Strongly self-absorbing property for inclusions of $C^*$-algebras with a finite Watatani index

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Motivation

In Elliott program to classify nuclear C*-algebras by K-theory data the systematic use of strongly self-absorbing C*-algebras plays a central role. In the purely infinite case the Cuntz algebra $O_{\infty}$ is a cornerstone of the Kirchberg-Phillips classification of simple purely infinite C*-algebras [17] [25]. In the stably finite case the Jiang-Su algebra $\mathbb{Z}$ plays a role similar to that of $O_{\infty}$. In fact Jiang-Su proved in [12] that simple, infinite dimensional AF algebras and Kirchberg algebras (simple, nuclear, purely infinite and satisfying the Universal Coefficient Theorem) are $\mathbb{Z}$-stable, that is, for any such an algebra $A$ one has an isomorphism $\alpha: A \to A \otimes \mathbb{Z}$. Gong, Jiang, and Su proved in [5] that $(K_0(A), K_0(A)^+)$ is isomorphic to $(K_0(A \otimes \mathbb{Z}), K_0(A \otimes \mathbb{Z})_+)$ if and only if $K_0(A)$ is weakly unperforated as an ordered group, when $A$ is a simple C*-algebra. Hence $A$ and $A \otimes \mathbb{Z}$ have isomorphic Elliott invariant if $A$ is simple with weakly unperforated $K_0$-group, that is, $A \cong A \otimes \mathbb{Z}$ whenever $A$ is classifiable. On the contrary, Rørdam and Toms in [31] and [33] presented examples which have the same Elliott invariant as, but are not isomorphic to, and not $\mathbb{Z}$-absorbing. So it appears plausible that the
Elliott conjecture, which is formulated in [30], holds for all simple, unital, nuclear, separable $\mathcal{Z}$-absorbing C*-algebras.

In this talk we reconsider the $\mathcal{D}$-absorbing property for crossed product of a C*-algebra $A$ with $\mathcal{D}$-absorbing by a finite group action with the Rokhlin property in the framework of inclusion of unital C*-algebras $P \subset A$ of Watatani index finite ([36]) and show that if a faithfule conditional expectation $E$ from $A$ to $P$ has the Rokhlin property in the sense of Kodaka-Osaka-Teruya [18], then $P$ is $\mathcal{D}$-absorbing.
**Stronly self-absorbing property**

**Definition 1.** A separable, unital C*-algebra $D$ is called *strongly self-absorbing* if it is infinite-dimensional and the map $\text{id}_D \otimes 1_D : D \to D \otimes D$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism $\varphi : D \to D \otimes D$, that is, there is a sequence $(v_n)_{n \in \mathbb{N}}$ of unitaries in $D$ satisfying

$$\| v_n^* (\text{id}_D \otimes 1_D(d)) v_n - \varphi(d) \| \to 0 \ (n \to \infty) \ \forall d \in D.$$ 

A C*-algebra $A$ is called *$D$-absorbing* if $A \otimes D \cong A$.

**Example 2.**

1. (Jiang-Su ’99) The Jiang-Su algebra $\mathcal{Z}$ is a direct limits of prime dimension drop algebras $I_{p,q} = \{ f \in C([0,1], M_{pq}) \mid f(0) \in 1_p \otimes M_q, f(1) \in M_p \otimes 1_q \}$ for relative prime integers $p, q \geq 2$. Then $\mathcal{Z}$ is strongly self-absorbing.

2. (Toms-Winter ’07) UHF algebras of infinite type (for example, an universal UHF algebra $\mathcal{U}_\infty = \prod_p M_{p\infty}$), Cuntz algebras $\mathcal{O}_2, \mathcal{O}_\infty, B \otimes \mathcal{O}_\infty$ (with $B$ UHF of infinite type) are strongly self-absorbing property.
Question 3. Let $P \subset A$ be an inclusion of unital C*-algebras and $E : A \to P$ be a conditional expectation of index finite type. That is, there is a quasi-basis $\{(w_i, w_i^*)\}_{i=1}^n \subset A \times A$ such that $x = \sum_{i=1}^n E(xw_i)w_i^* = \sum_{i=1}^n w_iE(w_i^*x)$ for any $x \in A$.

(1) If $A$ is strongly self-absorbing, when $P$ is strongly self-absorbing?

(2) Let $D$ is strongly self-absorbing and $A$ $D$-absorbing. When $P$ is $D$-absorbing?
In this talk we introduce the **finitely saturated property** for a class \( C \) of separable unital C*-algebras and **local C-property** for a unital C*-algebra.

**Answer 4**

(1) Let \( A \) be a unital C*-algebra which is a local C*-algebra and an action \( \alpha \) of a finite group \( G \). Suppose that \( \alpha \) has the Rokhlin property, then the crossed product algebra \( A \rtimes_\alpha G \) is a unital local C-algebra.

(2) Moreover, we introduce the Rokhlin property for a conditional expectation for a pair of unital C*-algebras \( A \supset P \) and show that

(a) if \( A \) is strongly self-absorbing and semiprjective, then \( P \) is strongly self-absorbing.

(b) if \( A \) is a unital local C-algebra, then so is \( P \).

Note that if \( \mathcal{C} \) is the set of all separable, unital, \( \mathcal{D} \)-absorbing C*-algebras, then \( \mathcal{C} \) is finitely saturated.
**Local $\mathcal{C}$-property**

**Definition 5.** (Osaka-Phillips 07) Let $\mathcal{C}$ be a class of separable unital C*-algebras. Then $\mathcal{C}$ is *finitely saturated* if the following closure conditions hold:

1. If $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.

2. If $A_1, A_2, \ldots, A_n \in \mathcal{C}$ then $\bigoplus_{k=1}^{n} A_k \in \mathcal{C}$.

3. If $A \in \mathcal{C}$ and $n \in \mathbb{N}$, then $M_n(A) \in \mathcal{C}$.

4. If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

Moreover, the *finite saturation* of a class $\mathcal{C}$ is the smallest finitely saturated class which contains $\mathcal{C}$.

**Example 6.**

1. Let $\mathcal{C}$ be the set of all unital C*-algebras such as $\bigoplus_{i=1}^{n} P_i M_{n_i}(C(X_i)) P_i$, where $P_1$ is a projection in $M_{n_i}(C(X_i))$. If all $X_i$ is a point $\{\cdot\}$, or an interval $[0, 1]$, or a torus $S^1$. Then $\mathcal{C}$ is finitely saturated.

2. Let $\mathcal{C}$ be the set of unital C*-algebras with stable rank one. Then $\mathcal{C}$ is finitely saturated.
3. Let $\mathcal{C}$ be the set of unital C*-algebras with real rank zero. Then $\mathcal{C}$ is finitely saturated.

4. Let $\mathcal{C}$ be the set of all separable, unital, $\mathcal{D}$-absorbing C*-algebras. Then $\mathcal{C}$ is finitely saturated.

**Definition 7.** (Osaka-Phillips 07) Let $\mathcal{C}$ be a class of separable unital C*-algebras. A **unital local C-algebra** is a separable unital C*-algebra $A$ such that for every finite set $S \subset A$ and every $\varepsilon > 0$, there is a C*-algebra $B$ in the finite saturation of $\mathcal{C}$ and a unital *-homomorphism $\varphi : B \to A$ (not necessarily injective) such that $\text{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in S$. 
Rokhlin property for an inclusion of unital C*-algebras

Let $A$ be a C*-algebra. Then we define

$$c_0(A) = \{ (a_n) \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n \to \infty} \|a_n\| = 0 \}$$

and

$$A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(A).$$

**Definition 8** (Izumi 04). Let $A$ be a unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if there are mutually orthogonal projections $e_g \in A^\infty$ for $g \in G$ such that:

1. $\alpha^\infty_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $e_g a = ae_g$ for all $g \in G$ and all $a \in A$.
3. $\sum_{g \in G} e_g = 1$. 

Example 9. Let $M_{n\infty} = \otimes_{k=1}^{\infty} M_n(C)$ and

$$\alpha = \otimes_{k=1}^{\infty} \text{Ad} \left( \begin{array}{cccccc}
\lambda_1 & 0 & \cdots & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
\vdots & \cdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \lambda_n
\end{array} \right),$$

where $\{\lambda_i\}_{i=1}^{n}$ is the root of the unit. Then $\alpha$ be the automorphism of order $n$ on $M_{n\infty}$, and $\alpha$ has the Rokhlin property.

More general, let $G$ be a finite group, $\lambda$ be the left regular representation of $G$. We identify $B(\ell^2(G))$ with $M_{|G|}$ and consider an action of $G$ on $M_{|G|\infty}$ by

$$\mu^G_g = \otimes_{n=1}^{\infty} \text{Ad}(\lambda(g)), \ g \in G.$$ 

Then $\mu^G$ has the Rokhlin property.
Proposition 10 (Phillips 06). Let $D$ be an infinite tensor product C*-algebra and let $\alpha \in \text{Aut}(D)$ be an automorphism of order 2, of the form

$$D = \bigotimes_{n=1}^{\infty} \mathbb{M}_{k(n)}(C) \quad \text{and} \quad \alpha = \bigotimes_{n=1}^{\infty} \text{Ad}(p_n - q_n),$$

with $k(n) \in \mathbb{N}$ and where $p_n, q_n \in \mathbb{M}_{k(n)}(C)$ are projections with $p_n + q_n = 1$ and $\text{rank}(p_n) \geq \text{rank}(q_n)$ for all $n \in \mathbb{N}$. Set

$$\lambda_n = \frac{\text{rank}(p_n) - \text{rank}(q_n)}{\text{rank}(p_n) + \text{rank}(q_n)}$$

for $n \in \mathbb{N}$ and, for $m \leq n \Lambda(m, n) = \lambda_{m+1}\lambda_{m+2}\cdots\lambda_n$ and $\Lambda(m, \infty) = \lim_{n \to \infty} \Lambda(m, n)$. Then the followings are equivalent:

1. The action $\alpha$ has the Roklin property.

2. There are infinitely many $n \in \mathbb{N}$ such that $\text{rank}(p_n) = \text{rank}(q_n)$, i.e. $\lambda_n = 0$.

3. $D \rtimes_{\alpha} \mathbb{Z}_2$ is a UHF algebra.

$\square$
Remark 11. A crossed product algebra \( M_{|G|\infty} \rtimes_{\mu} G \) is also an UHF algebra.

We also could construct an action which does not have the Rokhlin property.

Proposition 12 (Phillips 06). Let \( \alpha \in \text{Aut}(D) \) be a product type automorphism of order 2 as in Proposition 10. Then the followings are equivalent:

1. The action \( \alpha \) has the tracial Rokhlin property.
2. \( \Lambda(m, \infty) = 0 \) for all \( m \).

\( \square \)
The following observation is our motivation to introduce the Rokhlin property for the inclusion of unital \( C^* \)-algebras with a finite \( C^* \)-index.

**Proposition 13.** (Kodaka-Osaka-Teruya 08) Let \( \alpha \) be an action of a finite group \( G \) on a unital \( C^* \)-algebra \( A \) and \( E \) the canonical conditional expectation from \( A \) onto the fixed point algebra \( P = A^\alpha \) defined by

\[
E(x) = \frac{1}{|G|} \sum_{g \in G} \alpha_g(x) \quad \text{for } x \in A,
\]

where \( |G| \) is the order of \( G \). Then \( \alpha \) has the Rokhlin property if and only if there is a projection \( e \in A' \cap A^\infty \) such that \( E^\infty(e) = \frac{1}{|G|} \cdot 1 \), where \( E^\infty \) is the conditional expectation from \( A^\infty \) onto \( P^\infty \) induced by \( E \).

**Definition 14.** (Kodaka-Osaka-Teruya 08) A conditional expectation \( E \) of a unital \( C^* \)-algebra \( A \) with a finite index is said to have the Rokhlin property if there exists a projection \( e \in A' \cap A^\infty \) satisfying

\[
E^\infty(e) = (\text{Index}E)^{-1} \cdot 1
\]

and a map \( A \ni x \mapsto xe \) is injective. We call \( e \) a Rokhlin projection.
When $\alpha$ is an action of a finite group $G$ on $A$ and is saturated (i.e. $A \rtimes G = \text{span}\{xey \mid x, y \in A\}$), let $P$ denote the fixed point algebra $A^\alpha$. We know that the canonical conditional expectation $E : A \to A^\alpha$ is of finite index and we have the following basic construction:

$$A^\alpha \subset A \subset A \rtimes_\alpha G.$$

**Remark 15.** Let $\alpha$ be an action of a finite group $G$ on a unital $C^*$-algebra $A$ and $E$ the canonical conditional expectation from $A$ onto the fixed point algebra $P = A^\alpha$. Then $\alpha$ is outer. Hence $E$ is of a finite index with $\text{Index}E = |G|$. That is, there is a quasi-basis $\{(w_i, w_i^*)\}_{i=1}^n \subset A \times A$ such that

1. for any $x \in A$

$$x = \sum_{i=1}^n E(xw_i)w_i^* = \sum_{i=1}^n w_iE(w_i^*x)$$

2. $\sum_{i=1}^n w_iw_i^* = |G| = \text{Index}E$. 
The following is a key lemma to prove the main theorem

**Lemma 16.** (Kodaka-Osaka-Teruya 08)

Let $P \subset A$ be an inclusion of unital C*-algebras and $E$ a conditional expectation from $A$ onto $P$ with a finite index. If $E$ has the Rokhlin property with a Rokhlin projection $e \in A' \cap A^\infty$, then there is a unital linear map $\beta : A^\infty \to P^\infty$ such that for any $x \in A^\infty$ there exists the unique element $y$ of $P^\infty$ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty$. In particular, $\beta|_A$ is a unital injective *-homomorphism and $\beta(x) = x$ for all $x \in P$.

We have

$$A \hookrightarrow A^\infty \xrightarrow{\beta} P^\infty.$$
Theorem 17. (Kodaka-Osaka-Teruya 08) Let \( C \) be any saturated class of semiprojective, separable unital \( C^* \)-algebras. Let \( A \supset P \) be a finite index inclusion with the Rokhlin property. If \( A \) is a unital local \( C \)-algebra, then \( P \) is also a unital local \( C \)-algebra.
idea for the proof

Since $A$ is a unital local $C$-algebra, for finite set $S \subset P \subset A$ and $\varepsilon > 0$, there is a $C^*$-algebra $Q$ in the finite saturation of $C$ and a unital $*$-homomorphism $\rho : Q \to A$ such that $S$ is within $\varepsilon$ of an element of $\rho(Q)$.

\[
\begin{array}{c}
l^\infty(N, P)/I_n \\
\孙\downarrow \longrightarrow \beta \\
Q(\rho \hookrightarrow A) \quad \beta \quad \downarrow \quad P^\infty = l^\infty(N, P)/\bigcup_n I_n
\end{array}
\]

Using the semiprojectivity of $Q$, we can lift the $*$-homomorphism $\beta$ to a $*$-homomorphism $\bar{\beta} : Q \to l^\infty(N, P)/I_n$ for some $n$. (Note that $c_o(P) = \bigcup_n I_n$)

Take sufficient large $k \in \mathbb{N}$ such that $\beta_k : Q \to P$ is a $*$-homomorphism such that $S \subset \varepsilon \beta_k(Q)$, where $\bar{\beta} = (\beta_k)_{k \in \mathbb{N}} + I_n$. \hfill \Box
Corollary 18. Let $A \supset P$ be an inclusion of separable unital $C^*$-algebras with the Rokhlin property.

1. If $A$ is a unital AF algebra, then $P$ is a unital AF algebra.

2. If $A$ is a unital AI algebra, then $P$ is a unital AI algebra.

3. If $A$ is a unital AT algebra, then $P$ is a unital AT algebra.

4. If $A$ is a unital AD algebra, then $P$ is a unital AD algebra.
Rokhlin property and strongly self-absorbing

**Proposition 19.** Let $P \subset A$ be an inclusion of separable unital C*-algebras with index finite and $A$ have approximately inner half flip. Suppose that $E$ has the Rokhlin property and $A$ is semiprojective. Then $P$ has approximately inner half flip.

**Remark 20.** 1. Under the same condition for an inclusion of separable unital C*-algebras $P \subset A$ in Proposition 19 since $P$ has approximately inner half flip map we know that $P$ is nuclear and simple.

2. To deduce the simplicity of $P$ we need only the simplicity of $A$ and the Rokhlin condition for $E: A \to P$.

3. If $\mathcal{D}$ is a strongly self-absorbing inductive limit of recursive subhomogeneous algebras in the sense of Phillips [26], then $\mathcal{D}$ is either projectionless (i.e. the Jiang-Su algebra $\mathcal{Z}$) or a UHF algebra of infinite type by Toms and Winter [34, Corollary 5.10]. On the contrary, if $\mathcal{D}$ is a separable purely infinite strongly self-absorbing C*-algebra which satisfies the Universal Coefficients Theorem
(We write $\mathcal{D}$ is in the UCT class $\mathcal{N}$). Then $\mathcal{D}$ is either $\mathcal{O}_2$, $\mathcal{O}_\infty$ or a tensor product of $\mathcal{O}_\infty$ with a UHF algebra of infinite type by Toms and Winter [34, Corollary 5.2].

**Definition 21.** (Phillips 01) The class of *recursive subhomogeneous algebras* is the smallest class $\mathcal{R}$ of C*-algebras which is closed under isomorphism and such that

1. If $X$ is a compact Hausdorff space and $n \geq 1$, then $C(X, M_n) \in \mathcal{R}$.

2. $\mathcal{R}$ is closed under the following pull back construction: If $A \in \mathcal{R}$, if $X$ is a compact Hausdorff space, if $X^{(0)} \subset X$ is closed, $\phi: A \to C(X^{(0)}, M_n)$ any unital homomorphism and $\rho: C(X, M_n) \to C(X^{(0)}, M_n)$ is the restrict homomorphism, then the pullback

$$A \oplus_{C(X^{(0)}, M_n)} C(X, M_n)$$

$$= \{ (a, f) \in A \oplus C(X, M_n): \phi(a) = \rho(f) \}$$

is in $\mathcal{R}$. 
**Theorem 22.** Let $\mathcal{D}$ be $\mathcal{U}_\infty$ and let $\alpha$ be an action of a finite group $G$ on $\mathcal{D}$. Suppose that $\alpha$ has the Rokhlin property. Then the crossed product $\mathcal{U}_\infty \rtimes_\alpha G$ is isomorphic to $\mathcal{U}_\infty$. 
The following example implies that the Rokhlin property is essential in Theorem 22.

**Example 23.** Let $U_\infty$ be the universal UHF algebra and $A = M_{2\infty}$. Then $A \otimes U_\infty \cong U_\infty$.

Let $\alpha$ be an symmetry by Blackadar [1, Proposition 5.1.2]. Then $A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ is not a AF algebra. We note that $\alpha$ has the tracial Rokhlin property by Phillips [28, Proposition 3.4], but does not have the Rokhlin property, since the crossed product algebra $A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ is not AF algebra by Phillips [27, Theorem 2.2].

Then $\alpha \otimes id$ is a symmetry with the tracial Rokhlin property on $A \otimes U_\infty (\cong A)$, and the crossed product algebra

$$(A \otimes U_\infty) \rtimes_{\alpha \otimes id} \mathbb{Z}/2\mathbb{Z} \cong (A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}) \otimes U_\infty$$

$$\cong B \otimes U_\infty,$$

where $B$ is the Bunce-Dedens algebras of type $2^\infty$ by [1, Proposition 5.4.1]. Note that $K_1(B \otimes U_\infty) \neq 0$, that is, $B \otimes U_\infty$ is not a AF algebra. Since a strongly self-absorbing inductive limit of type I with real rank zero C*-algebra is a UHF algebra of infinite type by Toms and Winter [34, Corollary 5.9], $B \otimes U_\infty$ is not
a strongly self-absorbing algebra. Hence there is a symmetry \( \beta \) with the tracial Rokhlin property on \( \mathcal{U}_\infty \) such that \( \mathcal{U}_\infty \rtimes_\beta \mathbb{Z}/2\mathbb{Z} \) is not strongly self-absorbing.
**Theorem 24.** Let $P \subset A$ be an inclusion of unital separable C*-algebras with index finite. Suppose that a conditional expectation $E: A \to P$ has the Rokhlin property and $A$ is semiprojective and strongly self-absorbing. Then $P$ is strongly self-absorbing.

**Corollary 25.** Let $P \subset A$ be an inclusion of unital separable C*-algebras with index finite. Suppose that a conditional expectation $E: A \to P$ has the Rokhlin property. Suppose that $A$ is $O_2$ or $O_\infty$. Then $P \cong A$.

**Corollary 26.** (Izumi 2002 [9, Theorem 4.2]) Let $\alpha$ be an action of a finite group $G$ on $O_2$. Suppose that $\alpha$ has the Rohklin property. Then we have

1. $O_2^G \cong O_2$.

2. The crossed product algebra $O_2 \rtimes_\alpha G \cong O_2$.

**Remark 27.** (Izumi 2004) From [10, Theorem 3.6] there is no non-trivial finite group action with the Rokhlin property on $O_\infty$.
Rokhlin property and $\mathcal{D}$-absorbing

We use the following characterization of the $\mathcal{D}$-absorbing.

**Theorem 28.** (Rordam 02) Let $\mathcal{D}$ be a strongly self-absorbing and $A$ be any separable C*-algebra. $A$ is $\mathcal{D}$-absorbing (i.e. $A \otimes \mathcal{D} \cong A$) if and only if $\mathcal{D}$ admits a unital *-homomorphism to $A' \cap M(A)\infty$.

Using the above characterization and a basic Lemma 16 we have the following:

**Theorem 29.** Let $P \subset A$ be an inclusion of unital C*-algebras and $E$ a conditional expectation from $A$ onto $P$ with a finite index. Suppose that $\mathcal{D}$ is a separable unital self-absorbing C*-algebra, $A$ is a separable $\mathcal{D}$-absorbing, and $E$ has the Rokhlin property. Then $P$ is $\mathcal{D}$-absorbing.
Remark 30. If we replace the Rokhlin property by the tracial Rokhlin property, which is weaker than the Rokhlin property, then the $\mathcal{D}$-absorbing property fails. Indeed, Phillips constructed an symmetry $\alpha$ on a strongly self-absorbing UHF algebra $\mathcal{D}$ with the tracial Rokhlin property in the sense of Phillips such that $\mathcal{D} \rtimes_{\alpha} \mathbb{Z}/2\mathbb{Z}$ is not $\mathcal{D}$-absorbing. (See Example 4.11 of [28].)
Intermediate fixed point algebras

In this section we present an inclusion of unital C*-algebras $P \subset A$ which does not come from an action of finite group on $A$.

**Proposition 31.** Let $A$ be a separable unital C*-algebra, $\alpha$ an action of a finite group $G$ on $A$ and $E: A \to A^G$ a canonical conditional expectation. Suppose that $\alpha$ has the Rokhlin property. Then we have

1. For any subgroup $H$ of $G$ the restricted $E$ to $A^H$, which is a conditional expectation from $A^H$ onto $A^G$, has the Rokhlin property.

2. If $A$ is a unital local $C$-algebra, then for any subgroup $H$ of $G$ $A^H$ is a unital local $C$-algebra.

3. Let $D$ be a strongly self-absorbing C*-algebra and $A$ be $D$-absorbing. Then for any subgroup $H$ of $G$ $A^H$ is $D$-absorbing.

4. If $A = \mathcal{O}_2$, then for any subgroup $H$ of $G$ $A^H \cong \mathcal{O}_2$. 
Remark 32. Let $A$ be a unital C*-algebra and $\alpha$ be an action from a finite group $G$ on $A$. Let $H$ be a subgroup of $G$. Then the condition that an inclusion $A^G \subset A^H$ is isomorphic to $B^K \subset B$ for some C*-algebra $B$ and an action from a finite group $K$ on $B$ implies that $H$ is a normal subgroup of $G$ (c.f. [32]). Hence from Proposition 31 we have examples of conditional expectations for inclusions of unital C*-algebras with the Rokhlin property which do not come from finite group actions.
References


