Operator Space Approximation Properties for Group C*-algebras

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Operator Spaces-
Natural Quantization of Banach Spaces

Banach Spaces

A Banach space is a complete normed space $(V/\mathbb{C}, \|\cdot\|)$.

In Banach spaces, we consider

Norms and Bounded Linear Maps.

Classical Examples:

$$C_0(\Omega),\ \ M(\Omega) = C_0(\Omega)^*,\ \ \ell_p(I),\ \ L_p(X,\mu),\quad 1 \le p \le \infty.$$

Hahn-Banach Theorem: Let $V \subseteq W$ be Banach spaces. We have

$$\begin{array}{ccc} W & & \\ \uparrow & \searrow \tilde{\varphi} & \\ V & \xrightarrow{\varphi} & \mathbb{C} \end{array}$$

with $\|\tilde{\varphi}\| = \|\varphi\|$.

It follows from the Hahn-Banach theorem that for every Banach space $(V, \|\cdot\|)$ we can obtain an isometric inclusion

$$(V, \|\cdot\|) \hookrightarrow (\ell_\infty(I), \|\cdot\|_\infty)$$

where we may choose $I = V_1^*$ to be the closed unit ball of V^*.

So we can regard $\ell_\infty(I)$ as the home space of Banach spaces.

Classical Theory	**Noncommutative Theory**
$\ell_\infty(I)$	$B(H)$
Banach Spaces $(V, \|\cdot\|) \hookrightarrow \ell_\infty(I)$	Operator Spaces $(V, ??) \hookrightarrow B(H)$
	norm closed subspaces of $B(H)$?

Matrix Norm and Concrete Operator Spaces [Arveson 1969]

Let $B(H)$ denote the space of all bounded linear operators on H. For each $n \in \mathbb{N}$,

$$H^n = H \oplus \cdots \oplus H = \{[\xi_j] : \xi_j \in H\}$$

is again a Hilbert space. We may identify

$$M_n(B(H)) \cong B(H \oplus \ldots \oplus H)$$

by letting

$$\left[T_{ij}\right]\left[\xi_j\right] = \left[\sum_j T_{i,j}\xi_j\right],$$

and thus obtain an operator norm $\|\cdot\|_n$ on $M_n(B(H))$.

A concrete operator space is norm closed subspace V of $B(H)$ together with the canonical operator matrix norm $\|\cdot\|_n$ on each matrix space $M_n(V)$.

Examples of Operator Spaces

- C*-algebras A, i.e. norm closed *-subalgebras of some $B(H)$

- $A = C_0(\Omega)$ or $A = C_b(\Omega)$ for locally compact space

- Reduced group C*-algebras $C^*_\lambda(G)$, full group C*-algebras $C^*(G)$

- von Neumann algebras M, i.e. strong operator topology (resp. w,o.t, weak* topology) closed *-subalgebras of $B(H)$

- $L_\infty(X, \mu)$ for some measure space (X, μ)

- Group von Neumann algebras $L(G)$

Group C*-algebras and Group von Neumann Algebras

Let G be a discrete group. For each $s \in G$, there exists a unitary operator λ_s on $\ell_2(G)$ given by

$$\lambda_s \xi(t) = \xi(s^{-1}t)$$

We let

$$C^*_\lambda(G) = \overline{\lambda(\mathbb{C}[G])}^{\|\cdot\|} = \overline{\text{span}\,\{\lambda_s\}}^{\|\cdot\|}$$

denote the reduced group C*-algebra of G.

We let

$$L(G) = \overline{\lambda(\mathbb{C}[G])}^{s.o.t} \subseteq B(L_2(G))$$

be the left group von Neumann algebra of G.

If G is an abelian group, then we have

$$\lambda_s \circ \lambda_t = \lambda_{st} = \lambda_{ts} = \lambda_t \circ \lambda_s.$$

Then $C^*_\lambda(G)$ is a commutative C*-algebra and $L(G)$ is a commutative von Neuman algebra. In fact, we have

$$C^*_\lambda(G) = C_0(\widehat{G}) \text{ and } L(G) = L_\infty(\widehat{G}),$$

where $\widehat{G} = \{\chi : G \to \mathbb{T} : \text{continuous homo}\}$ is the dual group of G.

Example: Let $G = \mathbb{Z}$. Then $\widehat{\mathbb{Z}} = \mathbb{T}$ and we have

$$C^*_\lambda(\mathbb{Z}) = C(\mathbb{T}) \text{ and } L(\mathbb{Z}) = L_\infty(\mathbb{T}).$$

Therefore, for a general group G, we can regard $C^*_\lambda(G)$ and $L(G)$ as the dual object of $C_0(G)$ and $L_\infty(G)$, respectively.

Full Group C*-algebra

Let $\pi_u : G \to B(H_u)$ be the universal representation of G. We let

$$C^*(G) = \overline{\pi_u(L_1)}^{\|\cdot\|}$$

denote the full group C*-algebra of G.

It is known that we have a canonical C*-algebra quotient

$$\pi_\lambda : C^*(G) \to C^*_\lambda(G).$$

Completely Bounded Maps

Let $\varphi : V \to W$ be a bounded linear map. For each $n \in \mathbb{N}$, we can define a linear map

$$\varphi_n : M_n(V) \to M_n(W)$$

by letting

$$\varphi_n([v_{ij}]) = [\varphi(v_{ij})].$$

The map φ is called completely bounded if

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : \ n \in \mathbb{N}\} < \infty.$$

We let $CB(V, W)$ denote the space of all completely bounded maps from V into W.

In general $\|\varphi\|_{cb} \neq \|\varphi\|$. Let t be the transpose map on $M_n(\mathbb{C})$. Then

$$\|t\|_{cb} = n, \ \text{but} \ \|t\| = 1.$$

Theorem: If $\varphi : V \to W = C_b(\Omega)$ is a bounded linear map, then φ is completely bounded with

$$\|\varphi\|_{cb} = \|\varphi\|.$$

Proof: Given any contractive $[v_{ij}] \in M_n(V)$, $[\varphi(v_{ij})]$ is an element in

$$M_n(C_b(\Omega)) = C_b(\Omega, M_n) = \{[f_{ij}] : x \in \Omega \to [f_{ij}(x)] \in M_n\}.$$

Then we have

$$\begin{aligned}
\|[\varphi(v_{ij})]\|_{C_b(\Omega,M_n)} &= \sup\{\|[\varphi(v_{ij})(x)]\|_{M_n} : x \in \Omega\} \\
&= \sup\{|\sum_{i,j=1}^{n} \alpha_i \varphi(v_{ij})(x)\beta_j| : x \in \Omega, \|\alpha\|_2 = \|\beta\|_2 = 1\} \\
&= \sup\{|\varphi(\sum_{i,j=1}^{n} \alpha_i v_{ij}\beta_j)(x)| : x \in \Omega, \|\alpha\|_2 = \|\beta\|_2 = 1\} \\
&\leq \|\varphi\| \sup\{\|[\alpha_i][v_{ij}][\beta_j]\| : \|\alpha\|_2 = \|\beta\|_2 = 1\} \\
&\leq \|\varphi\|\|[v_{ij}]\| \leq \|\varphi\|.
\end{aligned}$$

This shows that $\|\varphi_n\| \leq \|\varphi\|$ for all $n = 1, 2, \cdots$. Therefore, we have

$$\|\varphi\| = \|\varphi_2\| = \cdots = \|\varphi_n\| = \cdots = \|\varphi\|_{cb}.$$

Arveson-Wittstock-Hahn-Banach Theorem

Let $V \subseteq W \subseteq B(H)$ be operator spaces.

$$
\begin{array}{ccc}
W & & \\
\uparrow & \searrow \tilde{\varphi} & \\
V & \xrightarrow{\varphi} & B(H)
\end{array}
$$

with $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$.

In particular, if $B(H) = \mathbb{C}$, we have $\|\varphi\|_{cb} = \|\varphi\|$. This, indeed, is a generalization of the classical Hahn-Banach theorem.

Column and Row Hilbert Spaces

Let $H = C^m$ be an m-dimensional Hilbert space .

H_c: There is a natural column operator space structure on H given by

$$H_c = M_{m,1}(C) \subseteq M_m(C).$$

H_r: Similarly, there is a row operator space structure given by

$$H_r = M_{1,m}(C) \subseteq M_m(C).$$

Moreover, Pisier introduced an OH structure on H by considering the complex interperlation over the matrix spaces

$$M_n(OH) = (M_n(H_c), M_n(H_r))_{\frac{1}{2}} = (M_n(MAX(H)), M_n(MIN(H)))_{\frac{1}{2}}.$$

All these matrix norm structures are distinct from $MIN(H)$ and $MAX(H)$.

Dual Operator Spaces

Let V be an operator space. Then the dual space

$$V^* = B(V, \mathbb{C}) = CB(V, \mathbb{C})$$

has a natural operator space matrix norm given by

$$M_n(V^*) = CB(V, M_n(\mathbb{C})).$$

We call V^* the operator dual of V.

More Examples

- $T(\ell_2(\mathbb{N})) = K(\ell_2(\mathbb{N}))^* = B(\ell_2(\mathbb{N}))_*$;
- $M(\Omega) = C_0(\Omega)^*$, operator dual of C*-algebras A^*;
- $L_1(X,\mu) = L_\infty(X,\mu)_*$, operator predual of von Neumann algebras R_*;
- Fourier algebra $A(G) = L(G)_*$
- Fourier-Stieltjes algebra $B(G) = C^*(G)^*$

Operator Space Structure on L_p spaces

- L_p-spaces $L_p(X, \mu)$

$$M_n(L_p(X,\mu)) = (M_n(L_\infty(X,\mu)), M_n(L_1(X,\mu)))_{\frac{1}{p}}.$$

- Non-commutative L_p-spaces $L_p(R, \varphi)$,

$$M_n(L_p(R,\varphi)) = (M_n(R), M_n(R_*^{op}))_{\frac{1}{p}},$$

where R_*^{op} is the operator predual of the opposite von Neumann algebra R^{op}.

Related Books

- E.G.Effros and Z-J.Ruan, *Operator spaces*, London Math. Soc. Monographs, New Series 23, Oxford University Press, New York, 2000

- Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002.

- Pisier, *An introduction to the theory of operator spaces*, London Mathematical Society Lecture Note Series 294, Cambridge University Press, Cambridge, 2003.

- Blecher and Le Merdy, *Operator algebras and their modules-an operator space approach*, London Mathematical Society Monographs. New Series, 30. Oxford University Press, New York 2004.

Grothendick's Approximation Property

Grothendick's Approximation Property

A Banach space is said to have Grothendicks' AP if there exists a net of bounded finite rank maps $T_\alpha : V \to V$ such that $T_\alpha \to id_V$ uniformly on compact subsets of V.

We note that a subset $K \subseteq V$ is compact if and only if there exists a sequence $(x_n) \in c_0(V)$ such that

$$K \subseteq \overline{conv\{x_n\}}^{\|\cdot\|} \subseteq V.$$

Therefore, V has Grothendick's AP if and only if there exists a net of finite rank bounded maps T_α on V such that

$$\|(T_\alpha(x_n)) - (x_n)\|_{c_0(V)} \to 0$$

for all $(x_n) \in c_0(V)$.

Operator Space Approximation Property

An operator space V is said to have the operator space approximation property (or simply, OAP) if there exists a net of finite rank bounded maps T_α on V such that

$$\|[T_\alpha(x_{ij})] - [x_{ij}]\|_{K_\infty(V)} \to 0$$

for all $[x_{ij}] \in K_\infty(V)$, where we let $K_\infty(V) = \overline{\cup_{n=1}^{\infty} M_n(V)}$.

In this case, we say that $T_\alpha \to id_V$ in the stable point-norm topology.

We say that $V \subseteq B(H)$ has the strong OAP if we can replace $K_\infty(V)$ by $B(\ell_2)\check{\otimes}V$, which is the norm closure of $B(\ell_2) \otimes V$ in $B(\ell_2 \otimes \ell_2(G))$.

For any discrete group C*-algebra $A = C^*_\lambda(G)$,

Nuclearity $\Rightarrow$ CBAP $\Rightarrow$ strong OAP $=$ OAP $\Rightarrow$ Exactness.

Nuclearity

An operator space (or a C*-algebra) V is said to be nuclear if there exists two nets of completely contractive maps

$$S_\alpha : V \to M_{n(\alpha)} \text{ and } T_\alpha : M_{n(\alpha)} \to V$$

such that

$$\|T_\alpha \circ S_\alpha(x) - x\| \to 0$$

for all $x \in V$.

CCAP and CBAP

An operator space V is to have the CBAP (resp. CCAP) if there exists a net of comletely bounded (resp. completely contractive) finite rank maps $T_\alpha : V \to V$ such that

$$\|T_\alpha(x) - x\| \to 0$$

for all $x \in V$.

Exact Operator Spaces

An operator space (or a C*-algebra) $V \subseteq B(H)$ is said to be exact if there exists two nets of completely contractive maps

$$S_\alpha : V \to M_{n(\alpha)} \text{ and } T_\alpha : M_{n(\alpha)} \to B(H)$$

such that

$$\|T_\alpha \circ S_\alpha(x) - x\| \to 0$$

for all $x \in V$.

For any discrete group C*-algebra $A = C_\lambda^*(G)$,

Nuclearity $\Rightarrow$ CBAP $\Rightarrow$ strong OAP $=$ OAP $\Rightarrow$ Exactness.

- $C_\lambda^*(\mathbb{F}_n)$ has CCAP, but not nuclear
- $C_\lambda^*(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ has the OAP, but not CBAP

Question: It has been an open question for a while that for any C*-algebra,

whether exactness implies OAP.

Theorem [J-R]: Let G be a discrete group.

1. G has the AP, i.e. C*-algebra $C_\lambda^*(G)$ has the OAP, if and only if $A(G) = L(G)_*$ has the OAP.

2. If G has the AP, the $L_p(L(G))$ has the OAP for any $1 < p < \infty$.

3. Suppose that G has the AP and is residually finite. Then $L_p(L(G))$ has the CCAP.

We wondered that $G = SL(3, \mathbb{Z})$ should be an example such that

$C_\lambda^*(G)$ is exact, but does not have the OAP.

Theorem [Lafforgue and de la Salle]: For $1 < p < \infty$, $L_p(L(SL(3,\mathbb{Z})))$ does not have the CBAP. Therefore,

$$C_\lambda^*(SL(3,\mathbb{Z})) \text{ does not have the OAP.}$$

More precisely, they proved that $SL(3,\mathbb{R})$ does not have the AP.

Theorem [Haagerup]: $Sp(2,\mathbb{R})$ does not have the AP.

It follows that all connected simple Lie groups with finite center and real rank greater or equal to two does not have the AP.

Exactness is a Local Operator Space Property !

Local Property of Banach Spaces

It is known from the Hahn-Banach theorem that given any finite dimensional Banach space V, there exists an isometric inclusion

$$V \hookrightarrow \ell_\infty(\mathbb{N}).$$

Question: If V is finite dimensinal, can we

"approximately embed" V into a finite dimensional $\ell_\infty(n)$

for some positive integer $n \in \mathbb{N}$?

Finite Representability in $\{\ell_\infty(n)\}$

Theorem: Let E be a f.d. Banach space. For any $\varepsilon > 0$, there exsit $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq \ell_\infty(n(\varepsilon))$ such that

$$E \overset{1+\varepsilon}{\cong} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\| \, \|T^{-1}\| < 1 + \varepsilon.$$

Therefore, we say that

- Every f.d. Banach space E is representable in $\{\ell_\infty(n)\}$;
- Every Banach space V is finitely representable in $\{\ell_\infty(n)\}$.

Proof: Since E^* is finite dim, the closed unit ball E_1^* is totally bounded. For arbitrary $1 > \varepsilon > 0$, there exists finitely many functionals $f_1, \cdots, f_n \in E_1^*$ such that for every $f \in E_1^*$, there exists some f_j such that

$$\|f - f_j\| < \frac{\varepsilon}{1+\varepsilon}.$$

Then we obtain a linear contraction

$$T : x \in E \to (f_1(x), \cdots, f_n(x_n)) \in \ell_n^\infty.$$

For any $f \in E_1^*$, we let f_j such that $\|f - f_j\| < \frac{\varepsilon}{1+\varepsilon}$. Then we get

$$\|T(x)\| \geq |f_j(x)| \geq |f(x)| - |f(x) - f_j(x)| \geq |f(x)| - \frac{\varepsilon\|x\|}{1+\varepsilon}.$$

This shows that

$$\|T(x)\| \geq \|x\| - \frac{\varepsilon\|x\|}{1+\varepsilon} = \frac{\|x\|}{1+\varepsilon}.$$

Therefore, $\|T^{-1}\| < 1 + \varepsilon$.

Finite Representatility of Operator Spaces in $\{M_n\}$

An operator space V is called finitely representable in $\{M_n\}$ if for every f.d. subspace E and $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq M_{n(\varepsilon)}$ such that

$$E \overset{1+\varepsilon}{\cong}_{cb} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.$$

It is natural to ask

whether every finite dim operator space is representable in $\{M_n\}$,

or

whether every operator space is finitely representable in $\{M_n\}$?

Theorem [Pisier 1995]: Let $\ell_1(n)$ be the operator dual of $\ell_\infty(n)$. If

$$T : \ell_1(n) \to F \subseteq M_k$$

is a linear isomorphism, then for $n \geq 3$

$$\|T\|_{cb}\|T^{-1}\|_{cb} \geq n/2\sqrt{n-1}.$$

Hence for $n \geq 3$,

$$\ell_1(n) \hookrightarrow C^*(\mathbb{F}_{n-1}) \subseteq B(H_\pi)$$

are note finitely represenetable in $\{M_n\}$.

So $C^*(\mathbb{F}_{n-1})$ and $B(H_\pi)$ are examples of non-exact C*-algebras.

Theorem [Pisier 1995]: An operator space (or C*-algebra) V is finitely representable in $\{M_n\}$ if and only if V is exact.

Finite Representability in $\{\ell_1(n)\}$

We say that a Banach space V is finitely representable in $\{\ell_1(n)\}$ if for any f.d. subspace $E \subseteq V$ and $\varepsilon > 0$, there exsit $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq \ell_1(n(\varepsilon))$ such that

$$E \overset{1+\varepsilon}{\cong} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\| \, \|T^{-1}\| < 1 + \varepsilon.$$

It is known that a Banach space V is finitely represebtable in $\{\ell_1(n)\}$ if and only if there exists an $L_1(\mu)$ space such that we have the isometric inclusion

$$V \hookrightarrow L_1(\mu).$$

Finite Representability in $\{T_n\}$

An operator space V is finitely representable in $\{T_n\}$ if for any f.d. subspace E and $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq T_{n(\varepsilon)}$ such that

$$E \overset{1+\varepsilon}{\cong}_{cb} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.$$

- If A is a nuclear C*-algebra, then A^* and A^{***} are finitely representable in $\{T_n\}$. For example

$$C(X)^*, \quad T(\ell_2), \quad , B(\ell_2)^*.$$

- $C^*_\lambda(\mathbb{F}_2)^*$ is finitely representable in $\{T_n\}$.

Question: Is the predual M_* of a von Neumann algebra is finitely representable in $\{T_n\}$?

Theorem [E-J-R 2000]: Let M be a von Neumann algebra. Then M_* is finitely representable in $\{T_n\}$ if and only if M has the QWEP, i.e. M is a quotient of a C*-algebra with Lance's weak expectation proeprty.

A C*-algebra has the WEP if for the universal representation $\pi : A \to B(H)$, there exists a completely positive and contraction $P : B(H) \to A^{**}$ such that $P \circ \pi = id_A$.

A. Connes' conjecture 1976: Every finite von Neumann algebra with separable predual is $*$-isomorphic to a von Neumann subalgebra of the ultrapower of the hyperfinite II_1 factor

$$M \hookrightarrow \prod_{\mathcal{U}} R_0.$$

E. Kirchberg 's conjecture 1993: Every C^*-algebra has QWEP.

Thank you for your attention !