Conformal Field Theory, von Neumann Algebras and Noncommutative Geometry

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Operator algebraic approach to conformal field theory

$\rightarrow$ Connecting subfactor theory and noncommutative geometry through superconformal field theory

(with S. Carpi, R. Hillier, R. Longo and F. Xu)

Outline of the talk:

1. Conformal symmetry and the Virasoro algebras
2. Analogy between conformal field theory and differential geometry
3. Supersymmetry and the Dirac operator
4. $N = 2$ supersymmetry, the Doplicher-Haag-Roberts theory (subfactors) and the Jaffe-Lesniewski-Osterwalder cocycle (noncommutative geometry)

Our spacetime is S^1 and the spacetime symmetry group is the infinite dimensional Lie group $\mathrm{Diff}(S^1)$. It gives a Lie algebra generated by $L_n = -z^{n+1}\dfrac{\partial}{\partial z}$ with $|z| = 1$.

The Virasoro algebra is a central extension of its complexification. It is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c with the following relations.

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c.$$

We have a good understanding of its irreducible unitary highest weight representations, where the central charge c is mapped to a positive scalar.

Fix a nice representation π of the Virasoro algebra, called a vacuum representation, and simply write L_n for $\pi(L_n)$.

Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, called the stress-energy tensor, for $z \in \mathbb{C}$ with $|z| = 1$. Regard it as a Fourier expansion of an operator-valued distribution on S^1. This is a typical example of a quantum field.

Fix an interval I and take a C^∞-function f with $\mathrm{supp}\, f \subset I$. We have an (unbounded) operator $\langle L, f\rangle$ as an application of an operator-valued distribution.

Let $A(I)$ be the von Neumann algebra of bounded linear operators generated by these operators with various f. The family $\{A(I)\}$ gives an example of a conformal field theory.

Operator algebraic axioms: (conformal field theory)

Motivation: Operator-valued distributions $\{T\}$ on S^1.

Fix an interval $I \subset S^1$, consider $\langle T, f\rangle$ with $\mathrm{supp}\, f \subset I$.

$A(I)$: the von Neumann algebra generated by these (possibly unbounded) operators

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (the commutator)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy
5. Vacuum vector

Such a family $\{A(I)\}$ is called a local conformal net.

Each $A(I)$ is usually an injective type III_1 factor (the unqiue Araki-Woods factor). Each $A(I)$ has no physical information, but the family $\{A(I)\}$ has.

Representation theory of local conformal nets:
Doplicher-Haag-Roberts theory of superselection sectors.

Each representation is given by an endomorphism of one factor $A(I_0)$ and its dimension is given by the square root of the Jones index of the image. A representation thus gives a subfactor. ($\to$ many applications to subfactor theory)

[K-Longo-Müger] Complete rationality: We have only finitely many irreducible representations and all have finite dimensions. ($\to$ finite depth subfactors)

Results in this operator algebraic approach:

Longo-Rehren, Xu, Böckenhauer-Evans-K: α-induction [Theory of induced representations]

K-Longo: Classification of local conformal nets with $c < 1$ (Ann. Math. 2004) [The first classification result with a completely new example]

K-Longo, K-Longo-Penning-Rehren: Classification of full/boundary conformal field theories with $c < 1$ [Studies of new cohomology]

K-Longo: Construction of the Moonshine net with the Monster automorphism group

K-Longo: An analogy to differential geometry [to be explained below]

Geometric aspects of local conformal nets

Classical geometry: Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. The classical Weyl formula gives an asymptotic expansion

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

as $t \to 0+$, where a_0 is the volume of the manifold, and if $n = 2$, then a_1 is (constant times) the Euler characteristic of the manifold.

So the coefficients in the asymptotic expansion have a geometric meaning. We look for their analogues in the setting of local conformal nets.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1.

For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given. (K-Longo)

This gives an analogy of the Laplacian Δ of a manifold and the conformal Hamiltonian L_0 of a local conformal net.

A "square root" of the Laplacian gives a classical Dirac operator. The Connes approach in noncommutative geometry uses its abstract axiomatization.

Noncommutative geometry:

Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.

In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: a spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the smooth algebra $C^\infty(M)$.
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator with compact resolvents, the Dirac operator.
4. We require $[D, x] \in B(H)$ for all $x \in \mathcal{A}$.

$N = 1$ super Virasoro algebras: (Adding a square root of L_0)

The infinite dimensional super Lie algebras generated by central element c, even elements L_n, $n \in \mathbb{Z}$, and odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}c.
\end{aligned}
$$

Ramond ⟦Neveu-Schwarz⟧ algebra, if $r \in \mathbb{Z}$ $[\![r \in \mathbb{Z} + 1/2]\!]$.
Note $G_0^2 = L_0 - c/24$ for $r = s = 0$ in a representation.

We again consider a unitary representation of (one of) the $N = 1$ super Virasoro algebras. Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ as before and $G(z) = \sum_r G_r z^{-r-3/2}$ as operator-valued distributions on S^1.

Using test functions supported in an interval I, they produce a family $\{A(I)\}$ of von Neumann algebras parametrized by $I \subset S^1$. This gives a superconformal net, for which now the bracket in the axioms means a graded commutator.

To make a further study in connection to noncommutative geometry, we work on $N = 2$ super Virasoro algebra and its unitary representations. Instead of one series $\{G_r\}$, we next have two series $\{G_r^\pm\}$ for the $N = 2$ case.

$N=2$ super Virasoro algebra: Generated by c, L_n, J_n and $G^{\pm}_{n\pm a}$, $n\in\mathbb{Z}$, with the following relations. (a: a real parameter)

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c, \\
[J_m, J_n] &= \frac{m}{3}\delta_{m+n,0}c \\
[L_n, J_m] &= -mJ_{m+n}, \\
[G^+_{n+a}, G^+_{m+a}] &= [G^-_{n-a}, G^-_{m-a}] = 0, \\
[L_n, G^{\pm}_{m\pm a}] &= \left(\frac{n}{2}-(m\pm a)\right)G^{\pm}_{m+n\pm a}, \\
[J_n, G^{\pm}_{m\pm a}] &= \pm G^{\pm}_{m+n\pm a}, \\
[G^+_{n+a}, G^-_{m-a}] &= 2L_{m+n} + (n-m+2a)J_{n+m} + \\
&\quad \frac{1}{3}\left((n+a)^2-\frac{1}{4}\right)\delta_{m+n,0}c.
\end{aligned}$$

It is known that an irreducible unitary representation maps c to a scalar in the set

$$\left\{ \frac{3m}{m+2} \middle| m = 1, 2, 3, \ldots \right\} \cup [3, \infty).$$

We consider only the case $c = 3m/(m+2)$ in the discrete series now.

We use $G^1_n = (G^+_n + G^-_n)/\sqrt{2}$ and $G^2_n = -i(G^+_n - G^-_n)/\sqrt{2}$. We fix a unitary representation and write L_n, G^1_n, G^2_n, J_n for their images in the representation. They are closed unbounded operators. We then use the four operator-valued distributions $L(z) = \sum_n L_n z^{-n-2}$, $G^j(z) = \sum_n G^j_n z^{-n-3/2}$ $(j = 1, 2)$ and $J(z) = \sum_n J_n z^{-n-1}$, where $z \in \mathbb{C}$ with $|z| = 1$.

As before, using these four operator-valued distributions and test functions supported in $I \subset S^1$, we obtain a family of von Neumann algebras $\{A(I)\}$ parametrized by the intervals I.

For the vacuum representation of the $N = 2$ super Virasoro algebra with $c = 3m/(m + 2)$ in the discrete series, we have a vacuum representation with the coset construction (relative commutant) arising from $U(1)_{2m+4} \subset SU(2)_m \otimes U(1)_4$ due to Di Vecchia-Petersen-Yu-Zheng.

However, while it is clear that the coset contains a representation of the $N = 2$ super Virasoro algebra, it is not clear whether this representation gives all of the coset or not. The equality here has been often taken as a mathematically established result, but we have been unable to find a complete proof in literature.

Operator algebraic methods give a proof of this equality for the coset.

The extensions of the cosets are $N = 2$ superconformal nets by definition. They are classified and listed completely. Typical methods to give such an extension are the coset construction and the mirror extension in the sense of Xu, which copies one extension to give another through a coset.

Now these two are mixed together, and we have a simple current extension with a finite cyclic group of an arbitrary order. This was not the case in the previous classification results for (super) conformal nets.

This is partly based on Gannon's classification and a kind of A-D-E classification.

We now connect these to noncommutative geometry by constructing a family of spectral triples parameterized by the intervals I. We need the Dirac operator, and have two candidates G_0^1 and G_0^2 in the Ramond representation, but they are unitarily equivalent, so we just choose G_0^1, and put $\delta(x) = [G_0^1, x]$ for a bounded linear operator x on the representation space.

We put

$$\mathcal{A}(I) = A(I) \cap \bigcap_{n=1}^{\infty} \mathrm{dom}(\delta^n).$$

Each $\mathcal{A}(I)$ is strongly dense in $A(I)$ and satisfies $\delta(\mathcal{A}(I)) \subset \mathcal{A}(I)$. That is, for each I, our spectral triple $(\mathcal{A}(I), H, G_0^1)$ gives a quantum algebra in the sense of Jaffe-Lesniewski-Osterwalder.

We now deal with entire cyclic cohomology introduced by Connes, a nice cohomology theory for an infinite dimensional noncommutative manifold. Our Dirac operator $D = G_0^1$ satisfies the condition $\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$, the θ-summability condition.

A JLO cocycle for a θ-summable spectral triple is defined by a sequence of multilinear functionals defined in terms of traces and integrals involving e^{-tD^2} and $[D, \cdot\,]$ on the $*$-algebra. This gives an element in the entire cyclic cohomology.

We are interested in spectral triples arising from the Ramond representations with the lowest conformal weight $h = c/24$. They produce subfactors and then projections in the universal von Neumann algebra for a local conformal net, and in turn elements in the K_0-group.

In general, we have the index pairing between the K_0-group and the entire cyclic cohomology, producing a number.

Within the universal von Neumann algebra, we define a $*$-subalgebra with different representations, and each image gives a spectral triple with an appropriate Dirac operator in each representation. We thus have different JLO-cocycles for the same $*$-algebra.

In the above, we have the K_0-elements depending on the Ramond representations, and also the JLO-cocycles in the entire cyclic cohomology given by the same Ramond representations.

Our result then says that the pairing between them give the Kronecker δ of the representations. In this way, subfactor theory and noncommutative geometry are connected.