GEORGE A. ELLIOTT
University of Toronto

Towards a theory of classification

Close to a century of progress in the understanding of operator algebras—both C*-algebras and von Neumann algebras—both their structure in general and the particular phenomena that arise in connection with the question of classification—suggest that it may be appropriate to consider what exactly the notion of classification means.

The question of when a category can be simplified, without losing information concerning isomorphism, is an important one. In general, it is not useful just to pass to isomorphism classes: not only is the natural topological or Borel structure that is usually present destroyed in non-trivial cases, but also, if one is interested in morphisms other than isomorphisms, the equivalence classes of morphisms do not in general form a category (the product of two equivalence classes will not be a single equivalence class).

Even if one does not coalesce objects, but just agrees to identify morphisms if they differ by an automorphism, one still does not obtain a category (for the same reason). In categories of algebras or groups, in which there is a very natural subgroup of the automorphism group, namely, the inner automorphisms, an interesting possibility appears: identify morphisms if they differ by an inner automorphism. The resulting equivalence classes in fact form a category (with the original objects—but recall that the objects are not very important in a category, and may differ widely between equivalent categories).

If the space of morphisms between two objects has a topology, as indeed it does for various categories of algebras or groups, then it is natural to consider the closures of equivalence classes of morphisms rather than the equivalence classes themselves. If the composition of morphisms is continuous, then one again obtains a category, by defining the product of the closures of two equivalence classes to be the closure of the product (provided that, as was observed above in the case of equivalence modulo inner automorphisms, the product of two equivalence classes is again an equivalence class).

It is an interesting question to what extent this topological quotient category will still distinguish the isomorphism classes of the original category. Simple axioms, which hold in the case of arbitrary separable C*-algebras, or countably generated algebras, or countable groups, among other familiar categories, ensure that it does. The question then is finding a suitable concrete realization of this abstract classifying category. (The history of the classification of C*-algebras can be described in this way.)