K Group and Similarity Invariants of Operators in type II_1 factors

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2. S. I. D. in direct integral form
   - Direct integral
   - Operators not similar to SI direct integrals
   - A necessary and sufficient condition about SI direct integral

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Background

Similarity invariants

An important question in operator theory is that how we can determine the complete similarity invariants for operators. People always have interest in similarity invariants of operators. Two operators A and B in $\mathscr{L}(\mathscr{H})$ are said to be similar if there is an invertible operator X in $\mathscr{L}(\mathscr{H})$ such that

$$XA = BX,$$

where $\mathscr{H}$ denotes a complex separable Hilbert space and $\mathscr{L}(\mathscr{H})$ is the set of bounded linear operators on $\mathscr{H}$.

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- Similarity invariant.
- Complete similarity invariant.

Background

Operators on finite dimensional Hilbert spaces

On finite dimensional Hilbert spaces, Jordan canonical form theorem shows that eigenvalues and generalized eigenspaces are complete similarity invariants of operators.

The complexity of infinite dimensional Hilbert spaces makes it impossible to find generally similarity invariants. The main difficulty is that it is impossible to find a representation theorem in $\mathscr{L}(\mathscr{H})$ as perfect as Jordan canonical form theorem.

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- J. B. Conway proved that two normal operators are similar if and only if they have the same scalar-valued spectral measure and their multiplicity functions are equal.
- For Cowen-Douglas operators of index 1, M. J. Cowen and R. G. Douglas introduced a curvature function and they showed that this function is a complete unitarily invariant of Cowen-Douglas operators of index 1.

Background

In the 1970's, F. Gilfeather [2] and Zejian Jiang introduced the concept of strongly irreducible operators independently.

Strongly irreducible operator

An operator T in $\mathscr{L}(\mathscr{H})$ is said to be strongly irreducible (short for SI) if its commutant $\{T\}' \equiv \{B \in \mathscr{L}(\mathscr{H}) : BT = TB\}$ contains no nontrivial idempotents. (Separability Assumption of $\mathscr{H}$)

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Examples

It is obvious that strong irreducibility is a similarity invariant. Cowen-Douglas operators of index 1 and unicellular operators are classical SI operators. It is easy to observe that the adjoint of a SI operator is still strongly irreducible and the spectrum of every SI operator is connected.

Background

Motivation

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Some progress

Many people did a lot of work around this subject.
In 1987, G. Gong obtained that SI operators form a nowhere dense subset of $\mathscr{L}(\mathscr{H})$. He also conjectured that SI operators form a dense subset of operators with connected spectra.
To answer this conjecture, in 1990, D. A. Herrero and C. Jiang proved that every operator in $\mathscr{L}(\mathscr{H})$ can be written as a direct sum of finitely many SI operators by a small perturbation [3].

Background

In 2002, Y. Cao, J. Fang and C. Jiang proved the following theorem [4].

Theorem CFJ

Let T in $\mathscr{L}(\mathscr{H})$, $T \sim \bigoplus_{i=1}^{k} {A_i}^{(n_i)}$, where A_i is SI and $A_i \nsim A_j$ for $i \neq j$. For every positive integer n, the finite strongly irreducible decomposition of $T^{(n)}$ is unique up to similarity if and only if the ordered K_0 group of $\{T\}'$ is isomorphic to the ordered group of integer-valued vectors i. e. $\{V(\{T\}'), K_0(\{T\}'), [I]\} \overset{\alpha}{\cong} \{\mathbb{N}^{(k)}, \mathbb{Z}^{(k)}, e\}$.

Background

In 2005, C. Jiang, X. Guo and K. Ji proved the following theorem [5].

Theorem JGJ

Ordered K_0 group is the completely similar invariant of two Cowen-Douglas operators.

In 2007, C. Jiang and K. Ji introduced the concept of the commutants of holomorphic curves E_f, and proved the following theorem [6].

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Theorem JJ

Completely similarity invariants of E_f are determined by the ordered K_0 group of the commutant of E_f.

Background

Minimal idempotent

We need to mention that Theorem CFJ plays a key role in proving both Theorem JGJ and Theorem JJ. However in the proof of Theorem CFJ, minimal idempotents are necessary.

In a unital Banach algebra $\mathscr{A}$, an idempotent P is said to be minimal if every nontrivial idempotent Q in the commutant of P satisfies $PQ = QP = P$ or $PQ = QP = 0$.

Background

Role of minimal idempotents

For an operator A in $\mathscr{L}(\mathscr{H})$ and a minimal idempotent P in $\{A\}' \equiv \{B \in \mathscr{L}(\mathscr{H}) : BA = AB\}$, we observe that $A|_{\mathrm{ran}P}$ is strongly irreducible.

Unfortunately, the commutants of many operators contain no minimal idempotents. Therefore, many operators in $\mathscr{L}(\mathscr{H})$ can not be written as direct sums of SI operators. Specially, the commutants of many operators in type II_1 factors contain no minimal idempotents.

Background

Type II_1 factor

A von Neumann algebra $\mathscr{M}$ is said to be a factor if the center of $\mathscr{M}$ is trivial, i. e. , the relation $\mathscr{M} \cap \mathscr{M}' = \mathbb{C}$ holds. A factor is said to be of type II_1 if there is no minimal projection in $\mathscr{M}$ and the identity is finite

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Finite projections in $\mathscr{M}$

In a type II_1 factor $\mathscr{M}$, every projection is finite. Equivalently, the identity is finite. (In a C^* algebra $\mathscr{A}$, a projection P is said to be finite if for every projection P_1, the relations $P_1 \leq P$ and $P_1 \sim_{M.v.} P$ must imply $P_1 = P$, where $\sim_{M.v.}$ stands for Murray-von Neumann equivalence.)

Background

Research methods

In this talk, we mainly consider characterizing similarity invariants of operators in type II_1 factors by applying terms of ordered K_0 group, where ordered K_0 group are usually called the Elliott invariant [7,8]. This invariant plays an important role in characterizing isomorphic invariant of C^* algebras.

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In this talk, we mainly consider characterizing similarity invariants of operators in type II_1 factors by applying terms of ordered K_0 group, where ordered K_0 group are usually called the Elliott invariant [7,8]. This invariant plays an important role in characterizing isomorphic invariant of C^* algebras.

Theorem CFJ is not a suitable trick to find similarity invariants of operators in type II_1 factors. Therefore we need to find new techniques. In the following, we introduce concepts about direct integrals to find new similarity invariants of operators in type II_1 factors.
Then, we introduce new representation theorem with direct integrals of strongly irreducible operators.

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Direct integral of Hilbert spaces

The following are some notations about direct integrals.

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- let $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \cdots \subset \mathscr{H}_\infty$ a sequence of Hilbert spaces, with $\mathscr{H}_n$ having dimension n, and $\mathscr{H}_\infty$ spanned by the union of Hilbert spaces $\mathscr{H}_n$, for $1 \leq n < \infty$.

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- let $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \cdots \subset \mathscr{H}_\infty$ a sequence of Hilbert spaces, with $\mathscr{H}_n$ having dimension n, and $\mathscr{H}_\infty$ spanned by the union of Hilbert spaces $\mathscr{H}_n$, for $1 \leq n < \infty$.
- Let $(\Lambda, \mu, \{\Lambda_n\}_{n=1}^{n=\infty})$ be a partitioned measure space, where Λ is a separable metric space, μ is (the completion of) a regular Borel measure on Λ, and $\{\Lambda_\infty, \Lambda_1, \Lambda_2, \ldots\}$ is a Borel partition of Λ. Assume that μ is σ-finite and Λ is almost σ-compact.

Direct integral of Hilbert spaces

Definition (Direct integral of Hilbert spaces)

We form the associated direct integral Hilbert space $\mathscr{H} = \int_\Lambda^\oplus \mathscr{H}(\lambda)\mu(d\lambda)$. This space consists of all (equivalence classes of) measurable functions f from Λ into $\mathscr{H}_\infty$ such that:

The element in $\mathscr{H}$ represented by the function $\lambda \to f(\lambda)$ is denoted by $\int_\Lambda^\oplus f(\lambda)\mu(d\lambda)$. For λ in Λ, the Hilbert space $\mathscr{H}(\lambda)$ is said to be the fiber space with respect to $\mathscr{H}$.

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- $f(\lambda) \in \mathscr{H}(\lambda) \equiv \mathscr{H}_n$ for $\lambda \in \Lambda_n$;
- $\int_\Lambda \|f(\lambda)\|^2 \mu(d\lambda) < \infty$.

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Direct integral of Hilbert spaces

Inner product on $\mathscr{H}$

We can verify that the map

$$u : \mathscr{H} \times \mathscr{H} \to \mathbb{C},\ u(f,g) = \int_\Lambda < f(\lambda), g(\lambda) > \mu(d\lambda)$$

is an inner product on $\mathscr{H}$. Therefore $\mathscr{H}$ is a complex Hilbert space.

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Example (Direct integral of Hilbert spaces)

Write $\Lambda = [0,1]$, the measure μ is regular Borel supported on Λ. The Hilbert space $L^2(\Lambda,\mu) \oplus L^2(\Lambda,\mu)$ can be considered as a direct integral of Hilbert spaces $\mathbb{C}^{(2)}$ with respect to (Λ,μ).

Direct integral of operators

Definition (Direct integral of operators)

An operator A in $\mathscr{L}(\mathscr{H})$ is said to be decomposable if there exists a strongly μ-measurable operator-valued function $A(\cdot)$ defined on Λ such that:

We write $A \equiv \int_\Lambda^\oplus A(\lambda)\mu(d\lambda)$ for the equivalence class corresponding to $A(\cdot)$. If $A(\lambda)$ is a scalar multiple of the identity on $\mathscr{H}(\lambda)$ for almost all λ, then A is said to be diagonal. The collection of all diagonal operators is said to be the diagonal algebra of Λ. It is an abelian von Neumann algebra.

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- $A(\lambda)$ is an operator in $\mathscr{L}(\mathscr{H}(\lambda))$ a. e. for λ in Λ;
- $(Af)(\lambda) = A(\lambda)f(\lambda)$, for all $f \in \mathscr{H}$.

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Direct integral of operators

Example (Diagonal operator)

Let $L^2(\Lambda, \mu)$ be as the example we mentioned just now. Let χ_{Λ_1} be some characteristic function on a measurable subset Λ_1 of Λ. This function induces multiplication operator $M_{\chi_{\Lambda_1}}$. And $M_{\chi_{\Lambda_1}}$ is diagonal with respect to $L^2(\Lambda, \mu)$.

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Operators not similar to SI direct integrals

Definition (Direct integrals of SI operators)

An operator A in $\mathscr{L}(\mathscr{H})$ is said to be a direct integral of strongly irreducible operators if there exists a partitioned measure space $(\Lambda, \mu, \{\Lambda_n\}_{n=1}^{n=\infty})$ for $\mathscr{H}$ such that A is decomposable with respect to $(\Lambda, \mu, \{\Lambda_n\}_{n=1}^{n=\infty})$ and $A(\lambda) \in \mathscr{L}(\mathscr{H}(\lambda))$ is strongly irreducible almost everywhere on Λ. This kind of integrals is short for "SI integrals".

By computing, we find that not every operator in $\mathscr{L}(\mathscr{H})$ is similar to a direct integral of strongly irreducible operators. In the following, we construct two operators of this type. One is in discrete case, the other is in continue case.

Operators not similar to SI direct integrals

Example (Discrete case)

Assume that operator A has the form

$$A = \bigoplus_{i=1}^{\infty} A_i. \quad A_i = \begin{pmatrix} \frac{1}{i} & 1 \\ 0 & -\frac{1}{2i} \end{pmatrix} \in M_2(\mathbb{C}).$$

Operators not similar to SI direct integrals

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Every idempotent P in $\{A\}'$ is in the form $P = \bigoplus_{i=1}^{\infty} P_i$, where P_i is in one of the following four forms

$$\begin{pmatrix} 1 & \frac{2i}{3} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\frac{2i}{3} \\ 0 & 1 \end{pmatrix}, \quad 0_2, \quad I_2.$$

Operators not similar to SI direct integrals

Since $P \in \{A\}'$ is bounded, P_i must be 0_2 or I_2 for all but finitely many $i \in \mathbb{N}$. Hence, all the idempotents in $\{A\}'$ form the only maximal abelian set of idempotents $\mathscr{P}$. But this set is unbounded.

Operators not similar to SI direct integrals

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If there exists an operator B similar to A and B can be written as a direct integral of strongly irreducible operators, then every idempotent in the commutant of B must be a projection. This is a contradiction because $\mathscr{P}$ is unbounded.

Operators not similar to SI direct integrals

Example (Continue case)

Let μ be a regular Borel measure supported on interval [0,1].
Denote by N_μ the multiplication operator on $L^2((0,1),\mu)$ defined as follows,

$$(N_\mu f)(t) = t \cdot f(t), \quad f \in L^2(0,1), t \in (0,1).$$

Notice that the set $\{N_\mu\}'$ contains no minimal idempotent. let A be of the form

$$A = \begin{pmatrix} N_\mu & I \\ 0 & -\frac{1}{2}N_\mu \end{pmatrix} \begin{matrix} L^2(0,1) \\ L^2(0,1) \end{matrix}.$$

The reason for this operator not similar to a direct integral of strongly irreducible operators is essentially the same as the above example.

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Operators similar to SI direct integrals

According to the above discussion, we obtain a necessary and sufficient condition for operators which are similar to direct integrals of strongly irreducible operators.

Theorem 2.1

Let $\mathscr{H}$ be a separable Hilbert space and let $A \in \mathscr{L}(\mathscr{H})$. The operator A is similar to a direct integral of strongly irreducible operators if and only if its commutant $\{A\}'$ contains a bounded maximal abelian set of idempotents $\mathscr{P}$.

Operators similar to SI direct integrals

Definition (n normal operator)

Suppose that $\mathscr{M}$ is a maximal abelian von Neumann algebra. An operator in the commutant of $\mathscr{M}^{(n)}$ is said to be n normal.

Operators similar to SI direct integrals

Definition (*n* normal operator)

Suppose that $\mathscr{M}$ is a maximal abelian von Neumann algebra. An operator in the commutant of $\mathscr{M}^{(n)}$ is said to be n normal.

Example

In the definition, we observe that n normal operators need not to be normal. And this class of operators is abundant. Many n normal operators are similar to direct integrals of strongly irreducible operators. But in the above examples, we find that not every n normal operator behaves as we hope. However, we find that operators similar to direct integrals of strongly irreducible operators form a dense subset in $\mathscr{L}(\mathscr{H})$.

A finer representation about SI integral

Theorem 2.2

If an operator A can be written as a direct integral of strongly irreducible operators then A has a finer upper triangular representation

$$A = \bigoplus_{m=1}^{\infty}\bigoplus_{j=1}^{s_m}\bigoplus_{k=1}^{t_{m_j}}\int_{\Lambda_m}\begin{pmatrix} N_{\nu_m} & M_{\phi_{12,m_j,k}} & \cdots & M_{\phi_{1m_j,m_j,k}} \\ 0 & N_{\nu_m} & \cdots & M_{\phi_{2m_j,m_j,k}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{\nu_m} \end{pmatrix}_{m_j\times m_j}(\lambda)\,\nu_m(d\lambda),$$

$$\bigoplus\int_{\Lambda_\infty} A(\lambda)\mu_\infty(d\lambda)$$

where $0 \le t_{m_j}, s_m \le \infty$ are integers and every N_{ν_m} in the main diagonal is *-cyclic. The measures ν_m are pairwise singular for different "m"s.

A finer representation about SI integral

Remark

$A(\lambda)$ acts on an infinite dimensional Hilbert space a. e. on Λ_∞. Generally, we call

$$\int_{\Lambda_\infty} A(\lambda)\mu_\infty(d\lambda)$$

the "infinite order part" in the SI integral representation.

In section 4, we show that operators in a type II_1 factor contains no "infinite order parts".

A finer representation about SI integral

Remark

In the above upper triangular representation, the operator-valued matrix

$$\begin{pmatrix} N_{\nu_m} & M_{\phi_{12,m_j,k}} & \cdots & M_{\phi_{1m_j,m_j,k}} \\ 0 & N_{\nu_m} & \cdots & M_{\phi_{2m_j,m_j,k}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{\nu_m} \end{pmatrix}_{m_j \times m_j} \qquad \text{(I)}$$

is considered as a basic building block in SI integrals.

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About uniqueness of S. I. D. up to similarity

In the following, we introduce the definition of the uniqueness of the strongly irreducible decomposition of operators up to similarity.

Definition (Uniqueness of S. I. D. up to similarity)

For any two bounded maximal abelian sets of idempotents $\mathscr{P}$ and $\mathscr{Q}$ in $\{A\}'$, if $\{A\}'$ contains an invertible operator X such that $X\mathscr{P}X^{-1} = \mathscr{Q}$, then A is said to have unique strongly irreducible decomposition (short for S. I. D.) up to similarity.

Unlike operators in $M_n(\mathbb{C})$, we observe that not every operator in $\mathscr{L}(\mathscr{H})$ has unique S. I. D. up to similarity. This is a characteristic of countably infinite dimensional Hilbert spaces.

On infinite dimensional Hilbert space

Nontrivial example w. r. t. $L^2(\Lambda, \mu)$

Write $\Lambda = [0, 1]$, the measure μ is regular Borel supported on Λ. We construct a nontrivial example with operators on $L^2(\Lambda, \mu)$ such that its S. I. D. is not unique up to similarity.

Write A in the form

$$A = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix},$$

where λI is a scalar multiple of the identity on $L^2(\Lambda, \mu)$.

On infinite dimensional Hilbert space

Two bounded maximal abelian sets of idempotents in $\{A\}'$

What we need to do is to find two bounded maximal abelian sets of idempotents ($\mathscr{P}_1$ and $\mathscr{P}_2$) in the commutant of A satisfying that they are not similar in the commutant of A.

let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in $L^2(\Lambda, \mu)$, every e_i spanning the range of the projection E_i. Let $\mathscr{P}_1$ be generated by the set $\{E_i \otimes I_2\}_{i=1}^{\infty}$.

Let Λ_α be a Borel subset of Λ. Every characteristic function f_α on Λ_α induces a projection P_α (multiplication operator) on $L^2(\Lambda, \mu)$. All projections as $P_\alpha \otimes I_2$ form the set $\mathscr{P}_2$. Notice that $\mathscr{P}_2$ may contain no minimal idempotent.

On infinite dimensional Hilbert space

Two bounded maximal abelian sets of idempotents in $\{A\}'$

We obtain that $\mathscr{P}_1$ and $\mathscr{P}_2$ are two bounded maximal abelian sets of idempotents in $\{A\}'$. We observe that $\mathscr{P}_1$ contains minimal idempotents while $\mathscr{P}_2$ may contain no minimal idempotents. Therefore they may not be similar in the commutant of A.

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About uniqueness of S. I. D. up to similarity

Remark

By the above example, we consider only operators without “infinite order parts”in the following discussion. More precisely, corresponding to operators stated in Theorem 2.2, we consider an operator A in the form

$$A=\bigoplus_{m=1}^{s}\bigoplus_{j=1}^{k_m}\int_{\Lambda_m}\begin{pmatrix} N_{\nu_m} & M_{\phi_{12,m_j}} & \cdots & M_{\phi_{1m_j,m_j}} \\ 0 & N_{\nu_m} & \cdots & M_{\phi_{2m_j,m_j}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{\nu_m} \end{pmatrix}^{(t_{m_j})}_{m_j\times m_j}(\lambda)\ \nu_m(d\lambda), \qquad \text{(II)}$$

where s, k_m, m_j and t_{m_j} are positive integers. The measures ν_m are pairwise singular for different “m”s.

About uniqueness of S. I. D. up to similarity

To discuss the uniqueness of S. I. D. of the operator A up to similarity, we need to introduce the concepts of "The order of S. I. D." and "minimal idempotent function".

Definition (The order of S. I. D.)

Assume that A is in $\mathscr{L}(\mathscr{H})$, and there is an invertible operator X in $\mathscr{L}(\mathscr{H})$, such that XAX^{-1} is a direct integral of SI operators. Write $\hat{A} \equiv XAX^{-1}$ and the corresponding measure space is $\{\Lambda_{\hat{A}}, \mu_{\hat{A}}, \{\Lambda_{\hat{A},n}\}_{n=1}^{n=\infty}\}$. Define $\Omega \equiv \{n : \mu_{\hat{A}}(\Lambda_{\hat{A},n}) > 0, n \in \mathbb{N} \cup \{\infty\}\}$. The supremum $\sup\{n\}_{n\in\Omega}$ is said to be an order of strongly irreducible decomposition of A.

About uniqueness of S. I. D. up to similarity

Definition (Minimal idempotent function)

Let operators A be a direct integral of strongly irreducible operators, and the order of S. I. D. of A is finite. If an idempotent P in $\{A\}'$ satisfying

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- $A(\Lambda_m)|_{\mathrm{ran}P(\Lambda_m)}$ is unitarily equivalent to a $n_m \times n_m$ Borel function-valued upper triangular matrix A_m. Each entry on the main diagonal is the same and *-cyclic. And $\bigcup_{m=1}^{s} \sigma(A_m) = \sigma(A)$.

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- $A_m(\lambda)$ is strongly irreducible a. e. on $\sigma(A_m)$ for $1 \leq m \leq s$,

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Uniqueness of S. I. D. up to similarity

Theorem 3.1

If an operator A is stated as in (II), then

Uniqueness of S. I. D. up to similarity

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1. The ordered K_0 group of $\{A\}'$, $(K_0(\{A\}'), V(\{A\}'), \alpha)$, is in the form

Uniqueness of S. I. D. up to similarity

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1. The ordered K_0 group of $\{A\}'$, $(K_0(\{A\}'), V(\{A\}'), \alpha)$, is in the form
   - $K_0(\{A\}') \cong \{f(\lambda) \in \mathbb{Z}^{(\sum_{m=1}^{s} k_m \chi_{\Lambda_m}(\lambda))} | f \text{ is bounded Borel on } \sigma(A).\}$.

Uniqueness of S. I. D. up to similarity

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2. The S. I. D. of A is unique up to similarity.

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Theorem 3.2

Assume that A and B are both as in (II), and $V(\{A\}')$ satisfies

If the following conditions hold

then A is similar to B.

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If the following conditions hold

- $(K_0(\{A \oplus B\}'), V(\{A \oplus B\}'), \beta) \cong (K_0(\{A\}'), V(\{A\}'), \alpha)$,

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Single generator in type II_1 factors

Operator theory in type II_1 factors

Type II_1 factors has received a lot of attention since V. Jones initiated subfactor theory of type II_1 factor in the 1980's.
Many people began to consider operator theory in type II_1 factors since then. To consider invariant subspace problem in type II_1 factors, K. Dykema and U. Haagerup introduced DT-operators [9]. They showed a DT-operator generates a type II_1 factor.
In this section, we pay attention to operators in type II_1 factors to find similarity invariants in type II_1 factors.

Single generator of type II_1 factor

Single generator of irrational rotation algebra

Let θ be an irrational number in $(0, 1)$.
Denote by $\mathscr{A}_\theta$ the universal von Neumann algebra generated by two unitary operators U and V satisfying the relation

$$VU = e^{2\pi i\theta} UV.$$

There is an invertible operator X in $\mathscr{A}_\theta$ such that the operator XUX^{-1} generates $\mathscr{A}_\theta$. By the way, $\mathscr{A}_\theta$ is a hyperfinite type II_1 factor, and $\{XUX^{-1}\}'$ contains no minimal idempotents.

Irreducibility relative to type II_1 factor

Definition (Irreducibility relative to type II_1 factors)

An operator A in a type II_1 factor $\mathscr{M}$ is said to be irreducible relative to $\mathscr{M}$, if the relative commutant $\{A\}' \cap \mathscr{M}$ of A contains no nontrivial projection.

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Example

We know that every hyperfinite type II_1 factor has a single generator. And every single generator is irreducible relative to the corresponding hyperfinite type II_1 factor.

Irreducibility relative to type II_1 factor

Theorem (Normal operators in $(\mathscr{M}, \tau)$)

Let N be a normal operator in a type II_1 factor $(\mathscr{M}, \tau)$. If there are three spectral projections P_i of N satisfies

Then there is an invertible operator X in $\mathscr{M}$ such that the operator XNX^{-1} is irreducible relative to $(\mathscr{M}, \tau)$.

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Multiplicities of operators in $\mathscr{M}$

Property (About Multiplicity)

Many operators in a type II_1 factor $\mathscr{M}$ are of multiplicity ∞, such as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I_\infty .$$

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SI integral in $\mathscr{M}$

In order to describe the operator properties of SI integrals in $\mathscr{M}$, we need to introduce the following theorem.

Multiplicities of operators in $\mathscr{M}$

Theorem

Let $A = \int_\Lambda^\oplus A(\lambda)\mu d(\lambda)$ be a strongly irreducible direct integral, and $A(\lambda)$ acts on infinite dimensional Hilbert space $\mathscr{H}(\lambda)$ a. e. on Λ. Then

$$W^*(A) \cong \bigoplus_1^{n_1}(\mathscr{L}(\mathscr{H}) \otimes I_\infty)\bigoplus_{i=1}^{n_2}(\mathscr{L}(\mathscr{H}) \otimes L^\infty(\Lambda_i, \mu_i)),$$

where $1 \leq n_1, n_2 \leq \infty$.

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SI integral in $\mathscr{M}$

By the above theorem, we obtain that every operator in $\mathscr{M}$ does not contain "the infinite order" part, following from that a type II_1 factor is finite.

About similarity invariant

SI integral form in a type II_1 factor $\mathscr{M}$

For convenience, we only consider the following forms of SI integrals in a type II_1 factor $\mathscr{M}$:

$$\begin{aligned} A &= \Big(\bigoplus_{m=1}^{s}\bigoplus_{j=1}^{k_m}\int_{\Lambda_m}\begin{pmatrix} N_{\nu_m} & M_{\phi_{12,m_j}} & \cdots & M_{\phi_{1m_j,m_j}} \\ 0 & N_{\nu_m} & \cdots & M_{\phi_{2m_j,m_j}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{\nu_m} \end{pmatrix}_{m_j\times m_j}(\lambda)\,\nu_m(d\lambda)\Big)\otimes I_\infty, \\ &= T\otimes I_\infty \end{aligned} \tag{III}$$

About similarity invariant

Theorem

Let operators A and B are in $\mathscr{M}$, and $A = T_1 \otimes I_\infty$ and $B = T_2 \otimes I_\infty$ are as in (III). If T_1 and T_2 satisfy the conditions:

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- $(K_0(\{T_1 \oplus T_2\}'), V(\{T_1 \oplus T_2\}'), \beta) \cong (K_0(\{T_1\}'), V(\{T_1\}'), \alpha)$ and $\beta(I) = 2t_{m_1} e_1 + \cdots + 2t_{m_{k_m}} e_{k_m}$ a. e. on $\mathrm{spt}(\nu_m)$ for $1 \leq i \leq k_m$, $1 \leq m \leq s$.

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End

Thank you !

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