Classification of extensions of $A\mathbb{T}$-algebras

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Extension

Let A and B be C^*-algebras. Recall that an extension of A by B is a short exact sequence

$$0 \to B \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0.$$

Denote this extension by e or (E, α, β) and the set of all such extensions by $\mathcal{E}\mathrm{xt}(A, B)$.

The extension (E, α, β) is called trivial, if the above sequence splits, i.e. if there is a homomorphism $\gamma : A \to E$ such that $\beta \circ \gamma = id_A$.

We call (E, α, β) essential, if $\alpha(B)$ is an essential ideal in E. We denote the set of all essential extensions by $\mathcal{E}\mathrm{xt}^e(A, B)$.

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The Busby invariant

The Busby invariant of (E, α, β) is a homomorphism τ from A into the corona algebra $\mathcal{Q}(B) = M(B)/B$ defined by $\tau(a) = \pi(\sigma(b))$ for $a \in A$, where $\pi : M(B) \to \mathcal{Q}(B)$ is the quotient map, and $b \in E$ such that $\beta(b) = a$.

Hence, we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \tau & & \\ 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q}(B) & \longrightarrow & 0. \end{array}$$

If A is unital and the Busby invariant is unital, then (E, α, β) is called unital.

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Equivalence

Let $e_i : 0 \to B \to E_i \to A \to 0$ be two extensions with Busby invariants τ_i for $i = 1, 2$.

Definition 1

e_1 and e_2 are called congruent, denoted by $e_1 \equiv e_2$, if there exists an isomorphism η making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow \eta & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

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Definition 2

e_1 and e_2 are called (strongly) unitarily equivalent, denoted by $e_1 \overset{s}{\sim} e_2$, if there exists a unitary $u \in M(B)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for all $a \in A$. Denote by $\mathrm{Ext}(A, B)$ or $\mathrm{Ext}_s(A, B)$ the set of (strongly) unitary equivalence classes of extensions of A by B.

Definition 3

Weakly unitarily equivalent, denoted by $e_1 \overset{w}{\sim} e_2$, if there exists a unitary $u \in \mathcal{Q}(B)$ such that $\tau_2(a) = u\tau_1(a)u^*$ for all $a \in A$. Denote by $\mathrm{Ext}_w(A, B)$ the set of weakly unitary equivalence classes of extensions of A by B.

Definition 4

e_1 and e_2 are called isomorphic, denoted by $e_1 \cong e_2$, if there exist isomorphisms β, η, α making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Denote the morphism of extensions by $(\beta, \eta, \alpha) : e_1 \to e_2$. Denote by $\mathrm{Ext}_I(A, B)$ the set of equivalence classes of extensions up to isomorphism.

Sum of extensions

Suppose that B is a stable C^*-algebra. Then the sum of two extensions τ_1 and τ_2 is defined to be the homomorphism $\tau_1 \oplus \tau_2$, where $\tau_1 \oplus \tau_2 : A \to \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq M_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B)$.

- $\mathrm{Ext}_s(A, B)$ and $\mathrm{Ext}_w(A, B)$ are semigroups
- Trivial extensions construct subsemigroups of $\mathrm{Ext}_s(A, B)$ and $\mathrm{Ext}_w(A, B)$, respectively

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Ext-group

The stable Ext-group $\mathcal{E}\mathrm{xt}(A, B)$ is the quotient of $\mathrm{Ext}_s(A, B)$ by the subsemigroup of trivial extensions. The equivalence class of an extension τ in $\mathcal{E}\mathrm{xt}(A, B)$ is denoted by $[\tau]$.

If $[\tau_1] = [\tau_2]$ in $\mathcal{E}\mathrm{xt}(A, B)$, then τ_1 and τ_2 are called stably unitarily equivalent, denoted by $\tau_1 \overset{ss}{\sim} \tau_2$.

- $[\tau_1] = [\tau_2]$ iff there are trivial extensions σ_i such that $\tau_1 \oplus \sigma_1 \overset{s}{\sim} \tau_2 \oplus \sigma_2$
- If A is a separable nuclear C^*-algebra and B is a σ-unital C^*-algebra, then $\mathcal{E}\mathrm{xt}(A, B)$ is an abelian group
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Relations of equivalences

- $\equiv \Longrightarrow \overset{s}{\sim} \Longrightarrow \overset{w}{\sim} \Longrightarrow \overset{ss}{\sim}$. Conversely, they do not hold.
- $\overset{s}{\sim} \Longrightarrow \cong \not\Rightarrow \overset{w}{\sim}$
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Note: In general, $\mathrm{Ext}_I(A, B)$ is not a semigroup since the isomorphism equivalence can not preserve the addition.

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Invariant

Suppose that A is a unital C^*-algebra. Denote by $T(A)$ the tracial state space of A and denote by $Aff(T(A))$ the space of all real affine continuous functions on $T(A)$.

Define $\rho_A : K_0(A) \to Aff(T(A))$ to be the positive homomorphism defined by $\rho_A([p])(\tau) = \tau(p)$ for each projection p in $M_k(A)$.

Let A be a unital simple separable C^*-algebra. Recall that the Elliott invariant of A is the 6-tuple: $(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A)$. We denote it by $Ell(A)$.

When A is non-unital, let $\mathcal{T}(A)$ be the set of lower-semicontinuous densely defined traces on A equipped with the weakest topology such that the functional $\tau \to \tau(a)$ is continuous for any $a \in A^+$ dominated by a projection. Let $Inv(A) = (K_0(A), K_0(A)^+, \Sigma(A), K_1(A), \mathcal{T}(A), r_A)$, where $\Sigma(A) = \{[p] : p \in P(A)\}$ is the scale and $P(A)$ is the set of projections in A.

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Let A and B be two unital simple separable amenable C^*-algebras with stable rank one. We write $Ell(A) \cong Ell(B)$ if

$$(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A) \cong (K_0(B), K_0(B)^+, [1_B], K_1(B), T(B)$$

that is, if there are an isomorphism $\alpha_1 : K_1(A) \to K_1(B)$, an order isomorphism $\alpha_0 : K_0(A) \to K_0(B)$ such that $\alpha_0([1_A]) = [1_B]$ and an affine homeomorphism $\gamma : T(B) \to T(A)$ such that

$$\begin{array}{ccc} T(B) & \xrightarrow{\gamma} & T(A) \\ \downarrow r_A & & \downarrow r_B \\ S(K_0(B)) & \xrightarrow{{\alpha_0}^*} & S(K_0(A)) \end{array}$$

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Let $e: 0 \to B \to E \to A \to 0$ be an extension of A by B. Denote by $K(e)$ the six term exact sequence of e in K-theory:

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B) \end{array}$$

Denote by $\mathcal{H}ext(A, B)$ all such $K(e)$ of extensions of A by B.

Let $e : 0 \to B \to E \to A \to 0$ be an extension of A by B. Denote by $K(e)$ the six term exact sequence of e in K-theory:

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ {\scriptstyle \delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B) \end{array}$$

Denote by $\mathcal{H}ext(A, B)$ all such $K(e)$ of extensions of A by B.

Let $e_i \in \mathcal{E}\mathrm{xt}(A_i, B_i)(i = 1, 2)$. We call $(\alpha_*, \beta_*, \lambda_*) : K(e_1) \to K(e_2)$ a morphism if there are homomorphisms $\alpha_* : K_*(A_1) \to K_*(A_2)$, $\beta_* : K_*(B_1) \to K_*(B_2)$, and $\lambda_* : K_*(E_1) \to K_*(E_2)$ making the obvious diagram commutative.

If α_*, β_* and λ_* are isomorphisms, then $K(e_1)$ and $K(e_2)$ are called isomorphic, written $K(e_1) \cong K(e_2)$. If $A_1 = A_2 = A$, $B_1 = B_2 = B$ and there is an isomorphism $(id_{K_*(A)}, id_{K_*(B)}, \lambda_*) : K(e_1) \to K(e_2)$, then they are called congruent, written $K(e_1) \equiv K(e_2)$.

Let $Hext(A, B)$ denote the set of congruent classes of six term exact sequences in $\mathcal{H}ext(A, B)$.

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Denote by $KK(A,B)^{++}$ those elements $x \in KK(A,B)$ such that

$$K_0(x)(K_0(A)_+ \backslash \{0\}) \subset K_0(B)_+ \backslash \{0\}.$$

Suppose that both A and B are unital. Denote by $KK_e(A,B)^{++}$ the set of those elements x in $KK(A,B)^{++}$ such that $K_0(x)([1_A]) = [1_B]$.

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Classification — —nonunital case

Some Results

Consider the extension $0 \to B \to E \to A \to 0$

- Rordam, 1997: When A, B are stable Kirchberg algebras, then $K(e)$ is a complete invariant for extensions up to stable isomorphism
- Eilers-Restorff-Ruiz, 2009: Suppose that A, B are in a certain class of C^*-algebras which are classified by $K_*^+(A) = (K_0(A), K_0(A)_+, K_1(A))$ and B has CFP, then $K_*^+(A) + K(e)$ is a complete invariant for full extensions being stably isomorphic.

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Definition

Let B be separable stable C^*-algebra. Then B is said to have the Corona Factorization Property (CFP) if every full projection in $M(B)$ is M-v equivalent to $1_{M(B)}$.

If B has CFP, then

- Note: every nonunital full extension is absorbing, and every unital full extension is unital-absorbing.
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Lemma (Ortega-Perera-Rordam)

Let B be a separable, unital C^*-algebra with finite decomposition rank. Then $B \otimes \mathcal{K}$ has the corona factorization property.

Corollary

Let B be a unital $A\mathbb{T}$-algebra, then $B \otimes \mathcal{K}$ has CFP.

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Corollary

Let B be a unital $A\mathbb{T}$-algebra, then $B \otimes \mathcal{K}$ has CFP.

Theorem

Let A be a simple $A\mathbb{T}$-algebra with unit. Suppose that $a \in KK_e(A, A)^{++}$ and $\gamma : T(A) \to T(A)$ is an affine homeomorphism such that

$$K_*(a) : (K_0(A), K_0(A)^+, [1_A], K_1(A)) \to (K_0(A), K_0(A)^+, [1_A], K_1(A))$$

is an isomorphism and γ is compatible with $K_0(a)$.
It follows that there is an automorphism $\phi : A \to A$ such that $KK(\phi) = a$ in $KK(A, A)$ and $\phi_T = \gamma$.

Lemma (Rordam)

Let A and B be separable nuclear C^*-algebras in $\mathcal{N}$ with B stable, and let $x_1, \ x_2 \in \mathrm{Ext}(A, B)$. Then $K(x_1) = K(x_2)$ in $\mathrm{Hext}(A, B)$ if and only if there exist elements a in $KK(A, A)$ and b in $KK(B, B)$ with $K_*(a) = K_*(id_A)$ and $K_*(b) = K_*(id_B)$ such that $x_1 b = a x_2$.

Lemma

Let A and B be simple $A\mathbb{T}$-algebras with A unital and B stable. Assume that $a \in KK(A, A)$, $b \in KK(B, B)$ such that $K_*(a) = id_{K_*(A)}$ and $K_*(b) = id_{K_*(B)}$. Then there are isomorphisms $\alpha : A \to A$, $\beta : B \to B$ such that $KK(\alpha) = a$ and $KK(\beta) = b$.

Theorem

Let A_i and B_i be simple $A\mathbb{T}$-algebras with A unital and B stable. Suppose that $e_i : 0 \to B_i \to E_i \to A_i \to 0$ are non-unital full extensions. Then the following are equivalent:

(1) E_1 is isomorphic to E_2.

(2) There is an extension isomorphism $(\beta, \eta, \alpha) : e_1 \to e_2$, i.e. $e_1 \cong e_2$.

(3) The six term exact sequences associated to e_1 and e_2 are isomorphic, i.e. there are isomorphisms $\beta_\sharp : Inv(B_1) \to Inv(B_2)$, $\eta_* : K_*(E_1) \to K_*(E_2)$ and $\alpha_\sharp : Ell(A_1) \to Ell(A_2)$ such that $(\beta_*, \eta_*, \alpha_*) : K(e_1) \to K(e_2)$ is an isomorphism.

Theorem

Suppose that A_i are simple $A\mathbb{T}$-algebras with units, and B_i are stabilizations of unital AF-algebras. Let $e_i : 0 \to B_i \to E_i \to A_i \to 0$ be non-unital full extensions. Then the following are equivalent:
(1) $E_1 \cong E_2$.
(2) $e_1 \cong e_2$.
(3) The six term exact sequences associated to e_1 and e_2 are isomorphic, i.e. there is an isomorphism $(\beta_*, \eta_*, \alpha_*) : K(e_1) \to K(e_2)$ for some isomorphisms $\beta_* : (K_0(B_1), K_0(B_1)^+) \to (K_0(B_2), K_0(B_2)^+)$ and $\alpha_\sharp : Ell(A_1) \to Ell(A_2)$.

When is an extension an $A\mathbb{T}$-algebra?

Given an extension $0 \to B \to E \to A \to 0$

Question: Let A, B be in a class $\mathcal{A}$ of C^*-algebras. Which condition will make E be in $\mathcal{A}$?

Brown-Effros-Elliott, 1980s

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$$\mathcal{A} = \{\text{AF-algebras}\} \Longrightarrow E \in \mathcal{A}.$$

Lin-Rordam, 1992

Let A and B be $A\mathbb{T}$-algebras with real rank zero and let e be an extension $0 \to B \to E \to A \to 0$. Then the following three conditions are equivalent:
(1) E is an $A\mathbb{T}$-algebra of real rank zero.
(2) E has real rank zero and stable rank one.
(3) The index maps $\delta_i : K_i(A) \to K_{1-i}(B)$, $i = 0, 1$ are both trivial.

Dadarlat-Loring, 1993

Assume that A,B are AD-algebras with real rank zero, $K_1(B) = 0$ or $K_1(A)$ torsion free. TFAE:

(1) E is an AD-algebra of real rank zero.

(2) $RR(E) = 0$, $st(E) = 1$.

(3) $\delta_i = 0$

Theorem

Suppose that A is an $A\mathbb{T}$-algebra and B is the stabilization of a unital $A\mathbb{T}$-algebra. Let $e: 0 \to B \to E \to A \to 0$ be a non-unital full extension of A by B. Then the following are equivalent.
(1) E is an $A\mathbb{T}$-algebra.
(2) The index maps of e are zero.
(3) The extension e is quasidiagonal.

Proof:

(2) $\Longleftrightarrow$ (3) and (1) $\Longrightarrow$ (3) are immediate.

We only need to show that (3) $\Longrightarrow$ (1).

Lemma 1

Suppose that A and B are $A\mathbb{T}$-algebras with B stable. Then there is an absorbing trivial extension which is also quasidiagonal.

Lemma 2

Suppose that A and B are $A\mathbb{T}$-algebras with B stable. Let $e: 0 \to B \to E \xrightarrow{\psi} A \to 0$ be an essential trivial extension of A by B. If e is quasidiagonal, then E is an $A\mathbb{T}$-algebra.

Proof:

(2) $\iff$ (3) and (1) $\implies$ (3) are immediate.

We only need to show that (3) $\implies$ (1).

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Suppose that e is a quasidiagonal extension. Let $A = \lim_{n\to\infty}(A_n, \iota_n)$, where A_n is isomorphic to a quotient of a circle algebra and ι_n are the inclusion maps. Set $\tau_n = \tau \circ \iota_n$ and $E_n = \pi^{-1}(\tau_n(A_n))$, where τ is the Busby invariant associated to e. Then we have an essential extension e_n of A_n by B

$$0 \to B \to E_n \to A_n \to 0$$

for every $n \in \mathbb{N}$. Hence, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_n & \longrightarrow & A_n & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Since $A = \lim_{n\to\infty}(A_n, \iota_n)$, it follows that $\tau(A) = \overline{\cup_{n=1}^{\infty}\tau_n(A_n)}$.

Therefore, it follows that

$$E = \overline{\bigcup_{n=1}^{\infty} E_n} = \lim_{n\to\infty} E_n.$$

For each A_n, there is an increasing sequence $\{A_{n,k}\}$ of C^*-subalgebras of A_n such that each $A_{n,k}$ is isomorphic to a finite direct sum of C^*-algebras of the form $M_m(C(X))$ and $\cup_{k=1}^{\infty} A_{n,k}$ is dense in A_n, where X is a connected compact subset of the unit circle. Set $\tau_{n,k} = \tau \circ \iota_{n,k}$ and $E_{n,k} = \pi^{-1}(\tau_{n,k}(A_{n,k}))$, where $\iota_{n,k} : A_{n,k} \to A$ is the inclusion map. Let $e_{n,k}$ be the essential extension of $A_{n,k}$ by B:

$$0 \to B \to E_{n,k} \to A_{n,k} \to 0.$$

Obviously, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_{n,k} & \longrightarrow & A_{n,k} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & E_n & \longrightarrow & A_n & \longrightarrow & 0. \end{array}$$

As the above proof, we have

$$E_n = \overline{\bigcup_{k=1}^{\infty} E_{n,k}} = \lim_{k\to\infty} E_{n,k}.$$

Since e is non-unital and full, then $e_{n,k}$ is a non-unital full extension. Hence $e_{n,k}$ is absorbing. By the above proof, the index maps $\delta_i : K_i(A) \to K_{1-i}(B)$ of e are trivial. Since $\tau_{n,k} = \tau \circ \iota_{n,k}$, then the index maps of $e_{n,k}$ are also trivial. From Lemma 1, it follows that A is quasidiagonal relative to B, so the subalgebra $A_{n,k}$ is also quasidiagonal relative to B. Note that $K_*(A_{n,k})$ is free. Hence, $e_{n,k}$ is a trivial and quasidiagonal extension. It follows from Lemma 2 that $E_{n,k}$ is an $A\mathbb{T}$-algebra. Therefore, E_n is an $A\mathbb{T}$-algebra. Consequently, E is an $A\mathbb{T}$-algebra.

Classification——unital case

Lemma

Suppose that A and B are C^*-algebras with A unital and B stable. Let $e_i : 0 \to B \xrightarrow{I_i} E_i \to A \to 0$ be essential unital extensions. Suppose $\tau_2 = \mathrm{Ad}\, u \circ \tau_1$ for some unitary u in $\mathcal{Q}(B)$. Let v be a partial isometry in $M(B)$ such that $\pi(v) = u$, and let $p = v^* v$ and $q = vv^*$. Then

$$(K(e_1), [1]_0) \equiv (K(e_2), [q]_0 + [1-p]_0).$$

Let $0 \to B \to E \to A \to 0$ be an extension with index maps δ_0 and δ_1 in its K-theory. We set $G' = \{f([1]_0) | f \in \mathrm{Hom}(\mathrm{Ker}\delta_0, \mathrm{Coker}\delta_1)\}$ and let $\pi : K_0(B) \to \mathrm{Coker}\delta_1$ be the quotient map.

Lemma

Let e_i be essential unital extensions with Busby invariant τ_i. If e_1 is weakly unitarily equivalent to e_2 by a unitary $u \in \mathcal{Q}(B)$. Then

$$(K(e_1), [1]_0) \equiv (K(e_2), [1]_0)$$

if and only if $\pi([u]_1)$ is in G'.

Lemma

Suppose e_i are essential unital extensions with Busby invariant τ_i and e_1 is weakly unitarily equivalent to e_2. If the index maps of e_i are trivial and

$$(K(e_1), [1]_0) \equiv (K(e_2), [1]_0),$$

then $[e_1] = [e_2]$ in $\mathrm{Ext}_s^u(A, B)$.

Theorem

Let A_i and B_i be simple $A\mathbb{T}$-algebras with A_i unital and B_i stable. Suppose that $e_i : 0 \to B_i \to E_i \to A_i \to 0$ are unital quasidiagonal extensions. Then the following are equivalent:

(1) $E_1 \cong E_2$.

(2) There is an extension isomorphism $(\beta, \eta, \alpha) : e_1 \to e_2$.

(3) There are isomorphisms $\beta_\sharp : Ell(B_1) \to Ell(B_2)$, $\eta_* : (K_*(E_1), [1]_0) \to (K_*(E_2), [1]_0)$ and $\alpha_\sharp : Ell(A_1) \to Ell(A_2)$ such that $(\beta_*, \eta_*, \alpha_*) : (K(e_1), [1]_0) \to (K(e_2), [1]_0)$ is an isomorphism.

Theorem

Let A_i and B_i be simple $A\mathbb{T}$-algebras with A unital and B stable. Suppose that $e_i : 0 \to B_i \to E_i \to A_i \to 0$ are unital essential extensions. Then the following are equivalent:

(1) $E_1 \otimes \mathcal{K}$ is isomorphic to $E_2 \otimes \mathcal{K}$.

(2) There is an extension isomorphism $(\beta, \eta, \alpha) : Se_1 \to Se_2$.

(3) The six term exact sequences associated to e_1 and e_2 are isomorphic, i.e. there are isomorphisms $\beta_\sharp : Ell(B_1 \otimes \mathcal{K}) \to Ell(B_2 \otimes \mathcal{K})$, $\eta_* : K_*(E_1 \otimes \mathcal{K}) \to K_*(E_2 \otimes \mathcal{K})$ and $\alpha_\sharp : Ell(A_1 \otimes \mathcal{K}) \to Ell(A_2 \otimes \mathcal{K})$ such that $(\beta_*, \eta_*, \alpha_*) : K(Se_1) \to K(Se_2)$ is an isomorphism.

Question

Suppose that $e : 0 \to B \to E \to A \to 0$ is essential extension of $A\mathbb{T}$-algebras.

- Without the condition "absorption or fullness", e is trivial $\Longrightarrow$ e is QD or E is an $A\mathbb{T}$-algebra?
- Without the condition "absorption or fullness", e is QD $\Longrightarrow$ E is an $A\mathbb{T}$-algebra?
- Is $(K(e), [1]_0)$ a complete invariant of unital extensions $A\mathbb{T}$-algebras?

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Thanks