Toeplitz and Hankel operators on Hardy/Bergman spaces and their Dixmier trace

Genkai Zhang

Chalmers University of Technology and Göteborg University, Göteborg, Sweden

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1 Questions

1.1 Trace formula for Toeplitz operators on the unit disk

D, unit disk in $\mathbb{C}$.

H, Bergman space $L^2_a(D) \subset L^2(D)$ or Hardy space $H^2(\partial D) \subset L^2(\partial D)$, and $P : L^2 \longrightarrow H$ the orthogonal projection.

$T_f = PM_fP$, $H_f = (I - P)M_fP$, Toeplitz and Hankel operator on H with symbol f.

Trace formula: The commutator $[T_f^*, T_f]$ is of trace class,

$$\mathrm{tr}[T_f^*, T_f] = |H_{\bar{f}}|_2^2 = \int_D |f'(z)|^2 \, dm(z) = \mathsf{Area} f(D),$$

for holomorphic f with appropriate growth condition (say continuous on the boundary)

1.2 Unit ball

$B = B^d$ unit ball in $\mathbb{C}^d$, $d \geq 2$.

$[T_f^*, T_f] = H_{\bar{f}}^* H_{\bar{f}}$ is never trace class. Trace Formula above does not make sense.

Precise statement: Let $\mathcal{L}^p$ be the Schatten - von Neumann class of p-summable operators. The commutator $[T_f^*, T_f]$, for holomorphic f, on the Hardy or Bergman is in

$$\mathcal{L}^p \qquad \text{iff } f \in B_p \quad \text{(holomorphic Besov space)}$$

for $p > d$,

$$\mathcal{L}^p \qquad \text{iff } f = const. \quad (i.e. H_{\bar{f}} = 0)$$

for $p \leq d$.

(Arazy-Fisher-Janson-Peetre, Rochberg, Zhu, Zhang ...),

Remark: Different behaviour in dimension $d = 1$. The Hankel operator $H_{\bar{f}}$ on the Bergman space is in the Schatten class $\mathcal{L}^p$ for smooth f for $p > 1$. On Hardy space H_f can be of finite rank and thus in any $\mathcal{L}^p$, $p > 0$. [Peller, Arazy-Fisher-Peetre, ...])

Variation of the Trace formula due to Helton and Howe.

Take smooth functions $f_1, \cdots, f_{2d}$ on the closed unit ball. Consider the anti-symmetrization $[T_{f_1}, T_{f_2}, \cdots T_{f_{2d}}]$ of the $2d$ operators $T_{f_1}, T_{f_2}, \cdots T_{f_{2d}}$. It is of trace class

$$\mathrm{tr}[T_{f_1}, T_{f_2}, \cdots T_{f_{2d}}] = \int_B df_1 \wedge df_2 \cdots \wedge df_{2d}.$$

1.3 Dixmier Trace

Refined statement on the Schatten - von Neumann properties of the Toeplitz: Let $\mathcal{L}^{1,\infty}$ be the weak trace class. (E.g. the diagonal operator with eigenvalues $\{1/n\}$ is in $\mathcal{L}^{1,\infty}$.) The commutator $[T_f^*, T_f]^d$ is in $\mathcal{L}^{1,\infty}$ (but not $\mathcal{L}^{1,\infty}$)

Dixmier trace on $\mathcal{L}^{1,\infty}$: Consider (as an motivating example) the following spaces

$$l^1 \subset l^{1,\infty} \subset l^\infty$$

of sequences $\{a_n\}$. By Hahn-Banach theorem there exist linear functionals on l^∞ which vanish on l^1 but not on $l^{1,\infty}$. (A kind of summation method.)

Recall the Macaev (or Dixmier) class $\mathcal{L}^{1,\infty}$, consists of all compact operators T such that the eigenvalues $\mu_1 \geq \mu_2 \geq \dots$ of $|T|$ satisfy

$$\sum_{n=1}^{N} \mu_n = O(\log N).$$

Dixmier traces tr_w is a linear functional on $\mathcal{L}^{1,\infty}$ (depending on a linear functional ω on $C_b(\mathbb{R}_+)$).

Example: $\frac{d}{d\theta}$ the differential operator on $L^2(\partial D)$. The eigenvalues of $|\frac{d}{d\theta}|$ are

$|n|$, $|\frac{d}{d\theta}|e^{in\theta} = |n|e^{in\theta}$. The Dixmier trace

$$\operatorname{tr}_w |\frac{d}{d\theta}|^{-1} = 1$$

Theorem 1.1. (Connes) Let $Q = p(x, \partial)$ be a differential pseudo-differential operator of order $-n$ on a compact manifold M. Then $p(x, \partial)$ is in the weak trace class and

$$\operatorname{tr}_\omega Q = \int_S p$$

where S in the unit sphere bundle in the cotangent bundle of M. ($S = \{x, \xi); \xi \in T^*_x(M), |\xi|_x = 1\}$).

1.4 Questions.

• Study Dixmier trace formulas on a strongly pseudo-convex smooth domain in $\mathbb{C}^d$.

Motivation: Find CR-invariants of the boundary $\partial\Omega$.

2 Results

2.1 Hardy space on the unit ball

Let

$$\partial_j^b = \partial_j - \bar{z}_j R, \quad \bar{\partial}_j^b = \bar{\partial}_j - z_j \bar{R},$$

be the boundary Cauchy-Riemann operators, where $R = \sum_{j=1}^{d} z_j \partial_j$ is the holomorphic radial derivative.

Definition 2.1. Let f and g be smooth functions on S. The function

$$\{f, g\}_b := \sum_{j=1}^{d} (\partial_j^b f \, \bar{\partial}_j^b g - \bar{\partial}_j^b f \, \partial_j^b g)$$

is called the boundary Poisson bracket.

Theorem 2.2. (Englis-Guo-Zhang, 2009) Let $f_1, g_1, \cdots, f_d, g_d$ be smooth functions on S, $\tilde{f}_1, \tilde{g}_1, \cdots, \tilde{f}_d, \tilde{g}_d$ their smooth extensions to B and $T_{\tilde{f}_1}, T_{\tilde{g}_1}, \cdots, T_{\tilde{f}_d}, T_{\tilde{g}_d}$ the associated Toeplitz operators on $\mathcal{H}_\nu$ for $\nu \geq d$. Then the product $\prod_{j=1}^{d} [T_{\tilde{f}_j}, T_{\tilde{g}_j}]$

is in the Macaev class and its Dixmier trace is given by

$$\mathrm{tr}_\omega \prod_{j=1}^{d} [T_{\tilde{f}_j}, T_{\tilde{g}_j}] = \int_S \prod_{j=1}^{d} \{f_j, g_j\}_b.$$

The result is proved by using Howe's trick realizing Toeplitz operators on Hardy and Bergman spaces as pseudo-Toeplitz operators on the Fock space $\mathbb{F}(\mathbb{C}^n)$ with homogeneous symbols on $\mathbb{C}^n$.

Theorem 2.3. (Englis-Rochberg, 2009) $n = 1$.

$$\mathrm{tr}_\omega |H_f| = \int_{\partial D} |\bar{\partial} f| d\sigma$$

In particular, if f is holomorphic

$$\mathrm{tr}_\omega |H_f| = \text{length of } f(\partial D)$$

Similar results expressing the Haussdorff measure of $f(\partial D)$ in terms traces studied by Connes.

3 Teoplitz operators and Hardy on SPCXS domains in $\mathbb{C}^n$

3.1 Psudo-differential operators

On $\mathbb{R}^d$: A linear partial differential operator is a linear combination of

$f(x) \to m(x)\partial_1^{n_1} \cdots \partial_d^{n_d} f(x)$

which can be written as

$$f(x) \to \mathcal{F}^{-1}_{\xi \to x}(m(x)\xi_1^{n_1} \cdots \xi_d^{n_d} \mathcal{F}_{x \to \xi} f)$$

with symbol

$$m(x)\xi_1^{n_1}\cdots\xi_d^{n_d}.$$

A pseudo-differential operator is of the form

$$p(x,\partial): f(x) \to \mathcal{F}^{-1}_{\xi\to x}(p(x,\xi)\mathcal{F}_{x\to\xi})$$

with symbol $p(x,\xi)$. (There are various classes of symbol functions, e.g., the Hörmander class S^m.)

Pseudo-differential operator on Compact manifold M. Given a function $p(x,\xi)$ on the cotangent bundle of M of class S^m. M locally given by a coordinate chart $U \in \mathbb{R}^n$. Can define an operator $p(x,\partial)$ for functions f with compact support on U. On whole M can define an operator (up to operator of lower order) on $C^\infty(M)$.

3.2 Boutet de Monvel theory of Toeplitz operator on Hardy spaces with pseudo-differential symbols on the boundary of a pseudo-convex domain

Let r be a defining function for Ω, that is, $r \in C^\infty(\bar{\Omega})$, $r < 0$ on Ω, and $r = 0$, $\|\partial r\| > 0$ on $\partial\Omega$.

Ω is called strongly pseudo-convex if (r can be chosen so that) the Hessian

$$(\frac{\partial^2 r}{\partial z_j \bar{\partial} z_k})_{jk}$$

is positive definite for $z \in \partial\Omega$.

η: the restriction to $\partial\Omega$ of the 1-form $\mathrm{Im}(\partial r) = (\partial r - \overline{\partial} r)/2i$.

Strict pseudoconvexity: η is a contact form, i.e. the half-line bundle

$$\Sigma := \{(x, \xi) \in T^*(\partial\Omega) :\ \xi = t\eta_x,\ t > 0\}$$

is a symplectic submanifold of $T^*(\partial\Omega)$.

Equip $\partial\Omega$ with a measure with smooth positive density, and let $L^2(\partial\Omega)$ be the Lebesgue space with respect to this measure. The Hardy space $H^2(\partial\Omega)$ is the subspace in $L^2(\partial\Omega)$ of functions whose Poisson extension is holomorphic in Ω;

For $Q \in S^m$, the *generalized Toeplitz operator* $T_Q : W^m_{hol}(\partial\Omega) \to H^2(\partial\Omega)$ is defined as

$$T_Q = \Pi Q,$$

where $\Pi : L^2(\partial\Omega) \to H^2(\partial\Omega)$ is the orthogonal projection (the Szegö projection). Alternatively, one may view T_Q as the operator

$$T_Q = \Pi Q \Pi$$

on all of $W^m(\partial\Omega)$. Actually, T_Q maps continuously $W^s(\partial\Omega)$ into $W^{s-m}_{hol}(\partial\Omega)$, for each $s \in$, because Π is bounded on $W^s(\partial\Omega)$ for any $s \in$ (see []).

Example: $\Omega = B$ unit ball in $\mathbb{C}^2$. $Q = \bar{z}_1\partial_2 - \bar{z}_2\partial_1$. The Toeplitz operator $T = PQP$ is a (unbounded) shift operator:

$$z_1^m z_2^n \mapsto \alpha(m,n) z_1^{m-1} z_2^{n-1}$$

General Properties of Pseudo-Toeplitz calculus:

- "(P1)" They form an algebra which is, modulo smoothing operators, locally isomorphic to the algebra of classical pseudo differential operators on ^{n}.
- "(P2)" If P, Q are of the same order and $T_P = T_Q$, then the principal symbols $\sigma(P)$ and $\sigma(Q)$ coincide on Σ. One can thus define unambiguously the *order* of a generalized Toeplitz operator as $(T_Q) := \min\{(P) : T_P = T_Q\}$, and

its *principal symbol* (or just "symbol") as $\sigma(T_Q) := \sigma(Q)|_\Sigma$ if $(Q) = (T_Q)$. (The symbol is undefined if $(T_Q) = -\infty$.)

- "(P3)" If $(T_Q) \leq 0$, then T_Q is a bounded operator on $L^2(\partial\Omega)$; if $(T_Q) < 0$, then it is even compact.

Hankel operator on Hardy or Bergman space:

$$H_Q = (I - P)QP$$

3.3 Dixmier trace of Toeplitz operator on Hardy spaces

Let T be a positive self-adjoint generalized Toeplitz operator on $\partial\Omega$ of order 1 with $\sigma(T) > 0$. Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ be the points of its spectrum (counting multiplicities) and let $N(\lambda)$ denote the number of λ_j's less than λ.

Weyl type theorem on spectral distribution (Guillemin and Boutet de Monvel):

As $\lambda \to +\infty$,

$$N(\lambda) = \frac{vol(\Sigma_T)}{(2\pi)^n}\lambda^n + O(\lambda^{n-1}),$$

where Σ_T is the subset of Σ where $\sigma(T) \leq 1$, and $vol(\Sigma_T)$ is its symplectic volume.

Formula for the Dixmier trace:

Theorem 3.1. Let T be a generalized Toeplitz operator on $H^2(\partial\Omega)$ of order $-n$. Then $T \in \mathcal{L}^{1,\infty}$, and

$$\mathrm{tr}_w(T) = \frac{1}{n!(2\pi)^n}\sigma(T)(x, \eta_x)\ \eta \wedge (d\eta)^{n-1}.$$

In particular, T is measurable.

4 Toeplitz operator on Bergman spaces via the Poisson operator

Difficulty with the Toeplitz operators on Bergman space on Ω: Ω is a Non-Compact manifold. Generally Pseudo-differntial operators of negative powers on $L^2(\Omega)$ are not compact operators.

Let K denote the Poisson extension operator on Ω, i.e. K solves the Dirichlet problem

$$\Delta K u = 0 \quad \text{on } \Omega, \qquad K u|_{\partial\Omega} = u.$$

(Thus K acts from functions on $\partial\Omega$ into functions on Ω. Here Δ is the ordinary Laplace operator.)

Standard elliptic regularity theory: K acts continuously from $W^s(\partial\Omega)$ onto the subspace $W^{s+1/2}(\Omega)$ of all harmonic functions in $W^{s+1/2}(\Omega)$.

The composition

$$K^*K =: \Lambda$$

is an elliptic positive pseudo differential operator on $\partial\Omega$ of order -1.

We have

$$\Lambda^{-1}K^*K = I_{L^2(\partial\Omega)},$$

while

$$K\Lambda^{-1}K^* = \Pi_{harm},$$

the orthogonal projection in onto the subspace $L^2(\Omega)$ of all harmonic functions.

The restriction

$$\gamma := \Lambda^{-1} K^* |_{L^2(\Omega)}$$

is the operator of "taking the boundary values" of a harmonic function. γ extends to a continuous operator from $W^s(\Omega)$ onto $W^{s-1/2}(\partial\Omega)$, for any $s \in \mathbb{R}$, which is the inverse of K.

Polar decomposition of K:

$$K = U(K^*K)^{1/2} = U\Lambda^{1/2},$$

U is a unitary operator from $L^2(\partial\Omega)$ onto $L^2_{harm}(\Omega)$.

Proposition 4.1. Let $w \in C^\infty()$ be of the form

$$w = r^m g, \qquad m = 0, 1, 2, \ldots, \; g \in C^\infty().$$

Then

$$U^*T_wU = W^{-1/2-1/2}W^* = WT_{Q_w}W^*,$$

where Q_w is a pseudo differential operator on $\partial\Omega$ of order $-m$ with

$$\sigma(Q_w)(x,\xi)|_\Sigma = \frac{(-1)^m m!}{|\xi|^m}\, g(x)\, \|\eta_x\|^m.$$

Thus trace formula for Teoplitz operators can be computed using the Hardy space case.

Typical case of Toeplitz operators T_f of Dixmier class on Bergman space: f on Ω vanishes of order n at $\partial\Omega$,

$$f(x) \sim r^n(x), \quad x \to \partial\Omega$$

Theorem 4.2. Assume that $f \in C^\infty(\Omega)$ vanishes at $\partial\Omega$ to order n. Then T_f

belongs to the Dixmier class, is measurable, and

$$\operatorname{tr}_w(T_f) = \frac{1}{n!(4\pi)^n} \mathfrak{N}^n f \; \frac{\eta \wedge (d\eta)^{n-1}}{\|\eta\|^n},$$

where $\mathfrak{N}$ denotes the interior unit normal derivative.

4.1 Hankel type operators on Hardy and Bergman spaces

Symbols of commutators of two generalized Toeplitz operators

$$\sigma([T_P, T_Q]) = \tfrac{1}{i}\{\sigma(T_P), \sigma(T_Q)\}_\Sigma$$

are given by the Poisson bracket (with respect to the symplectic structure of Σ) of their symbols:

Goal: Product of Hankel operators

$$H^*_{g_1} H_{f_1} H^*_{g_2} H_{f_2} \dots H^*_{g_n} H_{f_n}$$

Have to study semi-commutator $T_{PQ} - T_P T_Q$ of two generalized Toeplitz operators. It will be given, roughly speaking, by an appropriate "half" of the Poisson bracket.

T'': the anti-holomorphic complex tangent space to $\partial\Omega$.

$$E := \sum_{j=1}^{n} \frac{\partial r}{\partial z_j} \frac{\partial}{\partial_j} - \frac{\partial r}{\partial_j} \frac{\partial}{\partial z_j}$$

(the "complex normal" direction).

The boundary d-bar operator $\overline{\partial}_b : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega, T''^*)$:

$$\overline{\partial}_b f := df|_{T''},$$

Levi form L' is the Hermitian form on T' defined by

$$L'(X,Y) := \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial_k} X_j \overline{Y}_k \qquad \text{if } X = \sum_j X_j \frac{\partial}{\partial z_j},\ Y = \sum_k Y_k \frac{\partial}{\partial z_k}.$$

Levi form L'' on T'' defined by

$$L''(X,Y) := \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_k \partial_j} X_j \overline{Y}_k \qquad \text{if } X = \sum_j X_j \frac{\partial}{\partial_j},\ Y = \sum_k Y_k \frac{\partial}{\partial_k}.$$

$$L''(X,Y) = L'(\overline{Y}, \overline{X}) \qquad \forall X, Y \in T''.$$

L'' induces a positive definite Hermitian form on the dual space T''^{*} of T''.

H_f the Hankel operator on $L^2_a(\Omega)$, the Bergman space on Ω.

Theorem 4.3. (Englis-Z.) Let $f_1, g_1, \ldots, f_d, g_d \in C^\infty(\Omega)$. Then the operator

$$H = H_{g_1}^* H_{f_1} H_{g_2}^* H_{f_2} \ldots H_{g_d}^* H_{f_d}$$

on $L_a^2(\Omega)$ belongs to the Dixmier class, and

$$\mathrm{tr}_\omega(H) = \frac{1}{d!(2\pi)^d} \int_{\partial\Omega} \mathcal{L}(\overline{\partial}_b f_1, \overline{\partial}_b g_1) \ldots \mathcal{L}(\overline{\partial}_b f_d, \overline{\partial}_b g_d)\, \eta \wedge (d\eta)^{d-1}.$$

Remark: $\eta \wedge (d\eta)^{d-1}$ is a multiple of the area form on $\partial\Omega$, (depending on η). However the whole integration $\int_{\partial\Omega} \omega$ is biholomorphic invariant, i.e. if $F : \Omega \to \Omega_1$ is a bihomorphic mapping, then $\int_{\partial\Omega} \omega = \int_{\partial\Omega_1} \omega_1$ with $\omega = F^* \omega_1$.

Corollary 4.4. Let f be holomorphic on Ω and C^∞ on . Then $|H_{|}^{2n}$ is in the Dixmier class, measurable, and

$$\mathrm{tr}_w(|H_f|^{2n}) = \frac{1}{n!(2\pi)^n} \int (\overline{\partial}_b f, \overline{\partial}_b f)^n\, \eta \wedge (d\eta)^{n-1}.$$