero points of $\gamma(z)$ and let $\phi : \mathbb{T} \to \mathbb{T}$ be the rotation of the unit circle determined by $\theta$, i.e., $\phi(z) = e^{2\pi i \theta} z$. Denote by $\text{Orb}(\xi) = \{\phi^n(\xi) : n \in \mathbb{Z}\}$ for $\xi \in \mathbb{T}$. Then the following properties are equivalent:

1. $\mathcal{A}_{\theta, \gamma}$ is simple;
2. $\mathcal{A}_{\theta, \gamma}$ has a unique tracial state;
3. $\phi^n(Y) \cap Y = \emptyset$ for all integer $n \neq 0$;
4. For each $\xi \in \mathbb{T}$, $\text{Orb}(\xi) \cap Y$ contains at most one point.
Simple generalized universal irrational rotation $C^*$-algebras

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Riffel projections in generalized universal irrational rotation $C^*$-algebras

**Theorem**

If $m(\{z|\gamma(z) = 0\}) = 0$, e.g., the zero points of $\gamma(z)$ is countable, then for every $\alpha$ in $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, there is a projection $p$ in $A_{\theta, \gamma} = C^*(w, x)$ such that $\tau(p) = \alpha$. 

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**Theorem**

Let $Y$ be the set of zero points of $\gamma(z)$. If $Y$ is not empty, then $K_1(A_{\theta,\gamma}) \cong \mathbb{Z}$ and $K_0(A_{\theta,\gamma})$ is determined by the following splitting exact sequence

$$0 \to \mathbb{Z} \to K_0(A_{\theta,\gamma}) \to C(Y,\mathbb{Z}) \to 0.$$ 

We identify $A_{\theta,\gamma}$ with $C^*(u\gamma(v)^{1/2}, v) \subseteq C^*(u, v)$. Let $A = C^*(v)$ and

$$J_1 = \{ f(v) : f \in C(\mathbb{T}) \text{ and } f(\lambda) = 0 \text{ for } \lambda \in Y \},$$

and let $J_2 = uJ_1u^*$. Then $A_{\theta,\gamma}$ is $*$-isomorphic to the covariance algebra $C^*(A, \Theta)$ for the partial automorphism $\Theta = (Adu, J_1, uJ_1u^*)$ of $C^*(v)$ in the sense of Ruy Exel.
For a covariance algebra $C^*(A, \Theta)$ for the partial automorphism $\Theta = (\theta, J, \theta(J))$ of $A$, Ruy Exel proved the following generalized Pimsner-Voiculescu exact sequence

$$
\begin{align*}
K_0(J) & \xrightarrow{i_* - \theta_*^{-1}} K_0(A) \xrightarrow{i_*} K_0(A_{\theta, \gamma}) \\
\uparrow & \quad & \quad \downarrow \\
K_1(A_{\theta, \gamma}) & \xleftarrow{i_*} K_1(A) & \xleftarrow{i_* - \theta_*^{-1}} K_1(J)
\end{align*}
$$
Theorem

Let $A_{\theta,\gamma}$ be a unital simple $C^*$-algebra. Then $A_{\theta,\gamma}$ is a unital simple $A\mathbb{T}$-algebra of real rank zero. In particular, $A_{\theta,\gamma}$ has tracial rank zero and stable rank one.
Theorem

Let $A_{\theta,\gamma}$ be a unital simple $C^*$-algebra. Then $A_{\theta,\gamma}$ is a unital simple $\mathcal{AT}$-algebra of real rank zero. In particular, $A_{\theta,\gamma}$ has tracial rank zero and stable rank one.

- Recursive subhomogeneous algebras due to Huaxin Lin and N. Christopher Phillips.
Theorem

Let $A_{\theta,\gamma}$ be a unital simple $C^*$-algebra. Then $A_{\theta,\gamma}$ is a unital simple $AT$-algebra of real rank zero. In particular, $A_{\theta,\gamma}$ has tracial rank zero and stable rank one.

- Recursive subhomogeneous algebras due to Huaxin Lin and N. Christopher Phillips.
- Classification results of Huaxin Lin and Zhuang Niu.

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On generalized universal irrational rotation algebras
Classification of simple generalized universal irrational rotation \( C^* \)-algebras

**Theorem**

Let \( A_{\theta, \gamma} \) be a unital simple \( C^* \)-algebra. Then \( A_{\theta, \gamma} \) is a unital simple \( AT \)-algebra of real rank zero. In particular, \( A_{\theta, \gamma} \) has tracial rank zero and stable rank one.

- Recursive subhomogeneous algebras due to Huaxin Lin and N. Christopher Phillips.
- Classification results of Huaxin Lin and Zhuang Niu.
- Decomposition rank and \( \mathcal{Z} \)-stability of Winter W.
Theorem

Let $\theta_1$ and $\theta_2$ be two irrational numbers, $\gamma_1$ and $\gamma_2 \in C(\mathbb{T})$ be non-negative functions and let $Y_i$ be the set of zeros of $\gamma_i$, $i = 1, 2$. Suppose that $A_{\theta_1, \gamma_1}$ and $A_{\theta_2, \gamma_2}$ are simple. Then $A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2}$ if and only if the following hold:

$$\theta_1 = \pm \theta_2 \mod(\mathbb{Z}) \quad C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}.$$ 

In particular, when $\gamma_1$ has only finitely many zeros, then $A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2}$ if and only if $\theta_1 = \pm \theta_2 \mod \mathbb{Z}$ and $\gamma_2$ has the same number of zeros.
(a) $C^*(u + v)$ is a proper subalgebra of $A_\theta = C^*(u, v)$. Indeed, 
\[ \inf \{|u - x| : x \in C^*(u, v)\} = 1. \]

(b) $C^*(u + v) \cong A_{\theta, \gamma}$ for $\gamma(z) = |1 + z|^2$.

(c) $C^*(u + v)$ is a simple $\mathcal{A}\mathbb{T}$-algebra of real rank zero and stable rank one.

(d) For every $\alpha \in (\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, there exists a projection $p$ in $C^*(u + v)$ such that $\tau(p) = \alpha$.

(e) $K_0(C^*(u_\theta + v_\theta)) \cong \mathbb{Z} + \mathbb{Z}\theta$ and $K_1(C^*(u + v)) \cong \mathbb{Z}$. In particular, $C^*(u + v)$ is not $\ast$-isomorphic to $C^*(u, v)$.

(f) $C^*(u_\theta + v_\theta) \cong C^*(u_\eta + v_\eta) \Leftrightarrow \theta = \pm \eta (\text{mod}\mathbb{Z})$. 

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On generalized universal irrational rotation algebras
Spectrum of $uf(v)$

Let $f(z) \in L^\infty(S^1, m)$ and let $x = uf(v)$. Then the spectrum of $x$ is given as follows:

1. If $f(v)$ is invertible, then $\sigma(uf(v)) = \Delta(f(v))S^1$.
2. If $f(v)$ is not invertible, then $\sigma(uf(v)) = \mathbb{B}(0, \Delta(f(v)))$.

Here $\Delta(f(v))$ is the Fuglede-Kadison determinant of $f(v)$.

Let $M$ be a finite von-Neumann algebra with a faithful normal tracial state $\tau$. The Fuglede-Kadison determinant $\triangle : M \to [0, \infty]$ is given by $\triangle(T) = \exp\{\tau(\ln |T|)\}, \quad T \in M$, with $\exp\{-\infty\} := 0$.
Brown’s Spectrum distribution

For an arbitrary element $T$ in $M$, the function $\lambda \mapsto \ln(\Delta(T - \lambda I))$ is subharmonic on $\mathbb{C}$, and its Laplacian:

$$d\mu_T(\lambda) = \frac{1}{2\pi} \nabla^2 \ln \Delta(T - \lambda I)$$

In the distribution sense, defines a probability measure $\mu_T$ on $\mathbb{C}$, called the Brown’s spectral distribution or Brown’s measure of $T$.

If $T$ is normal, then $\mu_T$ is the trace $\tau$ composed with the spectral projection of $T$. If $M = M_n(\mathbb{C})$, then $\mu_T$ is the normalized counting measure $\frac{1}{n}(\delta_{\lambda_1} + \delta_{\lambda_2} + \ldots + \delta_{\lambda_n})$, where $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of $T$ repeated according to root multiplicity.

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Let $T \in M$. If the support set of Brown’s measure of $T$ contains more than one point, then $T$ has a nontrivial invariant subspace relative to $M$. 