The non-commutative Schwartz space

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Special Week on Operator Algebras,
Shanghai, China, 18-22 June, 2012.

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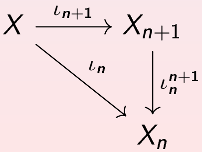

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$$\begin{array}{ccc} X & \xrightarrow{\iota_{n+1}} & X_{n+1} \\ & \searrow^{\iota_n} & \downarrow \iota_n^{n+1} \\ & & X_n \end{array} \qquad ||x||_n = ||\iota_n x||_{X_n}, \ U_n = \{x \in X\colon\ ||x||_n \leqslant 1\}.$$

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Theorem (Kōmura, Kōmura)

A Fréchet space X is nuclear if and only if $X \subset s^\mathbb{N}$.

If U is a 0-ngbd in s and B is bounded in s then
$W(U, B) := \{T \in L(s) \colon \ T(B) \subset U\}$ is a typical 0-ngbd in $L(s)$.

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Take $U_n := \{x \in s\colon ||x||_n \leqslant 1\}$, $U_n^\circ := \{\xi \in s'\colon |\xi(x)| \leqslant 1 \ \forall\, x \in U_n\}$
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Then $(W_n)_n$ is a countable basis of 0-ngbds in $L(s', s)$.

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Finally,

$$L(s', s) \subset B(\ell_2) \ \ \text{(as linear spaces)}.$$

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Duality of the pair $\langle s, s' \rangle$:

$$\langle u, \xi \rangle := \sum_{j=1}^{+\infty} u_j \overline{\xi_j} \ \ \forall\, u \in s, \xi \in s'.$$

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$$\langle x^* \xi, \eta \rangle := \langle \xi, x\eta \rangle \quad \forall\, x \in L(s', s),\ \xi, \eta \in s'.$$

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Conclusion: $\mathcal{S} := L(s', s)$ is an lmc Fréchet involutive algebra.

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Corollary

If $x \geqslant 0$ in $\mathcal{S}$ then $x^\theta \in \mathcal{S} \quad \forall \theta \in (0, 1]$.

Recall that

$$||x||_n = \sup\left\{\Big(\sum_{i=1}^{+\infty}\Big|\sum_{j=1}^{+\infty}x_{ij}\xi_j\Big|^2 i^{2n}\Big)^{\frac{1}{2}} : \sum_{j=1}^{+\infty}|\xi_j|^2 j^{-2n}\leqslant 1\right\}.$$

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By nuclearity, for arbitrary $1\leqslant p, q\leqslant +\infty$ the original topology of $\mathcal{S}$ is given by the norms:

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Fact

If $u_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ then $(u_n)_n$ is an approximate identity in $\mathcal{S}$.

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