Spectrum and Essential Spectrum of Toeplitz Operators

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This is a joint work with Carl Sundberg.
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The spectrum \( \sigma(T) \) of a bounded linear operator \( T \) acting on a Hilbert space \( H \) is the set of complex numbers \( \lambda \) such that \( \lambda I - T \) does not have an inverse that is a bounded linear operator.

If \( H = \mathbb{C}^n \), \( T \) can be viewed as a matrix

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

and so its spectrum consists of eigenvalues of the matrix. But if \( H \) is an infinite dimensional Hilbert space, the spectrum of its bounded operator \( T \) may have more numbers than its eigenvalues \( \sigma_p(T) \).
The essential spectrum of $T$, usually denoted $\sigma_e(T)$, is the set of all complex numbers $\lambda$ such that $\lambda I - T$ is not a Fredholm operator.

The Calkin algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the algebra of bounded linear operators on $H$ modulo the ideal of compact operators on $H$. 
The essential spectrum of $T$, usually denoted $\sigma_e(T)$, is the set of all complex numbers $\lambda$ such that $\lambda I - T$ is not a Fredholm operator.

Here, an operator is Fredholm if its range is closed and its kernel and cokernel are finite-dimensional.
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Here, an operator is Fredholm if its range is closed and its kernel and cokernel are finite-dimensional.

$$\sigma_e(T) = \{ \lambda \in \mathbb{C} : [\lambda I - T] \text{ is not invertible in } \mathcal{C}(H) \}$$

Calkin algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$

$\mathcal{B}(H)$: the algebra of bounded linear operators on $H$.

$\mathcal{K}(H)$: the ideal of compact operators on $H$. 

If $\lambda$ is not in $\sigma_e(T)$, $T - \lambda I$ is Fredholm. The Fredholm index is defined by

$$ind(T - \lambda I) = \dim Ker(T - \lambda I) - \dim Ker(T - \lambda I)^*.$$
If \( \lambda \) is not in \( \sigma_e(T) \), \( T - \lambda I \) is Fredholm. The Fredholm index is defined by

\[
\text{ind}(T - \lambda I) = \dim \ker(T - \lambda I) - \dim \ker(T - \lambda I)^*. 
\]


**Theorem**

Let \( \Omega \) be a connected component of \( C \setminus \sigma_e(T) \) such that \( \text{ind}(T - \lambda I) = 0 \) for each \( \lambda \in \Omega \). Then one of the following holds:

(a) \( \Omega \cap \sigma(T) \) is empty.

(b) \( \Omega \subset \sigma(T) \).

(c) \( \Omega \cap \sigma(T) \) is a countable set of isolated eigenvalues of \( T \), each having finite multiplicity.

Furthermore the intersection of \( \sigma(T) \) with the unbounded component of \( C \setminus \sigma_e(T) \) is a countable set of isolated eigenvalues of \( T \), each of which has finite multiplicity.
A Toeplitz operator on the Hardy space is the compression of a multiplication operator on the circle to the Hardy space.
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Let \( \partial D \) be the circle, with the standard Lebesgue measure, and \( L^2(\partial D) \) be the Hilbert space of square-integrable functions. A bounded measurable function \( \phi \) on \( \partial D \) defines a multiplication operator \( M_\phi \) on \( L^2(\partial D) \). Let \( P \) be the projection from \( L^2(\partial D) \) onto the Hardy space \( H^2 \). The Toeplitz operator with symbol \( \phi \) is defined by

\[
T_\phi = P M_\phi \big|_{H^2}
\]
A bounded operator on $H^2$ is Toeplitz if and only if its matrix representation, in the basis $\{z^n\}_{0}^{\infty}$, has constant diagonals:

$$
\begin{bmatrix}
a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots & \cdots \\
a_1 & a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\
a_2 & a_1 & a_0 & a_{-1} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
$$
Let $dA$ denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The *Bergman space* $L^2_a$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$:
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$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n + 1} < \infty.$$
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and

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$

Let $e_n = \sqrt{n+1}z^n$. Then $\{e_n\}_{0}^{\infty}$ form an orthonormal basis of the Bergman space $L^2_a$. 
For $\phi \in L^{\infty}(\mathbb{D}, dA)$ where $dA$ is normalized area measure on $\mathbb{D}$, the Toeplitz operator $T_\phi$ with symbol $\phi$ is the operator on $L^2_a$ defined by

$$T_\phi f = P(\phi f);$$

here $P$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a$. Note that if $\phi \in H^\infty$ (the set of bounded analytic functions on $\partial \mathbb{D}$), then $T_\phi$ is just the operator of multiplication by $\phi$ on $L^2_a$. 
Operator Theory in Function Spaces

Kehe Zhu
Let $e_n = \sqrt{n+1}z^n$ and $\phi(z) = \sum_{j=-\infty}^{-1} a_j \bar{z}^{|j|} + \sum_{j=0}^{\infty} a_j z^j$.

$$\langle T_\phi e_i, e_j \rangle = \sqrt{i + 1} \sqrt{j + 1} a_{j-i} \langle z^i, z^j \rangle = a_{j-i} \sqrt{\frac{i + 1}{j + 1}}.$$  

On the basis $\{e_n\}$, the matrix representation of the Toeplitz operator $T_\phi$ on the Bergman space is given by

$$
\begin{bmatrix}
a_0 & \sqrt{\frac{2}{1}} a_{-1} & \sqrt{\frac{3}{1}} a_{-2} & \sqrt{\frac{4}{1}} a_{-3} & \cdots & \cdots \\
\sqrt{\frac{1}{2}} a_1 & a_0 & \sqrt{\frac{3}{2}} a_{-1} & \sqrt{\frac{4}{2}} a_{-2} & \cdots & \cdots \\
\sqrt{\frac{1}{3}} a_2 & \sqrt{\frac{2}{3}} a_1 & a_0 & \sqrt{\frac{4}{3}} a_{-1} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}.$$

(a) \( T_{\alpha\phi + \beta\psi} = \alpha T_\phi + \beta T_\psi \).

(b) If \( \phi \) is in \( H^\infty \), then

\[
T_\psi T_\phi = T_{\psi\phi}.
\]

(c) If \( \overline{\psi} \) is in \( H^\infty \), then

\[
T_\psi T_\phi = T_{\psi\phi}.
\]

(d) \( T_\phi^* = T_{\overline{\phi}} \).

(e) If \( \phi \geq 0 \), then \( T_\phi \geq 0 \).
If $\phi$ is continuous on the unit circle $\partial D$ and does not vanish on $\partial D$, then $T\phi$ is Fredholm and

$$\text{ind}(T\phi) = n(\phi(\partial \mathbb{D}), 0).$$
Fredholm index for Toeplitz operator

If \( \phi \) is continuous on the unit circle \( \partial D \) and does not vanish on \( \partial D \), then \( T_\phi \) is Fredholm and

\[
\text{ind}(T_\phi) = n(\phi(\partial \mathbb{D}), 0).
\]

For a closed curve \( \gamma \) in the complex plane \( \mathbb{C} \) and \( a \in \mathbb{C} \setminus \gamma \), define the winding number of the curve \( \gamma \) with respect to \( a \) to be

\[
n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}.
\]
Bergman shift

On the basis \( \{ e_n = \sqrt{n+1}z^n \} \), the Toeplitz operator \( T_z \) with symbol \( z \) is a weighted shift operator, called the Bergman shift:

\[
T_z e_n = \sqrt{\frac{n+1}{n+2}} e_{n+1},
\]

and hence \( T_{\bar{z}} \) is a backward weighted shift:

\[
T_{\bar{z}} e_n = \begin{cases} 
0 & n = 0 \\
\sqrt{\frac{n}{n+1}} e_{n-1} & n > 0 
\end{cases}
\]

The matrix representation of the Toeplitz operators \( T_{1-|z|^2} = I - T_z^* T_z \) is given by

\[
\begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & \cdots & \cdots \\
0 & \frac{1}{3} & 0 & 0 & \cdots & \cdots \\
0 & 0 & \frac{1}{4} & 0 & \cdots & \cdots \\
\vdots & \cdots & \cdots & \ddots & \cdots & \cdots \\
\end{bmatrix}
\]
Differences between $H^2$ and $L^2_a$

**Theorem (Coburn Theorem)**

If $T_\phi \neq 0$ on the Hardy space, either $\ker T_\phi = \{0\}$ or $\ker T_\phi^* = \{0\}$.

**Question**

Does Coburn theorem hold on the Bergman space?

No! On the Bergman space, both $\ker T_1 - |z|^2 - 1/2$ and $\ker T_1^* - |z|^2 - 1/2$ contain the function 1.

But $1 - |z|^2$ is not harmonic in the unit disk and $T_1 - |z|^2$ is compact!

**Question**

Does Coburn theorem hold on the Bergman space for $T_\phi$ even if $\phi$ is harmonic on the unit disk?

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Does Coburn theorem hold on the Bergman space for $T_\phi$ even if $\phi$ is harmonic on the unit disk?
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Widom Theorem and Douglas Theorem

**Theorem (Widom Theorem)**

The spectrum $\sigma(T_\phi)$ of a Toeplitz operator on the Hardy space is connected.
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Theorem (Douglas Theorem)

The essential spectrum $\sigma_e(T_\phi)$ of a Toeplitz operator on the Hardy space is also connected.
Question

*Is the spectrum $\sigma(T_\phi)$ of a Toeplitz operator on the Bergman space connected?*
<table>
<thead>
<tr>
<th>Question</th>
</tr>
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<tbody>
<tr>
<td><strong>Is the spectrum</strong> $\sigma(T_\phi)$ <strong>of a Toeplitz operator on the Bergman space connected?</strong></td>
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$T_{1-|z|^2}$ is compact with the spectrum $\{\frac{1}{2}, \frac{1}{3}, \cdots\} \cup \{0\}$. Hence $\sigma(T_{1-|z|^2})$ is disconnected.
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Compact Toeplitz operators on the Hardy space and the Bergman space

**Theorem**

On the Hardy space, $T_\phi$ is compact if and only if $\phi = 0$.

**Theorem (Axler-Zheng)**

For $\phi \in L^\infty(D)$, $T_\phi$ is compact on the Hardy space if and only if

$$\lim_{|z| \to 1} \int_D \phi(w) \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} dA(w) = 0.$$
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If $\phi$ is harmonic on the unit disk, then

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\lim_{|z| \to 1} \int_D \phi(w) \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} dA(w) = 0
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implies that $\phi = 0$ on $\partial D$ and hence $\phi = 0$ on the unit disk.
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There is no nontrivial compact Toeplitz operator with bounded harmonic symbol on the Bergman space.
Is the spectrum $\sigma(T_\phi)$ of a Toeplitz operator on the Bergman space connected if $\phi$ is bounded and harmonic on the unit disk?
Revised Questions

Question

Is the spectrum $\sigma(T_\phi)$ of a Toeplitz operator on the Bergman space connected if $\phi$ is bounded and harmonic on the unit disk?

Sundberg’s conjecture: Yes! (Problem 7.10 in V.P. Havin and N.K. Nikolski (Eds), Linear and Complex Analysis Problem Book 3, Lecture notes in Mathematics 1573, 1994).

7.10

TOEPLITZ OPERATORS ON THE BERGMAN SPACE

C. Sundberg

Let $A^2$ denote the Bergman space of analytic functions in $L^2(D)$, and let $P$ be the orthogonal projection of $L^2(D)$ onto $A^2$. For $\varphi \in L^\infty(D)$ we define the Toeplitz operator with symbol $\varphi$ by $T_\varphi = P(\varphi f)$. In general the behaviour of these operators may be quite different from that of the Toeplitz operators on the Hardy space $H^2$. However it is shown in [1] that Toeplitz operators on $A^2$ with harmonic symbols behave quite similarly to those on $H^2$, and one can prove analogues for this class of many results about Toeplitz operators on $H^2$.

An important result about Toeplitz operators on $H^2$ is Widom’s Theorem, which states that the spectrum of such an operator is connected ([2]). This suggests our problem.

Conjecture. A Toeplitz operator on $A^2$ with harmonic symbol has a connected spectrum.
Question

Is the essential spectrum $\sigma_e(T_\phi)$ of a Toeplitz operator on the Bergman space connected if $\phi$ is bounded and harmonic on the unit disk?
Revised Questions

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Is the essential spectrum $\sigma_e(T_{\phi})$ of a Toeplitz operator on the Bergman space connected if $\phi$ is bounded and harmonic on the unit disk?

This was an open question in (G. McDonald and C. Sundberg, Indiana Univ. Math. J. 28 (1979)).
Supports for Sundberg’s conjecture

Let $\phi$ be in $H^\infty(D)$. 

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If $\lambda$ is not in the closure of $\phi(D)$, then $\frac{1}{\phi - \lambda}$ is in $H^\infty(D)$ and

$$T_{\phi - \lambda} T_{(\phi - \lambda)^{-1}} = T_{(\phi - \lambda)^{-1}} T_{\phi - \lambda} = I.$$
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If $\lambda = \phi(a)$ for some $a \in D$, then

$$T^*_{\phi - \lambda} k_a = 0.$$
Let \( \phi \) be in \( H^\infty(D) \).

If \( \lambda \) is not in the closure of \( \phi(D) \), then \( \frac{1}{\phi-\lambda} \) is in \( H^\infty(D) \) and

\[
T_{\phi-\lambda} T_{(\phi-\lambda)^{-1}} = T_{(\phi-\lambda)^{-1}} T_{\phi-\lambda} = I.
\]

If \( \lambda = \phi(a) \) for some \( a \in D \), then

\[
T_{\phi-\lambda}^* k_a = 0.
\]

Hence

(a) If \( \phi \) is analytic on the unit disk, then

\[
\sigma(T_{\phi}) = \text{clos} \phi(D).
\]
(b) If $\phi$ is real and harmonic on the unit disk, then

$$\sigma(T\phi) = [\inf \phi, \sup \phi].$$
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$$\sigma(T\phi) = [\inf \phi, \sup \phi].$$

(c) If $\phi$ is harmonic and has piecewise continuous boundary values, then $\sigma_e(T\phi)$ consists of the path formed boundary values of $\phi$ by joining the one-sided limits at discontinuities by straight line segments and hence $\sigma_e(T\phi)$ is connected.
(b) If $\phi$ is real and harmonic on the unit disk, then

$$\sigma(T\phi) = [\inf \phi, \sup \phi].$$

(c) If $\phi$ is harmonic and has piecewise continuous boundary values, then $\sigma_e(T\phi)$ consists of the path formed boundary values of $\phi$ by joining the one-sided limits at discontinuities by straight line segments and hence $\sigma_e(T\phi)$ is connected.

(b) and (c) are contained in (G. McDonald and C. Sundberg, *Indiana Univ. Math. J.* 28 (1979)).
We hope to construct $\phi$ having the following properties:

(a) $\phi(z)$ is a rational function with poles outside of the closure of the unit disk.
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(c) \( \sigma_e(T_h) = h(\partial D) \).

(d) 0 is an isolated eigenvalue of \( T_h \).
Eigenvectors of $T_h$ for the eigenvalue 0

Let $f$ be an eigenvector for $T_h$ for the eigenvalue 0. Then

\[
0 = T_h f(z) = T\bar{z} f(z) + T\phi(z) f(z) = \frac{1}{z^2} \int_0^z w f'(w) dw + \phi(z) f(z).
\]
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\]

**Lemma**

For $f$ in the Bergman space $L_a^2$,

\[
T\bar{z}f(z) = \frac{1}{z^2} \int_0^z w f'(w) dw.
\]
\[ \frac{1}{z^2} \int_0^z wf'(w)dw + \phi(z)f(z) = 0 \]

This is equivalent to the following first order differential equation

\[ (1 + z\phi(z))f'(z) = -(2\phi(z) + z\phi'(z))f(z). \quad (3) \]

For a fixed \( 0 < r < 1 \), we want

(a) a rational function \( \eta(z) \) with poles outside the closure of the unit disk such that

\[ 2\phi(z) + z\phi'(z) = (z - r)\eta(z); \]
\[ \frac{1}{z^2} \int_0^z wf'(w)dw + \phi(z)f(z) = 0 \]

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(b) \(1 + z\phi(z)\) has a simple zero at \(z = r\) and no other zeros in \(\overline{D}\).

Write

\[ \psi(z) = \frac{1 + z\phi(z)}{z - r}. \]

Then \(\psi\) is a rational function with poles outside of the closure of the unit disk.
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\]

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(3) becomes

\[
\frac{f'(z)}{f(z)} = -\frac{\eta(z)}{\psi(z)}.
\]
\[(1 + z\phi(z))f'(z) = -(2\phi(z) + z\phi'(z))f(z)\]

A solution of the above equation in the Bergman space $L^2_a$ is given by

\[f(z) = \exp\left[-\int_0^z \frac{\eta(w)}{\psi(w)} dw\right].\]

Thus $f$ is an eigenvector of $T_h$ for the eigenvalue equal to 0.
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\[f(z) = \exp\left[-\int^z_0 \frac{\eta(w)}{\psi(w)} \, dw\right].\]

Thus \(f\) is an eigenvector of \(T_h\) for the eigenvalue equal to 0.
Since \(\sigma_e(T_h) = h(\partial D)\) and

\[\text{ind}(T_h) = n(h(\partial \mathbb{D}), 0),\]

we want that 0 is an isolated eigenvalue of \(T_h\) to hope

(c) The winding number

\[n(h(\partial \mathbb{D}), 0) = 0.\]
\[(1 + z\phi(z))f'(z) = -(2\phi(z) + z\phi'(z))f(z)\]

A solution of the above equation in the Bergman space \(L^2_a\) is given by

\[f(z) = \exp\left[-\int_0^z \frac{\eta(w)}{\psi(w)} \, dw\right].\]

Thus \(f\) is an eigenvector of \(T_h\) for the eigenvalue equal to 0. Since \(\sigma_e(T_h) = h(\partial D)\) and

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(c) The winding number

\[n(h(\partial \mathbb{D}), 0) = 0.\]

\[\dim \ker T_h = \dim \ker T_h^*.\]
Lemma

For each $0 < r < 1$, there exists a rational function $\phi(z)$ with poles outside $\overline{D}$ such that

(a) $2\phi(r) + r\phi'(r) = 0$.

(b) $1 + z\phi(z)$ has a simple zero at $z = r$ and no other zeros in $\overline{D}$.

(c) The winding number

$$n(h(\partial D), 0) = 0$$

where $h = \overline{z} + \phi(z)$. 
Sketch of Proof

\[ \frac{1}{\sqrt{2}} \leq r \leq 1 \]

\[ -\Psi(z) \]

\[ z/(\sqrt{2}r) \]

\[ \psi \]

\[ \psi(0) = 1/r, \quad \psi(r) = -1/r \]
Disconnected Spectrum

Theorem

Let \( h(z) = \bar{z} + \phi(z) \) Then 0 is an eigenvalue of \( T_h \) and is an isolated point of \( \sigma(T_h) \). Hence \( \sigma(T_h) \) is disconnected.

(1) Since \( h \) is continuous on the closure of the unit disk, then

\[
\sigma_e(T_h) = h(\partial \mathbb{D}).
\]
Theorem

Let $h(z) = \bar{z} + \phi(z)$ Then $0$ is an eigenvalue of $T_h$ and is an isolated point of $\sigma(T_h)$. Hence $\sigma(T_h)$ is disconnected.

(1) Since $h$ is continuous on the closure of the unit disk, then

$$\sigma_e(T_h) = h(\partial\mathbb{D}).$$

(2) $0 \in \sigma_p(T_h) \cap \Omega$ where

$$\Omega = \{ \lambda \notin \sigma_e(T_h) : \text{ind}(T_h - \lambda I) = 0 \} = \{ \lambda \notin h(\partial\mathbb{D}) : n(h(\partial\mathbb{D}), \lambda) = 0 \}.$$
Disconnected Spectrum

**Theorem**

Let $h(z) = \bar{z} + \phi(z)$ Then 0 is an eigenvalue of $T_h$ and is an isolated point of $\sigma(T_h)$. Hence $\sigma(T_h)$ is disconnected.

(1) Since $h$ is continuous on the closure of the unit disk, then 

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(2) $0 \in \sigma_p(T_h) \cap \Omega$ where

$$\Omega = \{ \lambda \notin \sigma_e(T_h) : \text{ind}(T_h - \lambda I) = 0 \}$$

$$= \{ \lambda \notin h(\partial \mathbb{D}) : n(h(\partial \mathbb{D}), \lambda) = 0 \}.$$

Sketch of the proof

Want: $\Omega \cap \sigma_p(T_h)$ is countable.
Cauchy's argument principle

if \( f(z) \) is a meromorphic function inside and on some closed contour \( C \), and \( f \) has no zeros or poles on \( C \), then

\[
\oint_{C} \frac{f'(z)}{f(z)} \, dz = 2\pi i(N - P)
\]
\[ \lambda \in \Omega \cap \sigma_p(T_h), \quad n(h(\partial \mathbb{D}), \lambda) = 0 \]

Since \( \frac{1}{z} + \phi(z) \) has a simple pole at \( z = 0 \) and no other poles in the unit disk \( \mathbb{D} \), the argument principle tells us that if \( \lambda \) is in \( \Omega \cap \sigma_p(T_h) \), there is a unique point \( z_\lambda \) in \( \mathbb{D} \) such that

\[
\frac{1}{z_\lambda} + \phi(z_\lambda) = \lambda.
\]
\( \lambda \in \Omega \cap \sigma_p(T_h), \ n(h(\partial \mathbb{D}), \lambda) = 0 \)

Since \( \frac{1}{z} + \phi(z) \) has a simple pole at \( z = 0 \) and no other poles in the unit disk \( \mathbb{D} \), the argument principle tells us that if \( \lambda \) is in \( \Omega \cap \sigma_p(T_h) \), there is a unique point \( z_\lambda \) in \( \mathbb{D} \) such that

\[
\frac{1}{z_\lambda} + \phi(z_\lambda) = \lambda.
\]

As \( \lambda \) is an eigenvalue of \( T_h \), there is a nonzero function \( g \) in the Bergman space \( L^2_a \) such that

\[
\begin{align*}
\lambda g &= T_h g(z) \\
&= T_{\bar{z}} g(z) + T_{\phi(z)} g(z) \\
&= \frac{1}{z^2} \int_0^z wg'(w) dw + \phi(z)g(z).
\end{align*}
\]
\[ \lambda \in \Omega \cap \sigma_p(T_h), \; n(h(\partial \mathbb{D}), \lambda) = 0 \]

Since \( \frac{1}{z} + \phi(z) \) has a simple pole at \( z = 0 \) and no other poles in the unit disk \( \mathbb{D} \), the argument principle tells us that if \( \lambda \) is in \( \Omega \cap \sigma_p(T_h) \), there is a unique point \( z_\lambda \) in \( \mathbb{D} \) such that
\[
\frac{1}{z_\lambda} + \phi(z_\lambda) = \lambda.
\]

As \( \lambda \) is an eigenvalue of \( T_h \), there is a nonzero function \( g \) in the Bergman space \( L^2_a \) such that
\[
\lambda g = T_h g(z) = T_{\bar{z}} g(z) + T_{\phi(z)} g(z) = \frac{1}{z^2} \int_0^z \! w g'(w)dw + \phi(z)g(z).
\]

We solve the above equation to obtain
\[
\frac{g'(z)}{g(z)} = -\frac{2(\phi(z) - \lambda) + z\phi'(z)}{1 + z(\phi(z) - \lambda)}.
\]
\[ \frac{g'(z)}{g(z)} = \frac{-2(\phi(z) - \lambda) + z\phi'(z)}{1 + z(\phi(z) - \lambda)}. \]

This function has a simple pole at \( z = z_\lambda \) with residue

\[ -\frac{2(\phi(z_\lambda) - \lambda) + z_\lambda \phi'(z_\lambda)}{\phi(z_\lambda) - \lambda + z_\lambda \phi'(z_\lambda)} = -1 - \frac{1}{1 - z_\lambda^2 \phi'(z_\lambda)}. \]
\[
\frac{g'(z)}{g(z)} = -\frac{2(\phi(z) - \lambda) + z\phi'(z)}{1 + z(\phi(z) - \lambda)}.
\]

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\[
-\frac{2(\phi(z_\lambda) - \lambda) + z_\lambda\phi'(z_\lambda)}{\phi(z_\lambda) - \lambda + z_\lambda\phi'(z_\lambda)} = -1 - \frac{1}{1 - z^2_\lambda \phi'(z_\lambda)}.
\]

The regularity of \( g(z) \) at \( z = z_\lambda \) forces this residue to be in \( \mathbb{N} = \{0, 1, 2, 3, \cdots \} \) which leads to

\[
z^2_\lambda \phi'(z_\lambda) = 1 + \frac{1}{n + 1}
\]

for some \( n \in \mathbb{N} \).
\[
\frac{g'(z)}{g(z)} = \frac{-2(\phi(z) - \lambda) + z\phi'(z)}{1 + z(\phi(z) - \lambda)}.
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The regularity of \( g(z) \) at \( z = z_\lambda \) forces this residue to be in \( \mathbb{N} = \{0, 1, 2, 3, \cdots \} \) which leads to

\[
z_\lambda^2\phi'(z_\lambda) = 1 + \frac{1}{n + 1}
\]

for some \( n \in \mathbb{N} \). This restricts the set

\[
\Omega \cap \sigma_p(T_h) \subset \{ \lambda : \lambda = \frac{1}{z_\lambda} + \phi(z_\lambda), z_\lambda^2\phi'(z_\lambda) = 1 + \frac{1}{n + 1}, \text{ for some } n \in \mathbb{N} \}
\]

to be a countable set.
\[
g'(z) \frac{g(z)}{g(z)} = -2(\phi(z) - \lambda) + z\phi'(z) \frac{1}{1 + z(\phi(z) - \lambda)}.
\]

This function has a simple pole at \( z = z_\lambda \) with residue

\[
-2(\phi(z_\lambda) - \lambda) + z_\lambda\phi'(z_\lambda) \frac{1}{\phi(z_\lambda) - \lambda + z_\lambda\phi'(z_\lambda)} = -1 - \frac{1}{1 - z_\lambda^2\phi'(z_\lambda)}.
\]

The regularity of \( g(z) \) at \( z = z_\lambda \) forces this residue to be in \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \) which leads to

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\]

to be a countable set. Thus 0 is an isolated point in \( \sigma(T) \). Hence we conclude that the spectrum \( \sigma(T_h) \) is disconnected.
Unitary operator $U_z$

For $z \in \mathbb{D}$, let $\phi_z$ be the analytic map of $\mathbb{D}$ onto $\mathbb{D}$ defined by

$$
\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.
$$

(4)
For \( z \in \mathbb{D} \), let \( \phi_z \) be the analytic map of \( \mathbb{D} \) onto \( \mathbb{D} \) defined by

\[
\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.
\]

(4)

For \( z \in \mathbb{D} \), let \( U_z: L^2_a \to L^2_a \) be the unitary operator defined by

\[
U_zf = (f \circ \phi_z)\phi_z'.
\]
For $z \in \mathbb{D}$, let $\phi_z$ be the analytic map of $\mathbb{D}$ onto $\mathbb{D}$ defined by

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$  \hfill (4)

For $z \in \mathbb{D}$, let $U_z : L^2_a \to L^2_a$ be the unitary operator defined by

$$U_z f = (f \circ \phi_z)\phi_z'.$$

Notice that $U_z^* = U_z^{-1} = U_z$, so $U_z$ is actually a self-adjoint unitary operator.
**Unitary operator** $U_z$

For $z \in \mathbb{D}$, let $\phi_z$ be the analytic map of $\mathbb{D}$ onto $\mathbb{D}$ defined by

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$  \hspace{1cm} (4)

For $z \in \mathbb{D}$, let $U_z : L^2_a \to L^2_a$ be the unitary operator defined by

$$U_z f = (f \circ \phi_z) \phi_z'.$$

Notice that $U_z^* = U_z^{-1} = U_z$, so $U_z$ is actually a self-adjoint unitary operator.

For $S$ a bounded operator on $L^2_a$, define $S_z$ to be the bounded operator on $L^2_a$ given by conjugation with $U_z$:

$$S_z = U_z S U_z.$$
Let \( M \) be the maximal ideal space of \( H^\infty \), i.e., the set of complex homomorphisms of \( H^\infty \) with \( w^* \)-topology. Then \( M \) is a compact Hausdorff space.
Let $\mathcal{M}$ be the maximal ideal space of $H^\infty$, i.e., the set of complex homomorphisms of $H^\infty$ with $w^*$-topology. Then $\mathcal{M}$ is a compact Hausdorff space. If $z$ is a point in the unit disk $\mathbb{D}$, then point evaluation at $z$ is a multiplicative linear functional on $\mathcal{M}$. Thus we can think of $z$ as an element of $\mathcal{M}$ and the unit disk $\mathbb{D}$ as a subset of $\mathcal{M}$. Carleson’s corona theorem states that $\mathbb{D}$ is dense in $\mathcal{M}$. 
Let $\mathcal{M}$ be the maximal ideal space of $H^\infty$, i.e., the set of complex homomorphisms of $H^\infty$ with $w^*$-topology. Then $\mathcal{M}$ is a compact Hausdorff space. If $z$ is a point in the unit disk $\mathbb{D}$, then point evaluation at $z$ is a multiplicative linear functional on $\mathcal{M}$. Thus we can think of $z$ as an element of $\mathcal{M}$ and the unit disk $\mathbb{D}$ as a subset of $\mathcal{M}$. Carleson's corona theorem states that $\mathbb{D}$ is dense in $\mathcal{M}$.

Suppose $m \in \mathcal{M}$ and $z \mapsto \alpha_z$ is a mapping of $\mathbb{D}$ into some topological space $E$. Suppose also that $\beta \in E$. The notation

$$\lim_{z \to m} \alpha_z = \beta$$

means (as you should expect) that for each open set $X$ in $E$ containing $\beta$, there is an open set $Y$ in $\mathcal{M}$ containing $m$ such that $\alpha_z \in X$ for all $z \in Y \cap \mathbb{D}$. Note that with this notation $z$ is always assumed to lie in $\mathbb{D}$. 
For \( m \in \mathcal{M} \), let \( \phi_m : \mathbb{D} \to \mathcal{M} \) denote the Hoffman map. This is defined by setting
\[
\phi_m(w) = \lim_{z \to m} \phi_z(w)
\]
for \( w \in \mathbb{D} \); here we are taking a limit in \( \mathcal{M} \).
For $m \in \mathcal{M}$, let $\phi_m : \mathbb{D} \to \mathcal{M}$ denote the Hoffman map. This is defined by setting
\[
\phi_m(w) = \lim_{z \to m} \phi_z(w)
\]
for $w \in \mathbb{D}$; here we are taking a limit in $\mathcal{M}$. The existence of this limit, as well as many other deep properties of $\phi_m$, was proved by Hoffman (*Ann. Math.*, **103** (1967)).
The Toeplitz algebra $\mathcal{T}$ is the $C^*$-subalgebra of $B(L^2_a)$ generated by $\{T_g : g \in H^\infty\}$.

**Lemma**

If $S \in T$, the Toeplitz algebra and $m \in M$, then there exists $S_m \in T$ such that

$$\lim_{z \to m} \|Sz - Smf\| = 0$$

for every $f \in L^2_a$. If $S = T_{u_1} \ldots T_{u_n}$, where $u_1, \ldots, u_n \in U$, then

$$S_m = T_{u_1} \circ \phi_m \ldots T_{u_n} \circ \phi_m.$$
The *Toeplitz algebra* \( \mathcal{T} \) is the \( C^* \)-subalgebra of \( \mathcal{B}(L^2_a) \) generated by \( \{ T_g : g \in H^\infty \} \).

**Lemma**

If \( S \in \mathcal{T} \), the Toeplitz algebra and \( m \in \mathcal{M} \), then there exists \( S_m \in \mathcal{T} \) such that

\[
\lim_{z \to m} \| S_z f - S_m f \| = 0
\]

for every \( f \) in \( L^2_a \). If \( S = T_{u_1} \ldots T_{u_n} \), where \( u_1, \ldots, u_n \in \mathcal{U} \), then \( S_m = T_{u_1 \circ \phi_m} \ldots T_{u_n \circ \phi_m} \).
Essential spectrum

Using a similar argument as one in the proof of Theorem 10.3 in (D. Suarez, *Indiana Univ. Math. J.*, 56 (2007)), we have the following theorem.

**Theorem**

If $S \in \mathcal{T}$, the Toeplitz algebra, then

$$\mathbb{C} \setminus \sigma_e(S) = \{ \lambda \in \mathbb{C} : \lambda \notin \bigcup_{m \in \mathcal{M} \setminus \mathbb{D}} \sigma(S_m) \text{ and } \sup_{m \in \mathcal{M} \setminus \mathbb{D}} \| (S_m - \lambda I)^{-1} \| < \infty \}.$$
Thin Blaschke product

To a sequence \( \{z_n\} \) in \( \mathbb{D} \) with \( \sum_{n=1}^{\infty} (1 - |z_n|) < \infty \), there corresponds a Blaschke product

\[
b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in \mathbb{D}.
\]
To a sequence \( \{z_n\}_n \) in \( \mathbb{D} \) with \( \sum_{n=1}^{\infty} (1 - |z_n|) < \infty \), there corresponds a Blaschke product

\[
b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in \mathbb{D}.
\]

A sequence \( \{z_n\}_n \) and its associated Blaschke product are called thin if

\[
\lim_{n \to \infty} \prod_{k \neq n} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| = 1.
\]
Thin Blaschke Product

To a sequence \( \{z_n\}_n \) in \( \mathbb{D} \) with \( \sum_{n=1}^{\infty}(1 - |z_n|) < \infty \), there corresponds a Blaschke product

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A sequence \( \{z_n\}_n \) and its associated Blaschke product are called thin if

\[
\lim_{n \to \infty} \prod_{k \neq n} \left| \frac{z_n - z_k}{1 - \overline{z}_k z_n} \right| = 1.
\]

Hedenmalm (Proc. Amer. Math. Soc., 99 (1987)) showed that for each \( m \) in \( \mathcal{M} \setminus \mathbb{D} \), either

\[
b \circ \phi_m(z) = \lambda_m \quad \text{or} \quad b \circ \phi_m(z) \in \text{Aut}(\mathbb{D})
\]

for some unimodular constant \( \lambda_m \). The latter case actually occurs if \( m \) is in the Gleason part of some point in the closure of zeros of \( b \) in \( \mathbb{D} \).
Theorem

Let $F$ be a continuous function on the closure $\overline{D}$ of the unit disk, $b$ be an infinite thin Blaschke product and $F_b = F \circ b$. Then

$$\sigma_e(T_{F_b}) = \sigma(T_F).$$
**Theorem**

Let $F$ be a continuous function on the closure $\overline{D}$ of the unit disk, $b$ be an infinite thin Blaschke product and $F_b = F \circ b$. Then

$$\sigma_e(T_{F_b}) = \sigma(T_F).$$

**Proof**

Let $S = T_{F_b}$.

For each $m$ in $\mathcal{M}\setminus \mathbb{D}$,

$$S_m = T_{F \circ b \circ \phi_m}.$$
Theorem

Let $F$ be a continuous function on the closure $\overline{\mathbb{D}}$ of the unit disk, $b$ be an infinite thin Blaschke product and $F_b = F \circ b$. Then

$$\sigma_e(T_{F_b}) = \sigma(T_F).$$

Proof

Let $S = T_{F_b}$.

For each $m$ in $\mathcal{M}\setminus\mathbb{D}$,

$$S_m = T_{F \circ b \circ \phi_m}.$$ 

By Hedenmalm’s result above, we have that for each $m$ in $\mathcal{M}\setminus\mathbb{D}$, either

(a) $b \circ \phi_m(z) = \lambda_m$ for some unimodular constant $\lambda_m$ or
(b) $\tau_m = b \circ \phi_m(z) \in Aut(\mathbb{D})$.  

(A) $b \circ \phi_m(z) = \lambda_m$

$S_m$ equals the operator $F(\lambda_m)I$ and hence $\sigma(S_m)$ equals one point $F(\lambda_m)$. Thus

$$\sigma(S_m) \subset F(\partial \mathbb{D}) \subset \sigma(T_F),$$
\( \text{(A) } b \circ \phi_m(z) = \lambda_m \)

\( S_m \) equals the operator \( F(\lambda_m)I \) and hence \( \sigma(S_m) \) equals one point \( F(\lambda_m) \). Thus

\[ \sigma(S_m) \subset F(\partial \mathbb{D}) \subset \sigma(T_F), \]

and for each \( \lambda \) not in \( \sigma(T_F) \),

\[
\| (S_m - \lambda I)^{-1} \| = \frac{1}{|F(\lambda_m) - \lambda|} \leq \frac{1}{\text{dis}(\lambda, \sigma(T_F))}.
\]
\( \tau_m = b \circ \phi_m(z) \in Aut(\mathbb{D}) \)

\[ S_m = T_{F \circ \tau_m} = V_m T_F V_m^* \]

where \( V_m \) is the unitary operator on the Bergman space \( L^2_a \) given by

\[ V_m f(z) = f(\tau_m(z)) \tau'_m(z). \]
(B) \( \tau_m = b \circ \phi_m(z) \in \text{Aut}(\mathbb{D}) \)

\[
S_m = T_{F \circ \tau_m} = V_m T_F V_m^*
\]

where \( V_m \) is the unitary operator on the Bergman space \( L^2_a \) given by

\[
V_m f(z) = f(\tau_m(z)) \tau_m'(z).
\]

Thus \( \sigma(S_m) = \sigma(T_F) \) and for each \( \lambda \) in \( \mathbb{C} \setminus \sigma(S_m) \),

\[
\|(S_m - \lambda I)^{-1}\| = \|V_m T_F^{-1} V_m^*\| = \|T_F^{-1}\|.
\]
So we have

\[ \bigcup_{m \in M \setminus \mathbb{D}} \sigma(S_m) = \sigma(T_F) \]
So we have

\[ \bigcup_{m \in M \setminus \mathbb{D}} \sigma(S_m) = \sigma(T_F) \]

and for each \( \lambda \notin \sigma(T_F) \),

\[ \| (S_m - \lambda I)^{-1} \| \leq \max\{ \frac{1}{\text{dis}(\lambda, \sigma(T_F))}, \| T_{F-\lambda}^{-1} \| \} < \infty. \]
So we have
\[ \bigcup_{m \in \mathcal{M} \setminus \mathbb{D}} \sigma(S_m) = \sigma(T_F) \]
and for each \( \lambda \notin \sigma(T_F) \),
\[ \| (S_m - \lambda I)^{-1} \| \leq \max \left\{ \frac{1}{\text{dis}(\lambda, \sigma(T_F))}, \| T_{F-\lambda}^{-1} \| \right\} < \infty. \]

By the following theorem

**Theorem**

*If \( S \in \mathcal{T} \), the Toeplitz algebra, then*

\[ \mathbb{C} \setminus \sigma_e(S) = \{ \lambda \in \mathbb{C} : \lambda \notin \bigcup_{m \in \mathcal{M} \setminus \mathbb{D}} \sigma(S_m) \text{ and } \sup_{m \in \mathcal{M} \setminus \mathbb{D}} \| (S_m - \lambda I)^{-1} \| < \infty \}. \]

we have that
\[ \sigma_e(T_{F_b}) = \sigma_e(S) = \sigma(T_F). \]
Disconnected Essential Spectrum

**Theorem**

Let $h$ be $\bar{z} + \phi$ such that $\sigma(T_h)$ is disconnected. Let $b$ be an infinite thin Blaschke product and $h_b = h \circ b$. Then

\[ \sigma_e(T_{h_b}) = \sigma(T_h) \]

is disconnected.
Lemma

For each $0 < r < 1$, there exists a rational function $\phi(z)$ with poles outside $\overline{D}$ such that

(a) $2\phi(r) + r\phi'(r) = 0$.

(b) $1 + z\phi(z)$ has a simple zero at $z = r$ and no other zeros in $\overline{D}$.

(c) The winding number

$$n(h(\partial D), 0) = 0$$

where $h = \overline{z} + \phi(z)$.

Proof: For $\frac{1}{\sqrt{2}} < r < 1$, we are going to construct $\phi$ by some conformal mappings.
Proof of Lemma

\[ \frac{1}{\sqrt{2}} < r < 1 \]

\[ -i \Psi(z) \]

\[ \frac{z}{(\sqrt{2}r)} \]

\[ \psi (0) = \frac{1}{r}, \quad \psi (r) = -\frac{1}{r} \]
Proof of Lemma

Let $\lambda$ be the unimodular constant $i^{\frac{2+i}{2-i}} \frac{\sqrt{2}}{1+i}$. Define

$$\chi(z) = \frac{1}{2r} \left( \frac{1+z}{1-z} \right)^2,$$
Let 

\[ \Psi(z) = \chi \left( \frac{\lambda z - \frac{i}{2-i}}{1 + \frac{i}{2+i} \lambda z} \right). \]

Then

\[ \Psi(0) = -\frac{i}{r}, \quad \Psi \left( \frac{1}{\sqrt{2}} \right) = \frac{i}{r}. \]
Now define

$$\psi(z) = -i\psi\left(\frac{1}{\sqrt{2r}}z\right).$$

Since $r > \sqrt{2}/\sqrt{2}$, the poles of $\psi(z)$ are outside $D$. 

\[\psi(0) = 1/r, \quad \psi(r) = -1/r\]
Now define

$$\psi(z) = -i\psi\left(\frac{1}{\sqrt{2}r}\right).$$

Since $r > \frac{1}{\sqrt{2}}$, the poles of $\psi(z)$ are outside $\overline{D}$. 
Since $\chi$ is a conformal map of $\mathbb{D}$ onto $\mathbb{C}\setminus(-\infty, 0]$, $\psi$ is a conformal map of $\mathbb{D}$ onto a region bounded by a simple closed curve and 0 is outside the region. In particular $\psi(\partial \mathbb{D})$ does not wind around 0 and $\psi(z) \neq 0$ for all $z$ in $\overline{\mathbb{D}}$. 

Defining $\phi(z) = (z - r)\psi(z) - 1$ we see that (a) and (b) are satisfied:

(a) $2\phi'(r) + r\phi''(r) = 0$.

(b) $1 + z\phi(z)$ has a simple zero at $z = r$ and no other zeros in $\mathbb{D}$. 


Since $\chi$ is a conformal map of $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, 0]$, $\psi$ is a conformal map of $\mathbb{D}$ onto a region bounded by a simple closed curve and $0$ is outside the region. In particular $\psi(\partial \mathbb{D})$ does not wind around $0$ and $\psi(z) \neq 0$ for all $z$ in $\mathbb{D}$. Defining

$$\phi(z) = \frac{(z - r)\psi(z) - 1}{z},$$

we see that (a) and (b) are satisfied:

(a) $2\phi(r) + r\phi'(r) = 0$.

(b) $1 + z\phi(z)$ has a simple zero at $z = r$ and no other zeros in $\overline{\mathbb{D}}$. 
On $\partial \mathbb{D}$

\[
\overline{z} + \phi(z) = \frac{1}{z} + \phi(z) = \frac{1 + z\phi(z)}{z} = \frac{z - r}{z}\psi(z).
\]

So (c) is satisfied too.
Proof of $T_z f(z) = \frac{1}{z^2} \int_0^z w f'(w) dw$

Lemma

For $f$ in the Bergman space $L^2_a$, 

$$T_z f(z) = \frac{1}{z^2} \int_0^z w f'(w) dw.$$
Proof of \( T_{\bar{z}} f(z) = \frac{1}{z^2} \int_0^z wf'(w)dw \)

**Lemma**

For \( f \) in the Bergman space \( L_a^2 \),

\[
T_{\bar{z}} f(z) = \frac{1}{z^2} \int_0^z wf'(w)dw.
\]

Proof. Note that

\[
\{ e_n = \sqrt{n + 1}z^n \}_{n=0}^{\infty}
\]

is an orthonormal basis of the Bergman space. To prove this lemma, we need only verify the above equality for each \( f(z) = e_n \).
Proof of $T_{\bar{z}}f(z) = \frac{1}{z^2} \int_0^z wf'(w)dw$

**Lemma**

For $f$ in the Bergman space $L^2_a$,

$$T_{\bar{z}}f(z) = \frac{1}{z^2} \int_0^zwf'(w)dw.$$ 

Proof. Note that

$$\{e_n = \sqrt{n+1}z^n\}_{n=0}^{\infty}$$

is an orthonormal basis of the Bergman space. To prove this lemma, we need only verify the above equality for each $f(z) = e_n$. As $T_{\bar{z}}$ is the adjoint of the Bergman shift, we have

$$T_{\bar{z}}e_n = \begin{cases} 
0 & n = 0 \\
\sqrt{\frac{n}{n+1}}e_{n-1} & n > 0 
\end{cases}$$
On the other hand, since $e_n(w) = \sqrt{n + 1}w^n$, an easy calculation gives

$$\int_0^z we'_n(w)dw = \frac{nz^{n+1}}{\sqrt{n + 1}}$$
On the other hand, since \( e_n(w) = \sqrt{n+1}w^n \), an easy calculation gives

\[
\int_0^z we'_n(w)dw = \frac{nz^{n+1}}{\sqrt{n+1}}
\]

Thus we have

\[
\frac{1}{z^2} \int_0^z we'_n(w)dw = \frac{nz^{n-1}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}}e_{n-1},
\]

to obtain

\[
T_z e_n = \frac{1}{z^2} \int_0^z we'_n(w)dw.
\]

This completes the proof of the lemma.