Classification of extensions of $A\mathbb{T}$-algebras

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Extension

Let $A$ and $B$ be $C^*$-algebras. Recall that an extension of $A$ by $B$ is a short exact sequence

$$0 \to B \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0.$$ 

Denote this extension by $e$ or $(E, \alpha, \beta)$ and the set of all such extensions by $\mathcal{E}xt(A, B)$.

The extension $(E, \alpha, \beta)$ is called trivial, if the above sequence splits, i.e. if there is a homomorphism $\gamma : A \to E$ such that $\beta \circ \gamma = id_A$.

We call $(E, \alpha, \beta)$ essential, if $\alpha(B)$ is an essential ideal in $E$. We denote the set of all essential extensions by $\mathcal{E}xt^e(A, B)$. 
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The Busby invariant

The Busby invariant of \((E, \alpha, \beta)\) is a homomorphism \(\tau\) from \(A\) into the corona algebra \(Q(B) = M(B)/B\) defined by \(\tau(a) = \pi(\sigma(b))\) for \(a \in A\), where \(\pi : M(B) \to Q(B)\) is the quotient map, and \(b \in E\) such that \(\beta(b) = a\).

Hence, we have the commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & Q(B) & \longrightarrow & 0.
\end{array}
\]

If \(A\) is unital and the Busby invariant is unital, then \((E, \alpha, \beta)\) is called unital.
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Equivalence

Let \( e_i : 0 \to B \to E_i \to A \to 0 \) be two extensions with Busby invariants \( \tau_i \) for \( i = 1, 2 \).

Definition 1

\( e_1 \) and \( e_2 \) are called congruent, denoted by \( e_1 \equiv e_2 \), if there exists an isomorphism \( \eta \) making the following diagram commute:

\[
\begin{array}{cccccc}
0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
\| & & | & & \downarrow\eta & & | & & \\
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Definition 2

e_1 and e_2 are called (strongly) unitarily equivalent, denoted by e_1 \sim e_2, if there exists a unitary u \in M(B) such that \tau_2(a) = \pi(u)\tau_1(a)\pi(u)^* for all a \in A. Denote by \text{Ext}(A, B) or \text{Ext}_s(A, B) the set of (strongly) unitary equivalence classes of extensions of A by B.
Definition 3

Weakly unitarily equivalent, denoted by $e_1 \sim^w e_2$, if there exists a unitary $u \in Q(B)$ such that $\tau_2(a) = u\tau_1(a)u^*$ for all $a \in A$. Denote by $\text{Ext}_w(A, B)$ the set of weakly unitary equivalence classes of extensions of $A$ by $B$. 
Definition 4

$e_1$ and $e_2$ are called isomorphic, denoted by $e_1 \cong e_2$, if there exist isomorphisms $\beta, \eta, \alpha$ making the following diagram commute:

$$
\begin{array}{c}
0 \longrightarrow B \longrightarrow E_1 \longrightarrow A \longrightarrow 0 \\
\downarrow \beta \quad \downarrow \eta \quad \downarrow \alpha \\
0 \longrightarrow B \longrightarrow E_2 \longrightarrow A \longrightarrow 0.
\end{array}
$$

Denote the morphism of extensions by $(\beta, \eta, \alpha) : e_1 \rightarrow e_2$. Denote by $\text{Ext}_I(A, B)$ the set of equivalence classes of extensions up to isomorphism.
Sum of extensions

Suppose that $B$ is a stable $C^*$-algebra. Then the sum of two extensions $\tau_1$ and $\tau_2$ is defined to be the homomorphism $\tau_1 \oplus \tau_2$, where

$$\tau_1 \oplus \tau_2 : A \to Q(B) \oplus Q(B) \subseteq M_2(Q(B)) \cong Q(B).$$

- $\text{Ext}_s(A, B)$ and $\text{Ext}_w(A, B)$ are semigroups
- Trivial extensions construct subsemigroups of $\text{Ext}_s(A, B)$ and $\text{Ext}_w(A, B)$, respectively
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The stable Ext-group $\text{Ext}(A, B)$ is the quotient of $\text{Ext}_s(A, B)$ by the subsemigroup of trivial extensions. The equivalence class of an extension $\tau$ in $\text{Ext}(A, B)$ is denoted by $[\tau]$.

If $[\tau_1] = [\tau_2]$ in $\text{Ext}(A, B)$, then $\tau_1$ and $\tau_2$ are called stably unitarily equivalent, denoted by $\tau_1 \sssim \tau_2$.

- $[\tau_1] = [\tau_2]$ iff there are trivial extensions $\sigma_i$ such that $\tau_1 \oplus \sigma_1 \sssim \tau_2 \oplus \sigma_2$
- If $A$ is a separable nuclear $C^*$-algebra and $B$ is a $\sigma$-unital $C^*$-algebra, then $\text{Ext}(A, B)$ is an abelian group
- $\text{Ext}(A, B) \cong KK^1(A, B)$
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Ext-group

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Relations of equivalences

- $\equiv \implies \sim \implies w \implies ss$. Conversely, they do not hold.
- $s \implies \sim \neq w$
- $w \neq \sim$

Note: In general, $\text{Ext}_I(A, B)$ is not a semigroup since the isomorphism equivalence can not preserve the addition.
Relations of equivalences

- $\equiv \Rightarrow \sim \Rightarrow \sim \Rightarrow \sim$. Conversely, they do not hold.
- $s \Rightarrow \sim \Rightarrow \not\sim \wedge s$
- $w \Rightarrow \not\sim \Rightarrow \sim$

Note: In general, $\text{Ext}_I(A, B)$ is not a semigroup since the isomorphism equivalence can not preserve the addition.
Invariant

Suppose that $A$ is a unital $C^*$-algebra. Denote by $T(A)$ the tracial state space of $A$ and denote by $\text{Aff}(T(A))$ the space of all real affine continuous functions on $T(A)$.

Define $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$ to be the positive homomorphism defined by $\rho_A([p])(\tau) = \tau(p)$ for each projection $p$ in $M_k(A)$.

Let $A$ be a unital simple separable $C^*$-algebra. Recall that the Elliott invariant of $A$ is the 6-tuple: $(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A)$. We denote it by $\text{Ell}(A)$.

When $A$ is non-unital, let $T(A)$ be the set of lower-semicontinuous densely defined traces on $A$ equipped with the weakest topology such that the functional $\tau \rightarrow \tau(a)$ is continuous for any $a \in A^+$ dominated by a projection. Let $\text{Inv}(A) = (K_0(A), K_0(A)^+, \Sigma(A), K_1(A), T(A), r_A)$, where $\Sigma(A) = \{[p] : p \in P(A)\}$ is the scale and $P(A)$ is the set of projections in $A$. 
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When $A$ is non-unital, let $\mathcal{T}(A)$ be the set of lower-semicontinuous densely defined traces on $A$ equipped with the weakest topology such that the functional $\tau \to \tau(a)$ is continuous for any $a \in A^+$ dominated by a projection. Let $\text{Inv}(A) = (K_0(A), K_0(A)^+, \Sigma(A), K_1(A), \mathcal{T}(A), r_A)$, where $\Sigma(A) = \{[p] : p \in P(A)\}$ is the scale and $P(A)$ is the set of projections in $A$. 
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Let $A$ and $B$ be two unital simple separable amenable $C^*$-algebras with stable rank one. We write $Ell(A) \cong Ell(B)$ if

$$(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A) \cong (K_0(B), K_0(B)^+, [1_B], K_1(B), T(B))$$

that is, if there are an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$, an order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$ such that $\alpha_0([1_A]) = [1_B]$ and an affine homeomorphism $\gamma : T(B) \to T(A)$ such that

$$
\begin{array}{c}
T(B) \\ r_A \downarrow \\
S(K_0(B)) \\ \alpha_0^* \\
\end{array} \xrightarrow{\gamma} \begin{array}{c}
T(A) \\ r_B \downarrow \\
S(K_0(A)) \\ \end{array}
$$

commutes.

Similarly, one can define an isomorphism $Inv(A) \cong Inv(B)$ when $A$ and $B$ are non-unital.
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that is, if there are an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$, an order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$ such that $\alpha_0([1_A]) = [1_B]$ and an affine homeomorphism $\gamma : T(B) \to T(A)$ such that

$$\begin{array}{ccc}
T(B) & \xrightarrow{\gamma} & T(A) \\
\downarrow r_A & & \downarrow r_B \\
S(K_0(B)) & \xrightarrow{\alpha_0^*} & S(K_0(A))
\end{array}$$

commutes.

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$$\begin{align*}
T(B) & \xrightarrow{\gamma} T(A) \\
\downarrow r_A & \\
S(K_0(B)) & \xrightarrow{\alpha_0^*} S(K_0(A)) \\
\downarrow r_B & \\
\end{align*}$$

commutes.

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Let $A$ and $B$ be two unital simple separable amenable $C^*$-algebras with stable rank one. We write $Ell(A) \cong Ell(B)$ if

$$(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A) \cong (K_0(B), K_0(B)^+, [1_B], K_1(B), T(B))$$

that is, if there are an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$, an order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$ such that $\alpha_0([1_A]) = [1_B]$ and an affine homeomorphism $\gamma : T(B) \to T(A)$ such that

$$\begin{array}{ccc}
T(B) & \xrightarrow{\gamma} & T(A) \\
\downarrow r_A & & \downarrow r_B \\
S(K_0(B)) & \xrightarrow{\alpha_0^*} & S(K_0(A))
\end{array}$$

commutes.

Similarly, one can define an isomorphism $Inv(A) \cong Inv(B)$ when $A$ and $B$ are non-unital.
Let $e : 0 \to B \to E \to A \to 0$ be an extension of $A$ by $B$. Denote by $K(e)$ the six term exact sequence of $e$ in $K$-theory:

$$
\begin{array}{cccc}
K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\
\delta_1 & \uparrow & & \downarrow \delta_0 \\
K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B)
\end{array}
$$

Denote by $\mathcal{H}ext(A, B)$ all such $K(e)$ of extensions of $A$ by $B$. 
Let \( e : 0 \to B \to E \to A \to 0 \) be an extension of \( A \) by \( B \). Denote by \( K(e) \) the six term exact sequence of \( e \) in \( K \)-theory:

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\delta_1 & \uparrow & & & \delta_0 \\
K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B)
\end{array}
\]

Denote by \( \mathcal{H}_{\text{ext}}(A, B) \) all such \( K(e) \) of extensions of \( A \) by \( B \).
Let $e_i \in \mathcal{E}xt(A_i, B_i)(i = 1, 2)$. We call $(\alpha_*, \beta_*, \lambda_*) : K(e_1) \to K(e_2)$ a morphism if there are homomorphisms $\alpha_* : K_*(A_1) \to K_*(A_2)$, $\beta_* : K_*(B_1) \to K_*(B_2)$, and $\lambda_* : K_*(E_1) \to K_*(E_2)$ making the obvious diagram commutative.

If $\alpha_*$, $\beta_*$ and $\lambda_*$ are isomorphisms, then $K(e_1)$ and $K(e_2)$ are called isomorphic, written $K(e_1) \cong K(e_2)$. If $A_1 = A_2 = A$, $B_1 = B_2 = B$ and there is an isomorphism $(\text{id}_{K_*(A)}, \text{id}_{K_*(B)}, \lambda_*) : K(e_1) \to K(e_2)$, then they are called congruent, written $K(e_1) \equiv K(e_2)$.

Let $H_{\text{ext}}(A, B)$ denote the set of congruent classes of six term exact sequences in $H_{\text{ext}}(A, B)$. 
Let \( e_i \in \mathcal{E}xt(A_i, B_i)(i = 1, 2) \). We call \((\alpha_*, \beta_*, \lambda_*) : K(e_1) \to K(e_2)\) a morphism if there are homomorphisms \( \alpha_* : K_*(A_1) \to K_*(A_2) \), \( \beta_* : K_*(B_1) \to K_*(B_2) \), and \( \lambda_* : K_*(E_1) \to K_*(E_2) \) making the obvious diagram commutative.

If \( \alpha_*, \beta_* \) and \( \lambda_* \) are isomorphisms, then \( K(e_1) \) and \( K(e_2) \) are called isomorphic, written \( K(e_1) \cong K(e_2) \). If \( A_1 = A_2 = A \), \( B_1 = B_2 = B \) and there is an isomorphism \((\text{id}_{K_*(A)}, \text{id}_{K_*(B)}, \lambda_*) : K(e_1) \to K(e_2)\), then they are called congruent, written \( K(e_1) \equiv K(e_2) \).

Let \( \text{Hext}(A, B) \) denote the set of congruent classes of six term exact sequences in \( \mathcal{H}ext(A, B) \).
Let \( e_i \in \mathcal{E}xt(A_i, B_i)(i = 1, 2) \). We call \((\alpha_*, \beta_*, \lambda_*) : K(e_1) \to K(e_2)\) a morphism if there are homomorphisms \(\alpha_* : K_*(A_1) \to K_*(A_2)\), \(\beta_* : K_*(B_1) \to K_*(B_2)\), and \(\lambda_* : K_*(E_1) \to K_*(E_2)\) making the obvious diagram commutative.

If \(\alpha_*, \beta_*\) and \(\lambda_*\) are isomorphisms, then \(K(e_1)\) and \(K(e_2)\) are called isomorphic, written \(K(e_1) \cong K(e_2)\). If \(A_1 = A_2 = A\), \(B_1 = B_2 = B\) and there is an isomorphism \((id_{K_*(A)}, id_{K_*(B)}, \lambda_*) : K(e_1) \to K(e_2)\), then they are called congruent, written \(K(e_1) \equiv K(e_2)\).

Let \(Hext(A, B)\) denote the set of congruent classes of six term exact sequences in \(\mathcal{H}ext(A, B)\).
Let $e_i \in \mathcal{E}xt(A_i, B_i)(i = 1, 2)$. We call $(\alpha_*, \beta_*, \lambda_*) : K(e_1) \to K(e_2)$ a morphism if there are homomorphisms $\alpha_* : K_*(A_1) \to K_*(A_2)$, $\beta_* : K_*(B_1) \to K_*(B_2)$, and $\lambda_* : K_*(E_1) \to K_*(E_2)$ making the obvious diagram commutative.

If $\alpha_*, \beta_*$ and $\lambda_*$ are isomorphisms, then $K(e_1)$ and $K(e_2)$ are called isomorphic, written $K(e_1) \cong K(e_2)$. If $A_1 = A_2 = A$, $B_1 = B_2 = B$ and there is an isomorphism $(id_{K_*(A)} , id_{K_*(B)} , \lambda_*) : K(e_1) \to K(e_2)$, then they are called congruent, written $K(e_1) \equiv K(e_2)$.

Let $H_{\text{ext}}(A, B)$ denote the set of congruent classes of six term exact sequences in $\mathcal{H}_{\text{ext}}(A, B)$. 
Denote by $KK(A, B)^{++}$ those elements $x \in KK(A, B)$ such that

$$K_0(x)(K_0(A)_+\setminus\{0\}) \subset K_0(B)_+\setminus\{0\}.$$ 

Suppose that both $A$ and $B$ are unital. Denote by $KK_e(A, B)^{++}$ the set of those elements $x$ in $KK(A, B)^{++}$ such that $K_0(x)([1_A]) = [1_B]$. 
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Classification — nonunital case

Some Results

Consider the extension $0 \to B \to E \to A \to 0$

- Rordam, 1997: When $A, B$ are stable Kirchberg algebras, then $K(e)$ is a complete invariant for extensions up to stable isomorphism.

- Eilers-Restorff-Ruiz, 2009: Suppose that $A, B$ are in a certain class of $C^*$-algebras which are classified by $K^+_*(A) = (K_0(A), K_0(A)_+, K_1(A))$ and $B$ has CFP, then $K^+_*(A) + K(e)$ is a complete invariant for full extensions being stably isomorphic.
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Definition

Let $B$ be separable stable $C^*$-algebra. Then $B$ is said to have the Corona Factorization Property (CFP) if every full projection in $M(B)$ is $M$-v equivalent to $1_M(B)$.

If $B$ has CFP, then

- Note: every nonunital full extension is absorbing, and every unital full extension is unital-absorbing.
- Lin-Kucerovsky-Ng: $KK^1(A, B) = \{[\tau]_u : \tau \text{ is nonunital, full} \}$ for separable nuclear $C^*$-algebra $A$. 
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Lemma (Ortega-Perera-Rordam)

Let $B$ be a separable, unital $C^*$-algebra with finite decomposition rank. Then $B \otimes K$ has the corona factorization property.

Corollary

Let $B$ be a unital $A\mathbb{T}$-algebra, then $B \otimes K$ has CFP.
Lemma (Ortega-Perera-Rordam)

Let $B$ be a separable, unital $C^*$-algebra with finite decomposition rank. Then $B \otimes \mathcal{K}$ has the corona factorization property.

Corollary

Let $B$ be a unital $A\mathbb{T}$-algebra, then $B \otimes \mathcal{K}$ has CFP.
Theorem

Let \( A \) be a simple \( \mathcal{A}\mathcal{T} \)-algebra with unit. Suppose that \( a \in KK_e(A, A)^{++} \) and \( \gamma : T(A) \to T(A) \) is an affine homeomorphism such that

\[
K_*(a) : (K_0(A), K_0(A)^+, [1_A], K_1(A)) \to (K_0(A), K_0(A)^+, [1_A], K_1(A))
\]
is an isomorphism and \( \gamma \) is compatible with \( K_0(a) \).

It follows that there is an automorphism \( \phi : A \to A \) such that \( KK(\phi) = a \) in \( KK(A, A) \) and \( \phi_T = \gamma \).
Lemma (Rordam)

Let $A$ and $B$ be separable nuclear $C^*$-algebras in $\mathcal{N}$ with $B$ stable, and let $x_1, x_2 \in \text{Ext}(A, B)$. Then $K(x_1) = K(x_2)$ in $\text{Hext}(A, B)$ if and only if there exist elements $a$ in $KK(A, A)$ and $b$ in $KK(B, B)$ with $K_*(a) = K_*(id_A)$ and $K_*(b) = K_*(id_B)$ such that $x_1 b = ax_2$. 
Lemma

Let $A$ and $B$ be simple $A\mathbb{T}$-algebras with $A$ unital and $B$ stable. Assume that $a \in KK(A, A)$, $b \in KK(B, B)$ such that $K_*(a) = id_{K_*(A)}$ and $K_*(b) = id_{K_*(B)}$. Then there are isomorphisms $\alpha : A \to A$, $\beta : B \to B$ such that $KK(\alpha) = a$ and $KK(\beta) = b$. 
Theorem

Let $A_i$ and $B_i$ be simple $A\mathbb{T}$-algebras with $A$ unital and $B$ stable. Suppose that $e_i : 0 \to B_i \to E_i \to A_i \to 0$ are non-unital full extensions. Then the following are equivalent:

1. $E_1$ is isomorphic to $E_2$.
2. There is an extension isomorphism $(\beta, \eta, \alpha) : e_1 \to e_2$, i.e. $e_1 \cong e_2$.
3. The six term exact sequences associated to $e_1$ and $e_2$ are isomorphic, i.e. there are isomorphisms $\beta^\#: \text{Inv}(B_1) \to \text{Inv}(B_2)$, $\eta_* : K_*(E_1) \to K_*(E_2)$ and $\alpha^\#: \text{Ell}(A_1) \to \text{Ell}(A_2)$ such that $(\beta_*, \eta_*, \alpha_*) : K(e_1) \to K(e_2)$ is an isomorphism.
Theorem

Suppose that $A_i$ are simple $AT$-algebras with units, and $B_i$ are stabilizations of unital $AF$-algebras. Let $e_i : 0 \to B_i \to E_i \to A_i \to 0$ be non-unital full extensions. Then the following are equivalent:

1. $E_1 \cong E_2$.
2. $e_1 \cong e_2$.
3. The six term exact sequences associated to $e_1$ and $e_2$ are isomorphic, i.e. there is an isomorphism $(\beta_*, \eta_*, \alpha_*) : K(e_1) \to K(e_2)$ for some isomorphisms $\beta_* : (K_0(B_1), K_0(B_1)^+) \to (K_0(B_2), K_0(B_2)^+)$ and $\alpha^\# : El^l(A_1) \to El^l(A_2)$. 
When is an extension an $A\mathbb{T}$-algebra?

Given an extension $0 \to B \to E \to A \to 0$

**Question:** Let $A, B$ be in a class $\mathcal{A}$ of $C^*$-algebras. Which condition will make $E$ be in $\mathcal{A}$?

Brown-Effros-Elliott, 1980s

\[ \mathcal{A} = \{ \text{AF-algebras} \} \implies E \in \mathcal{A}. \]
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$\mathcal{A} = \{\text{AF-algebras}\} \implies E \in \mathcal{A}$. 
Let $A$ and $B$ be $A\mathbb{T}$-algebras with real rank zero and let $e$ be an extension $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$. Then the following three conditions are equivalent:

1. $E$ is an $A\mathbb{T}$-algebra of real rank zero.
2. $E$ has real rank zero and stable rank one.
3. The index maps $\delta_i : K_i(A) \rightarrow K_{1-i}(B)$, $i = 0, 1$ are both trivial.
Dadarlat-Loring, 1993

Assume that $A, B$ are $AD$-algebras with real rank zero, $K_1(B) = 0$ or $K_1(A)$ torsion free. TFAE:

1. $E$ is an $AD$-algebra of real rank zero.
2. $RR(E) = 0$, $st(E) = 1$.
3. $\delta_i = 0$.
Theorem

Suppose that $A$ is an $A\mathbb{T}$-algebra and $B$ is the stabilization of a unital $A\mathbb{T}$-algebra. Let $e : 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ be a non-unital full extension of $A$ by $B$. Then the following are equivalent.

1. $E$ is an $A\mathbb{T}$-algebra.
2. The index maps of $e$ are zero.
3. The extension $e$ is quasidiagonal.
Proof:

(2) \iff (3) and (1) \implies (3) are immediate.

We only need to show that (3) \implies (1).

Lemma 1

Suppose that $A$ and $B$ are $\mathcal{A}\mathcal{T}$-algebras with $B$ stable. Then there is an absorbing trivial extension which is also quasidiagonal.

Lemma 2

Suppose that $A$ and $B$ are $\mathcal{A}\mathcal{T}$-algebras with $B$ stable. Let $e : 0 \to B \to E \xrightarrow{\psi} A \to 0$ be an essential trivial extension of $A$ by $B$. If $e$ is quasidiagonal, then $E$ is an $\mathcal{A}\mathcal{T}$-algebra.
Proof:

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Suppose that $A$ and $B$ are $A\mathbb{T}$-algebras with $B$ stable. Then there is an absorbing trivial extension which is also quasidiagonal.

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Lemma 2

Suppose that $A$ and $B$ are $A\mathbb{T}$-algebras with $B$ stable. Let $e : 0 \to B \to E \xrightarrow{\psi} A \to 0$ be an essential trivial extension of $A$ by $B$. If $e$ is quasidiagonal, then $E$ is an $A\mathbb{T}$-algebra.
Suppose that $e$ is a quasidiagonal extension. Let $A = \lim_{n \to \infty} (A_n, \iota_n)$, where $A_n$ is isomorphic to a quotient of a circle algebra and $\iota_n$ are the inclusion maps. Set $\tau_n = \tau \circ \iota_n$ and $E_n = \pi^{-1}(\tau_n(A_n))$, where $\tau$ is the Busby invariant associated to $e$. Then we have an essential extension $e_n$ of $A_n$ by $B$

$$0 \to B \to E_n \to A_n \to 0$$

for every $n \in \mathbb{N}$. Hence, there is a commutative diagram

$$
\begin{array}{c}
0 & \longrightarrow & B & \longrightarrow & E_n & \longrightarrow & A_n & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0.
\end{array}
$$

Since $A = \lim_{n \to \infty} (A_n, \iota_n)$, it follows that $\tau(A) = \bigcup_{n=1}^{\infty} \tau_n(A_n)$.

Therefore, it follows that

$$E = \bigcup_{n=1}^{\infty} E_n = \lim_{n \to \infty} E_n.$$
For each $A_n$, there is an increasing sequence $\{A_{n,k}\}$ of $C^*$-subalgebras of $A_n$ such that each $A_{n,k}$ is isomorphic to a finite direct sum of $C^*$-algebras of the form $M_m(C(X))$ and $\bigcup_{k=1}^{\infty} A_{n,k}$ is dense in $A_n$, where $X$ is a connected compact subset of the unit circle. Set $\tau_{n,k} = \tau \circ \iota_{n,k}$ and $E_{n,k} = \pi^{-1}(\tau_{n,k}(A_{n,k}))$, where $\iota_{n,k} : A_{n,k} \to A$ is the inclusion map. Let $e_{n,k}$ be the essential extension of $A_{n,k}$ by $B$:

$$0 \to B \to E_{n,k} \to A_{n,k} \to 0.$$  

Obviously, there is a commutative diagram

$$\begin{array}{c}
0 \to B \to E_{n,k} \to A_{n,k} \to 0 \\
\| \quad \downarrow \quad \downarrow \\
0 \to B \to E_n \to A_n \to 0.
\end{array}$$
As the above proof, we have

\[ E_n = \bigcup_{k=1}^{\infty} E_{n,k} = \lim_{k \to \infty} E_{n,k}. \]

Since \( e \) is non-unital and full, then \( e_{n,k} \) is a non-unital full extension. Hence \( e_{n,k} \) is absorbing. By the above proof, the index maps \( \delta_i : K_i(A) \to K_{1-i}(B) \) of \( e \) are trivial. Since \( \tau_{n,k} = \tau \circ \iota_{n,k} \), then the index maps of \( e_{n,k} \) are also trivial. From Lemma 1, it follows that \( A \) is quasidiagonal relative to \( B \), so the subalgebra \( A_{n,k} \) is also quasidiagonal relative to \( B \). Note that \( K_*(A_{n,k}) \) is free. Hence, \( e_{n,k} \) is a trivial and quasidiagonal extension. It follows from Lemma 2 that \( E_{n,k} \) is an \( A_T \)-algebra. Therefore, \( E_n \) is an \( A_T \)-algebra. Consequently, \( E \) is an \( A_T \)-algebra.
Lemma

Suppose that $A$ and $B$ are $C^*$-algebras with $A$ unital and $B$ stable. Let $e_i : 0 \to B \xrightarrow{l_i} E_i \to A \to 0$ be essential unital extensions. Suppose $\tau_2 = \Ad u \circ \tau_1$ for some unitary $u$ in $Q(B)$. Let $\nu$ be a partial isometry in $M(B)$ such that $\pi(\nu) = u$, and let $p = \nu^* \nu$ and $q = \nu \nu^*$. Then

$$(K(e_1), [1]_0) \equiv (K(e_2), [q]_0 + [1 - p]_0).$$
Let $0 \to B \to E \to A \to 0$ be an extension with index maps $\delta_0$ and $\delta_1$ in its $K$-theory. We set $G' = \{ f([1]_0) \mid f \in \text{Hom}(\text{Ker}\delta_0, \text{Coker}\delta_1) \}$ and let $\pi : K_0(B) \to \text{Coker}\delta_1$ be the quotient map.

**Lemma**

Let $e_i$ be essential unital extensions with Busby invariant $\tau_i$. If $e_1$ is weakly unitarily equivalent to $e_2$ by a unitary $u \in Q(B)$. Then

$$(K(e_1), [1]_0) \equiv (K(e_2), [1]_0)$$

if and only if $\pi([u]_1)$ is in $G'$. 
Lemma

Suppose \( e_i \) are essential unital extensions with Busby invariant \( \tau_i \) and \( e_1 \) is weakly unitarily equivalent to \( e_2 \). If the index maps of \( e_i \) are trivial and

\[
(K(e_1), [1]_0) \equiv (K(e_2), [1]_0),
\]

then \([e_1] = [e_2]\) in \(\mathrm{Ext}^u_s(A, B)\).
Theorem

Let $A_i$ and $B_i$ be simple $A\mathcal{T}$-algebras with $A_i$ unital and $B_i$ stable. Suppose that $e_i : 0 \to B_i \to E_i \to A_i \to 0$ are unital quasidiagonal extensions. Then the following are equivalent:

1. $E_1 \cong E_2$.
2. There is an extension isomorphism $(\beta, \eta, \alpha) : e_1 \to e_2$.
3. There are isomorphisms $\beta^\#: \Ell(B_1) \to \Ell(B_2)$, $\eta_* : (K_*(E_1), [1]_0) \to (K_*(E_2), [1]_0)$ and $\alpha^\#: \Ell(A_1) \to \Ell(A_2)$ such that $(\beta^*, \eta^*, \alpha^*) : (K(e_1), [1]_0) \to (K(e_2), [1]_0)$ is an isomorphism.
Theorem

Let $A_i$ and $B_i$ be simple $A\mathbb{T}$-algebras with $A$ unital and $B$ stable. Suppose that $e_i : 0 \to B_i \to E_i \to A_i \to 0$ are unital essential extensions. Then the following are equivalent:

1. $E_1 \otimes \mathcal{K}$ is isomorphic to $E_2 \otimes \mathcal{K}$.
2. There is an extension isomorphism $(\beta, \eta, \alpha) : S e_1 \to S e_2$.
3. The six term exact sequences associated to $e_1$ and $e_2$ are isomorphic, i.e. there are isomorphisms $\beta^\#: \text{Ell}(B_1 \otimes \mathcal{K}) \to \text{Ell}(B_2 \otimes \mathcal{K})$, $\eta^* : K_*(E_1 \otimes \mathcal{K}) \to K_*(E_2 \otimes \mathcal{K})$ and $\alpha^\#: \text{Ell}(A_1 \otimes \mathcal{K}) \to \text{Ell}(A_2 \otimes \mathcal{K})$ such that $(\beta^*, \eta^*, \alpha^*) : K(S e_1) \to K(S e_2)$ is an isomorphism.
Question

Suppose that \( e : 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0 \) is essential extension of \( \mathcal{A}_\mathbb{T} \)-algebras.

- Without the condition "absorption or fullness", \( e \) is trivial \( \implies e \) is QD or \( E \) is an \( \mathcal{A}_\mathbb{T} \)-algebra?

- Without the condition "absorption or fullness", \( e \) is QD \( \implies E \) is an \( \mathcal{A}_\mathbb{T} \)-algebra?

- Is \((K(e), [1]_0)\) a complete invariant of unital extensions \( \mathcal{A}_\mathbb{T} \)-algebras?
Question

Suppose that $e : 0 \to B \to E \to A \to 0$ is essential extension of $A\mathcal{T}$-algebras.

- Without the condition "absorption or fullness", $e$ is trivial $\implies e$ is QD or $E$ is an $A\mathcal{T}$-algebra?

- Without the condition "absorption or fullness", $e$ is QD $\implies E$ is an $A\mathcal{T}$-algebra?

- Is $(K(e), [1]_0)$ a complete invariant of unital extensions $A\mathcal{T}$-algebras?
Question

Suppose that \( e : 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0 \) is essential extension of \( AT\)-algebras.

- Without the condition "absorption or fullness", \( e \) is trivial \( \implies e \) is QD or \( E \) is an \( AT\)-algebra?

- Without the condition "absorption or fullness", \( e \) is QD \( \implies E \) is an \( AT\)-algebra?

- Is \( (K(e), [1]_0) \) a complete invariant of unital extensions \( AT\)-algebras?
Thanks