Asymptotic unitary equivalence in $C^*$-algebras

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Abstract

Let $C = C(X)$ be the unital $C^*$-algebra of all continuous functions on a finite CW complex $X$ and let $A$ be a unital simple $C^*$-algebra with tracial rank at most one. We show that two unital monomorphisms $\varphi, \psi : C \to A$ are asymptotically unitarily equivalent, i.e., there exists a continuous path of unitaries $\{u_t : t \in [0, 1]\} \subset A$ such that

$$\lim_{t \to 1} u_t^* \varphi(f) u_t = \psi(f) \text{ for all } f \in C(X),$$

if and only if

$$[\varphi] = [\psi] \text{ in } KK(C, A),$$
$$\tau \circ \varphi = \tau \circ \psi \text{ for all } \tau \in T(A), \text{ and}$$
$$\varphi^\dagger = \psi^\dagger,$$

where $T(A)$ is the simplex of tracial states of $A$ and $\varphi^\dagger, \psi^\dagger : U(M_\infty(A))/DU(M_\infty(A)) \to U(M_\infty(A))/DU(M_\infty(A))$ are induced homomorphisms and where $U(M_\infty(A))$ and $U(M_\infty(C))$ are groups of unitary groups of $M_k(A)$ and $M_k(C)$ for all integer $k \geq 1$, $DU(M_\infty(A))$ and $DU(M_\infty(C))$ are commutator subgroups of $U(M_\infty(A))$ and $U(M_\infty(C))$, respectively.

We actually prove a more general result for the case that $C$ is any general unital AH-algebra.

1 Introduction

In the study of topology, it is fundamentally important to study continuous maps between topological spaces. In the study of $C^*$-algebras, or sometime called the non-commutative topological space, it is essential to study homomorphisms from one $C^*$-algebra to another.

One of the central problems in classification of amenable $C^*$-algebras is to determine how certain equivalence classes of homomorphisms between $C^*$-algebras can be determined by their K-theoretical invariants. In this note, we will study the unital monomorphisms from a unital commutative $C^*$-algebra $C$, or, more general, arbitrary unital AH-algebras, to a simple $C^*$-algebra $A$ with finite tracial rank (see [2,6] below) and consider the question when two given unital monomorphisms $\varphi, \psi : C \to A$ are asymptotically unitarily equivalent, that is, when does there exist a continuous path of unitaries $\{u_t : t \in [0, 1]\} \subset A$ such that

$$\lim_{t \to 1} u_t^* \varphi(f) u_t = \psi(f) \text{ for all } f \in C.$$

If one considers approximately unitary equivalence (recall that the maps $\varphi$ and $\psi$ are approximately unitarily equivalent if there exists a sequence of unitaries $\{u_n\} \subset A$ such that $\lim_{n \to \infty} u_n^* \varphi(f) u_n = \psi(f)$ for all $f \in C$), there are already several results recently:

Let $C$ be a unital AH-algebra and let $A$ be a unital simple $C^*$-algebra with tracial rank zero. It has been shown in [6] by the first author that $\varphi$ and $\psi$ are approximately unitarily equivalent if and only if

$$[\varphi] = [\psi] \text{ in } KL(C, A) \text{ and } \tau \circ \varphi = \tau \circ \psi \text{ for all } \tau \in T(A).$$
And in [17], Ng and Winter showed that the result above still holds if \( C = C(X) \) with \( X \) a second countable, path connected, compact metric space and \( A \) is any simple unital separable nuclear C*-algebra which is real rank zero and \( \mathcal{Z} \)-stable, where \( \mathcal{Z} \) is the Jiang-Su algebra.

Beyond the real rank zero case, in a more recent paper ([14]), it was shown that, if \( A \) is a unital simple C*-algebra with tracial rank at most one, then \( \varphi \) and \( \psi \) are approximately unitarily equivalent if and only if

\[
[\varphi] = [\psi] \quad \text{in} \quad KL(C, A),
\]

\[
\tau \circ \varphi = \tau \circ \psi \quad \text{for all} \quad \tau \in T(A) \quad \text{and} \quad \varphi^\dagger = \psi^\dagger, \tag{1.1}
\]

where \( \varphi^\dagger, \psi^\dagger : U(M_\infty(C)/DU(M_\infty(C))) \to U(M_\infty(A)/DU(M_\infty(A))) \) are induced homomorphisms and \( DU(M_\infty(C)) \) and \( DU(M_\infty(A)) \) are commutator subgroups of \( \bigcup_{k=1}^\infty U(M_k(C)) \) and \( \bigcup_{k=1}^\infty U(M_k(A)) \), respectively.

These results play important roles in the recent progress of the Elliott program of the classification of amenable C*-algebras. It is natural to ask whether approximate unitary equivalence is the same as asymptotic unitary equivalence. It turns out, from a result of Kishimoto and Kumjian ([4]), that, in general, asymptotic unitary equivalence is different from approximate unitary equivalence. In particular, they studied the case that both \( A \) and \( C \) are unital simple \( \mathbb{A}\mathbb{T} \)-algebras of real rank zero.

Then, in [10], the following criterion for asymptotically unitarily equivalent was developed for any unital AH-algebra \( C \) and any simple C*-algebra \( A \) with tracial rank zero: Suppose that \( \varphi, \psi : C \to A \) are two unital monomorphisms. Then \( \varphi \) and \( \psi \) are asymptotically unitarily equivalent if and only if

\[
[\varphi] = [\psi] \quad \text{in} \quad KK(C, A),
\]

\[
\tau \circ \varphi = \tau \circ \psi \quad \text{for all} \quad \tau \in T(A), \tag{1.2}
\]

\[
R_{\varphi, \psi} = 0, \tag{1.3}
\]

where \( R_{\varphi, \psi} \) is the rotation map, which will be defined in 2.8.

It worth to point out that one application of this result is to the study of Voiculescu’s AF-embedding problem: Let \( \Omega \) be a compact metric space and let \( G \) be a finitely generated abelian group. Suppose that \( \Lambda \) is a \( G \) action on \( X \). Then the above mentioned result can be used to prove that \( C(\Omega) \rtimes_\Lambda G \) can be embedded into a unital simple AF-algebra if and only if \( \Omega \) has a faithful \( \Lambda \)-invariant Borel probability measure.

There are other applications. With a method developed by Winter ([21]), the above mentioned asymptotic unitary equivalence result was also used to give an important advance in the Elliott program (see [21], [7] and [13]) for the C*-algebras which might be projectionless. An even further advance was made which allows the class of unital separable amenable simple C*-algebras classified by the conventional Elliott invariant to include C*-algebras which are so-called rationally finite tracial rank and their \( K_0 \)-groups may not have the Riesz interpolation property. The technical key of this advance was the following asymptotic unitary equivalence theorem.

**Theorem 1.1** (Theorem 7.2, [12]). Let \( C \) be a unital simple AH-algebra of slow dimension growth and let \( A \) be any unital simple C*-algebra with tracial rank at most one. Suppose that \( \varphi, \psi : C \to A \) are two unital monomorphisms. Then \( \varphi \) and \( \psi \) are asymptotically unitarily equivalent if and only if

\[
[\varphi] = [\psi] \quad \text{in} \quad KK(C, A),
\]

\[
\tau \circ \varphi = \tau \circ \psi \quad \text{for all} \quad \tau \in T(A),
\]

\[
\varphi^\dagger = \psi^\dagger, \quad \text{and} \quad R_{\varphi, \psi} = 0.
\]
However, while in [12], the theorem above also was proved for certain non-simple AH-algebras, it only includes those unital AH-algebras whose K-theory behave as low dimensional topological spaces. In this paper we will generalize the theorem above so that it will apply to all unital AH-algebras (with no restriction on dimension growth). In particular, it holds for $C = C(X)$ for any compact metric space $X$.

Moreover, in the case that $K_1(C)$ is finitely generated, we also find that the invariant could be simplified. In fact, in Theorem 4.3 below, the conditions that $R\varphi,\psi = 0$ and $\varphi^\dagger = \psi^\dagger$ can be simplified to the condition that $\varphi^\dagger = \psi^\dagger$, i.e., $\varphi$ and $\psi$ induce the same homomorphisms on $\bigcup_{k=1}^\infty U(M_k(C))/DU(M_\infty(C))$. However, we also point out that, in general, this simplification is not possible. A specific example will be presented.

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## 2 Preliminaries

### 2.1. Let $A$ be a unital stably finite C*-algebra. Denote by $T(A)$ the simplex of tracial states of $A$ and denote by $\text{Aff}(T(A))$ the space of all real affine continuous functions on $T(A)$. Suppose that $\tau \in T(A)$ is a tracial state. We will also denote by $\tau$ the trace $\tau \otimes \text{Tr}$ on $M_k(A) = A \otimes M_k(\mathbb{C})$ (for every integer $k \geq 1$), where $\text{Tr}$ is the standard trace on $M_k(\mathbb{C})$.

Denote by $M_\infty(A)$ the set $\bigcup_{k=1}^\infty M_k(A)$, where $M_k(A)$ is regarded as a C*-subalgebra of $M_{k+1}(A)$ by the embedding $M_k(A) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_{k+1}(A)$.

For any projection $p \in M_\infty(A)$, the evaluation $\tau \mapsto \tau(p)$ defines a positive affine function on $T(A)$. This induces a canonical positive homomorphism $\rho_A : K_0(A) \to \text{Aff}(T(A))$.

Denote by $S(A) := C_0((0,1)) \otimes A$ the suspension of $A$, denote by $U(A)$ the unitary group of $A$, and denote by $U(A)_0$ the connected component of $U(A)$ containing the identity.

Let $C$ be another unital C*-algebra and let $\varphi : C \to A$ be a unital *-homomorphism. Denote by $\varphi_T : T(A) \to T(C)$ the continuous affine map induced by $\varphi$, i.e.,

$$\varphi_T(\tau)(c) = \tau \circ \varphi(c)$$

for all $c \in C$ and $\tau \in T(A)$. Denote by $\varphi_2 : \text{Aff}(T(C)) \to \text{Aff}(T(A))$ the map defined by

$$\varphi_2(f)(\tau) = f(\varphi_T(\tau))$$

for all $\tau \in T(A)$.

**Definition 2.2.** Let $A$ be a unital C*-algebra. Denote by $DU(A)$ the subgroup of generated by the commutators of $U(A)$ and denote by $CU(A)$ the closure of $DU(A)$. If $u \in U(A)$, its image in the quotient $U(A)/CU(A)$ will be denoted by $\overline{u}$.

Let $B$ be another unital C*-algebra and let $\varphi : A \to B$ be a unital homomorphism. It is clear that $\varphi$ maps $CU(A)$ into $CU(B)$. Let $\varphi^\dagger$ denote the induced homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$. It is also clear that $\varphi$ maps $DU(A)$ into $DU(B)$. Denote by $\varphi^\dagger : U(A)/DU(A) \to U(B)/DU(B)$ the homomorphism induced by $\varphi$.

Let $n \geq 1$ be any integer. Denote by $U_n(A)$ the unitary group of $M_n(A)$, and denote by $DU_n(A)$ and $CU_n(A)$ the commutator subgroup of $U_n(A)$ and its closure, respectively. Regard $U_n(A)$ as a subgroup of $U_{n+1}(A)$ via the embedding $U_k(A) \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U_{k+1}(A)$, and denote by $U_\infty(A)$ the union of all $U_n(A)$. 


Consider the union \( CU_\infty(A) := \bigcup_n CU_n(A) \). It is then a normal subgroup of \( U_\infty(A) \), and the quotient \( U(\infty)_\infty/CU_\infty(A) \) is in fact isomorphic to the inductive limit of \( U_n(A)/CU_n(A) \) (as abelian groups). Similarly, \( DU_\infty(A) := \bigcup_n DU_n(A) \) is a normal subgroup of \( U_\infty(A) \). We will use \( \varphi^\dagger \) for the homomorphism induced by \( \varphi \) from \( U(\infty)/CU_\infty(A) \) into \( U(\infty)/CU_\infty(B) \), and we will use \( \varphi^\dagger \) for the homomorphism induced by \( \varphi \) from \( U(\infty)/DU_\infty(A) \).

**Remark 2.3.** By Corollary 3.5 of [11], if \( A \) has tracial rank at most one (see 2.6 below), the map natural map

\[
U(A)/CU(A) \to U(M_n(A))/CU(M_n(A))
\]

is an isomorphism for any integer \( n \geq 1 \).

**Definition 2.4.** Let \( A \) be a unital C*-algebra, and let \( u \in U(A)_0 \). Let \( u(t) \in C([0,1],A) \) be a piecewise-smooth path of unitaries such that \( u(0) = u \) and \( u(1) = 1 \). Then the de la Harpe–Skandalis determinant of \( u(t) \) is defined by

\[
\text{Det}(u(t))(\tau) = \frac{1}{2\pi i} \int_0^1 \tau(\frac{du(t)}{dt} - u(t)^*)dt \quad \text{for all } \tau \in T(A),
\]

which induces a homomorphism

\[
\overline{\text{Det}} : U(A)_0 \to \text{Aff}(T(A))/\rho_A(K_0(A)).
\]

The determinant \( \overline{\text{Det}} \) can be extended to a map from \( U(\infty)_0 \) into \( \text{Aff}(T(A))/\rho_A(K_0(A)) \). It is easy to see that the determinant vanishes on the closure of commutator subgroup of \( U(\infty)(A) \).

In fact, by 3.1 of [20], the closure of the commutator subgroup is exactly the kernel of this map, that is, it induces an isomorphism \( \overline{\text{Det}} : U(\infty)_0/CU_\infty(A) \to \text{Aff}(T(A))/\rho_A(K_0(A)) \). Moreover, by (20), one has the following short exact sequence

\[
0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U_\infty(A)/CU_\infty(A) \xrightarrow{\Pi} K_1(A) \to 0 \tag{e 2.1}
\]

which splits (where the embedding of \( \text{Aff}(T(A))/\rho_A(K_0(A)) \) induced by \( (\overline{\text{Det}})^{-1} \). We will fix a splitting map \( s_1 : K_1(A) \to U_\infty(A)/CU_\infty(A) \). The notation \( \Pi \) and \( s_1 \) will be used late without further warning. For each \( \bar{u} \in s_1(K_1(A)) \), select and fix one element \( u_c \in \bigcup_{n=1}^\infty M_n(A) \) such that \( \overline{u_c} = \bar{u} \). Denote this set by \( U_c(A) \). Moreover, in the case that \( A \) is unital, simple and \( TR(A) \leq 1 \) (see 2.6 below), one has that \( U(A)/U_0(A) \to K_1(A) \) is an isomorphism and \( \overline{\text{Det}} : U_0(A)/CU(A) \to \text{Aff}(T(A))/\rho_A(K_0(A)) \) is also an isomorphism. Then one has

\[
0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(A)/CU(A) \to K_1(A) \to 0 \tag{e 2.2}
\]

**Definition 2.5.** Let \( A \) be a unital C*-algebra and let \( C \) be a separable C*-algebra which satisfies the Universal Coefficient Theorem. Recall that \( KL(C,A) \) is the quotient of \( KK(C,A) \) modulo pure extensions. By a result of Dădărlat and Loring in [2], one has

\[
KL(C,A) = \text{Hom}_A(K(C),K(A)), \tag{e 2.3}
\]

where

\[
\overline{K}(B) = (K_0(B) \oplus K_1(B)) \oplus \bigoplus_{n=2} K_0(B,\mathbb{Z}/n\mathbb{Z}) \oplus K_1(B,\mathbb{Z}/n\mathbb{Z})
\]

for any C*-algebra \( B \). Then, in the rest of the paper, we will identify \( KL(C,A) \) with \( \text{Hom}_A(K(C),K(A)) \). Let \( \kappa \in KL(C,A) \). Denote by \( \kappa_i : K_i(C) \to K_i(A) \) the homomorphism given by \( \kappa \) with \( i = 0,1 \).
Definition 2.6. Let $k \geq 0$ be an integer. A unital simple C*-algebra $A$ has \textit{tracial rank at most} $k$, denoted by $\TR(A) \leq k$, if for any finite subset $F \subset A$, any $\epsilon > 0$, and nonzero $a \in A^+$, there exist a nonzero projection $p \in A$ and a C*-subalgebra $I \cong \bigoplus_{i=1}^m C(X_i) \otimes M_r(i)$ with $1_I = p$ for some finite CW-complexes $X_i$ with dimension at most $k$ such that

1. $\|xp - xp\| < \epsilon$ for any $x \in F$,
2. for any $x \in F$, there is $x' \in I$ such that $\|xp - x'\| \leq \epsilon$, and
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Moreover, if the C*-subalgebra $I$ above can be chosen to be a finite dimensional C*-algebra, then $A$ is said to have \textit{tracial rank zero}, and in such case, we write $\TR(A) = 0$. It is a theorem of Guihua Gong [3] that every unital simple AH-algebra with no dimension growth has tracial rank at most one. It has been proved in [12] that every $Z$-stable unital simple AH-algebra has tracial rank at most one. It is shown recently ([15]) that if a unital separable simple C*-algebra $A$ satisfying the UCT has $\TR(A) \leq k$, then $\TR(A) \leq 1$.

Definition 2.7. Let $A$ and $B$ be two unital C*-algebras, and let $\psi$ and $\varphi$ be two unital monomorphisms from $B$ to $A$. Then the \textit{mapping torus} $M_{\varphi,\psi}$ is the C*-algebra defined by

$$M_{\varphi,\psi} := \{f \in C([0,1], A); \; f(0) = \varphi(b) \text{ and } f(1) = \psi(b) \text{ for some } b \in B\}.$$  

For any $\psi, \varphi \in \text{Hom}(B,A)$, denoting by $\pi_0$ the evaluation of $M_{\varphi,\psi}$ at 0, we have the short exact sequence

$$0 \to S(A) \to M_{\varphi,\psi} \to \pi_0 B \to 0.$$  

If $\varphi_{s_i} = \psi_{s_i}$ ($i = 0,1$), then the corresponding six-term exact sequence breaks down to the following two extensions:

$$\eta_i(M_{\varphi,\psi}) : 0 \to K_{i+1}(A) \to K_1(M_{\varphi,\psi}) \to K_i(B) \to 0 \quad (i = 0,1).$$

2.8. Suppose that, in addition,

$$(\epsilon 2.4) \quad \tau \circ \varphi = \tau \circ \psi \text{ for all } \tau \in T(A).$$

For any continuous piecewise smooth path of unitaries $u(t) \in M_{\varphi,\psi}$, consider the path of unitaries $w(t) = u^*(0)u(t)$ in $A$. Then it is a continuous and piecewise smooth path with $w(0) = 1$ and $w(1) = u^*(0)u(1)$. Denote by $R_{\varphi,\psi}(u) = \det(w)$ the determinant of $w(t)$. It is clear with the assumption of $([2,4])$ that $R_{\varphi,\psi}(u)$ depends only on the homotopy class of $u(t)$. Therefore, it induces a homomorphism, denoted by $R_{\varphi,\psi}$, from $K_1(M_{\varphi,\psi})$ to $\text{Aff}(T(A))$. One has the following lemma.

Lemma 2.9 (3.3 of [2], also see [4]). When $([2,4])$ holds, the following diagram commutes:

$$\begin{array}{ccc}
K_0(A) & \xrightarrow{[1]} & K_1(M_{\varphi,\psi}) \\
\rho_A \searrow & & \nearrow R_{\varphi,\psi} \\
& \text{Aff}(T(A)) &
\end{array}$$

Definition 2.10. Fix two unital C*-algebras $A$ and $B$ with $\TR(A) \neq O$. Define $\mathcal{R}_0$ to be the subset of $\text{Hom}(K_1(B), \text{Aff}(T(A)))$ consisting of those homomorphisms $h \in \text{Hom}(K_1(B), \text{Aff}(T(A)))$ for which there exists a homomorphism $d : K_1(B) \to K_0(A)$ such that

$$h = \rho_A \circ d.$$  

It is clear that $\mathcal{R}_0$ is a subgroup of $\text{Hom}(K_1(B), \text{Aff}(T(A)))$.  

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2.11. If $[\varphi] = [\psi]$ in $KK(B, A)$, then the exact sequences $\eta_i(M_{\varphi, \psi})$ ($i = 0, 1$) split. In particular, there is a lifting $\theta : K_1(B) \to K_1(M_{\varphi, \psi})$. Consider the map

$$R_{\varphi, \psi} \circ \theta : K_1(B) \to \text{Aff}(T(A)).$$

If a different lifting $\theta'$ is chosen, then, $\theta - \theta'$ maps $K_1(B)$ into $K_0(A)$. Therefore

$$R_{\varphi, \psi} \circ \theta - R_{\varphi, \psi} \circ \theta' \in \mathcal{R}_0.$$  

Then define

$$\overline{R}_{\varphi, \psi} = [R_{\varphi, \psi} \circ \theta] \in \text{Hom}(K_1(B), \text{Aff}(T(A))) / \mathcal{R}_0.$$  

See 3.4 of [12] for more details.

3 A basic homotopy lemma

The following is taken from Lemma 2.8 of [11].

**Lemma 3.1.** Let $C$ be a unital nuclear $C^*$-algebra. Let $\mathcal{F} \subseteq C$ be a finite subset, $N \in \mathbb{N}$, and $\epsilon > 0$. There then exist a finite subset $\mathcal{G} \subseteq C$ and $\delta > 0$ such that for any unital $C^*$-algebra $A$, any unitary $u \in A$ and any unital homomorphism $\varphi : C \to A$ with

$$\| [\varphi(c), u] \| < \delta, \quad \forall c \in \mathcal{G},$$

there is a unital completely positive linear map $L : C \otimes C(T) \to A$ such that

$$\| L(f \otimes z^n) - \varphi(f)u^n \| < \epsilon, \quad \forall f \in \mathcal{F}, \quad -N \leq n \leq N.$$  

Let $X$ be a metric space. In the rest of the paper, we fix the metric on $X \times T$ to be

$$\text{dist}((x, t), (y, s)) = \sqrt{\text{dist}(x, y)^2 + \text{dist}(t, s)^2}, \quad \forall x, y \in X, \quad s, t \in T.$$  

**Definition 3.2** (5.2 of [8]). Recall that a unital simple $C^*$-algebra $A$ is said to be tracially approximately divisible if for any finite subset $\mathcal{F} \subseteq A$, any $\epsilon > 0$, any natural number $N$, and any $a \in A^+$, there is a $C^*$-subalgebra $B \subseteq A$ with $B \cong M_k(\mathbb{C})$ for some $k \geq N$ such that if $p = 1_B$, then

1. $\mathcal{F} \subseteq_\epsilon B' \cap A$, and
2. $1 - p$ is Murray-von Neumann equivalent to a projection in $aAa$,

where $B' \cap A$ is the relative commutant of $B$ in $A$.

**Remark 3.3.** The definition above is slightly different—but equivalent—to the original definition in [8], in which the first condition is replaced by

1'. $\| cf - fc \| < \epsilon$ for any $f \in \mathcal{F}$ and any $c$ in the unit ball of $B$.

Indeed, as in [11], for any finite dimensional $C^*$-algebra $B \subseteq A$, one considers the conditional expectation

$$\mathbb{E}_B : A \ni a \mapsto \int_{U(B)} u^*a u d\mu,$$

where $\mu$ is the Haar measure on the unitary group $U(B)$. It is clear that $\mathbb{E}_B(a)$ commutes with $B$. Now, if $f \in A$ satisfies $\| fc - cf \| < \epsilon$ for any $c$ in the unit ball of $B$, one has that

$$\| \mathbb{E}_B(f) - f \| < \epsilon.$$  

In particular, this implies that $f \in_\epsilon B' \cap A$, and shows that the two definitions of tracially approximate divisibility are equivalent.
Similar to [11], for any nondecreasing function $\Delta : (0, 1) \to (0, 1)$ with $\lim_{t \to 0} \Delta(t) = 0$, define
\[
\Delta_0(t) = \Delta\left(\frac{1}{2^n + t}\right), \quad \text{if } t \in \left[\frac{1}{2^n + 1}, \frac{1}{2^n}\right),
\]
and
\[
\Delta_0(t) = \frac{\sqrt{2}}{48} \Delta_0(t^2/6)t
\]
Then $\Delta_{00}$ and $\Delta_0$ are also nondecreasing and satisfy $\lim_{t \to 0} \Delta_{00}(t) = 0$, $\lim_{t \to 0} \Delta_0(t) = 0$.

**Definition 3.4.** Let $X$ be a compact metric space and $P \in M_r(C(X))$ be a projection, where $r \geq 1$ is an integer. Put $C = PM_r(C(X))P$. Suppose $\tau \in T(C)$. It is known that there exists a probability measure $\mu_\tau$ on $X$ such that
\[
\tau(f) = \int_X t_x(f(x))d\mu_\tau(x),
\]
where $t_x$ is the normalized trace on $P(x)M_rP(x)$ for all $x \in X$.

**Remark 3.5.** Regard $C(X)$ as the center of $C = PM_r(C(X))P$, and denote by $\iota : C(X) \to C$ the embedding. Then the measure $\mu_\tau$ is in fact induced by the trace $\tau \circ \iota$ on $C(X)$.

**Remark 3.6.** The C*-algebra $(PM_r(C(X))P) \otimes C(\mathbb{T})$ is isomorphic to the homogeneous C*-algebra $PM_r(C(X \times \mathbb{T}))\hat{P}$ with the projection $\hat{P}$ given by $\hat{P}(x, z) = P(x)$. Hence there is a natural embedding of $C(X \times \mathbb{T})$ into $(PM_r(C(X))P) \otimes C(\mathbb{T})$ as the center.

**Lemma 3.7.** Let $C = PM_r(C(X))P$ for some compact metrizable space $X$, and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing function and $\eta > 0$ such that
\[
\mu_{\tau \circ \phi}(O_a) > \Delta(a) \quad \text{for all } \tau \in T(A)
\]
and for any open ball $O_a$ of $X$ with radius $a > \eta$.

Let $F \subseteq C$, $G' \subseteq C \otimes C(\mathbb{T})$, $H \subseteq C \otimes C(\mathbb{T})$ be finite subsets, and let $\epsilon > 0$. Then there are $\delta > 0$ and a finite subset $G \subseteq C$ such that for any C*-algebra $A$ which is tracially approximately divisible, any homomorphism $\varphi : C \to A$, any unitary $u \in A$ with
\[
\|[\varphi(c), u]\| < \delta \quad \forall c \in G,
\]
there exist unitaries $w_1, w_2 \in A$, a path of unitaries $\{w(t); t \in [0, 1]\} \subseteq A$ with $w(0) = 1$ and $w(1) = w_1w_2w_1^*w_2^* =: w$, and a completely positive $G'$-multiplicative linear maps $L_1, L_2 : C \otimes C(\mathbb{T}) \to A$ such that
\[
\|[w_i, \varphi(a)]\| < \epsilon \quad \text{for all } a \in F, \quad i = 1, 2, \quad (e.3.5)
\]
\[
\|[w(t), \varphi(a)]\| < \epsilon, \quad \text{for all } a \in F \cup \{u\} \quad \text{and} \quad t \in [0, 1], \quad (e.3.6)
\]
\[
\|L_1(a \otimes z) - (\varphi(a)uw)\| < \epsilon, \quad \|L_1(a \otimes 1) - \varphi(a)\| < \epsilon, \quad \text{for all } a \in F, \quad (e.3.7)
\]
\[
\|L_2(a \otimes z) - (\varphi(a)w)\| < \epsilon, \quad \|L_2(a \otimes 1) - \varphi(a)\| < \epsilon, \quad \text{for all } a \in F, \quad (e.3.8)
\]
\[
|\tau \circ L_1(g) - \tau \circ L_2(g)| < \epsilon, \quad \text{for all } g \in H, \quad \text{for all } \tau \in T(A), \quad (e.3.9)
\]
and
\[
\mu_{\tau \circ L_i}(B_a) > \Delta_0(a), \quad i = 1, 2, \quad \text{for all } \tau \in T(A)
\]
and for any open ball $B_a$ of $X \times \mathbb{T}$ with radius $a > 3\sqrt{2}\eta$. 

\[7\]
Proof. Let $\mathcal{H} \subseteq C(X \times T)$ (in the place of $\mathcal{G}$) and $\bar{v} > 0$ (in the place of $\delta$) be the finite subset and constant of Lemma 3.4 of [11] with respect to $\Delta_{00}(a^{\sqrt{2}/2})_{2}a^{\sqrt{2}/8}$, $\eta$ and $\lambda_{1} = \lambda_{2} = 1/2$. Regarding $C(X \times T)$ as the center of $C \otimes C(T)$, the subset $\mathcal{H}$ is inside $C \otimes C(T)$.

Then without loss of generality, one may assume that $\mathcal{H} \subseteq H$ and $\epsilon < \bar{v}$, and one may also assume

$$G' = \{f'_{i} \otimes z^{m_{i}}; f'_{i} \in C, m_{i} \in \mathbb{Z}, i = 1, ..., N\},$$

$$H = \{f_{i} \otimes z^{n_{i}}; f_{i} \in C, n_{i} \in \mathbb{Z}, i = 1, ..., N\},$$

$1 \in F$ and $|f_{i}|, |f'_{i}| \leq 1$. Choose $M \in \mathbb{N}$ so that $|m_{i}|, |n_{i}| < M$ for any $i = 1, ..., N$, and denote by

$$F_{1} = \{f'_{i}, f_{i}; i = 1, ..., N\}.$$

Let the natural number $N_{1}$ satisfies

$$\eta \in \left[\frac{1}{2N_{1}+1}, \frac{1}{2N_{1}}\right].$$

For each $1 \leq j \leq N_{1}$, by a compactness argument, choosing $O_{j}$ to be a finite collection of open balls of $X$ with radius $1/2^{j+2}$ which has the following property: for any open ball $O_{a}$ of $X$ with radius $a \in [1/2^{j+1}, 1/2^{j})$, there is an open ball $O' \in O_{j}$ such that $O' \subset O_{a}$.

Put $O = \bigcup_{j=1}^{N_{1}} O_{j}$. For each open ball $O' \in O_{j}$, fix a norm-one positive function $g$ such that the support of $gO'$ is in $O'$, and is constant one if restricted to the open ball with the same center of $O'$ and with the radius $\frac{1}{2^{j+1}}$. Then $gO'P$ is a central element of $C$. Put $T = \{gO'P : O' \in O\}$.

By Lemma 3.11 for any $\min\{\Delta(\frac{1}{2N_{1}+1}), 2^{N_{1}+7}, \epsilon/2\} > \epsilon' > 0$, there are $\delta' > 0$ and a finite subset $\mathcal{G} \subseteq C$ such that for any C*-algebra $A$, any unitary $v \in A$ with

$$||[\varphi(c), v]|| < \delta', \quad \forall c \in \mathcal{G},$$

there exists a unital contractive completely positive linear map $L : C \otimes C(T) \to A$ with

$$||L(f \otimes z^{n}) - \varphi(f)u^{n}|| < \epsilon' < \epsilon/16, \quad \forall f \in F \cup F_{1}, \quad -M \leq n \leq M.$$  

By choosing $\epsilon'$ sufficiently small, the resulting map $L$ is $\mathcal{G}'$-$\epsilon$-multiplicative. Without loss of generality, one may assume that $\delta' < \epsilon$.

One then asserts that $\delta := \delta'/2$ and $\mathcal{G}$ satisfy the lemma. Let $\varphi : C \to A$ be a homomorphism and $u \in A$ be a unitary with

$$||[\varphi(c), u]|| < \delta, \quad \forall c \in \mathcal{G}.$$  

Choose an integer $K \geq \max\{2^{6}/\pi, 4(M + 1)\}$. Since $A$ is tracially approximately divisible, for any $\min\{\Delta(\frac{1}{2N_{1}+1}), 2^{N_{1}+7}, \epsilon/32M\} > \epsilon'' > 0$ (which will be fixed later), there is a projection $p \in A$, a unital C*-subalgebra $B \subset A$ with $B \cong M_{k}(\mathbb{C})$, with $1_{B} = p$ and $k \geq K$ such that

1. $\tau(1 - p) < \epsilon''/16$ for any $\tau \in T(A)$,

2. $\varphi(F \cup F_{1} \cup \mathcal{G} \cup T) \subseteq e^{\epsilon''} B' \cap A$ and $u \in e^{\epsilon''} B' \cap A$,

where $B' \cap A$ is the relative commutant of $B$ in $A$. Let $w' \in B \cong M_{k}(\mathbb{C})$ which has the following matrix form

$$w' = \begin{pmatrix} e^{2\pi i/k} & 0 & 0 & \cdots \\ 0 & e^{2\pi i2/k} & 0 & \cdots \\ & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & e^{2\pi ik/k} \end{pmatrix}.$$  

(e 3.10)
We compute that
\[ t(w') = 0, \]
where \( t \in T(B) \) is the tracial state. Moreover, for any \( 0 < |n| \leq M \),
\[ t((w')^n) = 1 + \sum_{j=1}^{k-1} e^{\frac{2\pi n j i}{k}} = \frac{1 - e^{2\pi n i/k}}{1 - e^{2\pi n i/k}} = 0. \] (e 3.12)

In particular, \( w' \in DU(B) \). Note that, since \( B \cong M_k \), there exist two unitaries \( w_1', w_2' \in B \) such that \( w' = w_1'w_2'(w_1')^*(w_2')^* \). Let \( \{w'(t); t \in [0, 1] \} \subseteq B \) be a continuous path of unitaries such that \( w'(0) = 1_B = p \) and \( w'(1) = w' \). Denote by \( w_1 = (1 - p) + w_1' \), \( w_2 = (1 - p) + w_2' \) and
\[ w = (1 - p) + w' \quad \text{and} \quad w(t) = (1 - p) + w(t). \]

It is clear that (e 3.5) holds when \( \epsilon'' \) sufficiently small. By choosing \( \epsilon'' \) smaller, it follows from (2) above that
\[ \|[w(t), \varphi(a)]\| < \delta/2 < \epsilon, \quad \forall a \in \mathcal{F}, \forall t \in [0, 1], \]
which also verifies (e 3.6).

One also assume that \( \epsilon'' \) is even sufficiently small so that for any \( c \in \mathcal{G} \)
\[ \|[\varphi(c), uw]\| < \delta, \quad \|[\varphi(c), w]\| < \delta, \quad \text{and} \quad \|(uw)^n - u^n w^n\| < \epsilon/16, \quad -M \leq n \leq M. \] (e 3.13)

Then there are \( \mathcal{G}'\)-multiplicative linear maps \( L_1, L_2 : C \otimes C(\mathbb{T}) \to A \) such that
\[ \|L_1(f \otimes z^n) - \varphi(f)(uw)^n\| < \epsilon < \epsilon/16, \quad \forall f \in \mathcal{F} \cup \mathcal{F}_1, \quad -M \leq n \leq M, \]
and
\[ \|L_2(f \otimes z^n) - \varphi(f)w^n\| < \epsilon < \epsilon/16, \quad \forall f \in \mathcal{F} \cup \mathcal{F}_1, \quad -M \leq n \leq M. \]

Since \( 1 \in \mathcal{F} \), the maps \( L_1 \) and \( L_2 \) satisfy (e 3.7) and (e 3.8). Let us verify (e 3.9). Let \( \tau \) be any tracial state of \( A \). Note that, for any \( a \in B \subseteq pAp \) and any \( b \in B' \cap pAp \), one has that \( \tau(ba) = \tau(b)\tau(a) = \tau(b)\tau(p)t(t(a)) \), where \( t \) is the unique tracial state on \( B \).

For any \( i = 1, \ldots, N \), choose \( a_i', u'' \in (1 - p)A(1 - p) \) and \( a_i, u' \in B' \cap pAp \), where \( u', u'' \) are unitaries and \( \|a_i\|, \|a_i'\| \leq 1 \) such that
\[ \|(a_i + a_i') - \varphi(f_i)\| < \epsilon'' < \epsilon/32 \quad \text{and} \quad \|(u' + u'') - u\| < \epsilon'' < \epsilon/32M. \]

Then
\[ \tau \circ L_1(f_i \otimes z^{ni}) \approx \frac{\epsilon}{16} \left\{ \begin{array}{ll} \tau(\varphi(f_i)(uw)^{ni}) & \text{if } n_i = 0 \\ 0 & \text{if } n_i \neq 0 \end{array} \right\}, \]
and
\[ \tau \circ L_2(f_i \otimes z^{ni}) \approx \frac{\epsilon}{16} \left\{ \begin{array}{ll} \tau(\varphi(f_i)w^{ni}) & \text{if } n_i = 0 \\ 0 & \text{if } n_i \neq 0 \end{array} \right\}. \]
Consider a subgroup $\mathbb{Z}$

Denote by $\{ g_t \}$ linear functional $\forall \eta$ for some integer $I$ for any arc $\eta$.

One then computes that

$|\mu_t(I) - |I|| < \frac{1}{2N_1+3}$

for any arc $I$ with length at least $\eta$, where $\mu_t$ is the Borel probability measure induced by positive linear functional $t \circ f(w)$ on $C(\mathbb{T})$, where $t$ is the tracial state on $B$.

Now, let $B_a$ be any open ball on $X \times \mathbb{T}$ with radius $a$. Denote by $(a_0, b_0)$ the center of $B_a$.

Denote by $O_a\sqrt{2}/2$ the open ball of $X$ with radius $a\sqrt{2}/2$ and center $a_0$, and denote by $J_{a\sqrt{2}/2}$ the open ball of $\mathbb{T}$ with radius $a\sqrt{2}/2$ and center $b_0$. Note that $O_{a\sqrt{2}/2} \times J_{a\sqrt{2}/2} \subseteq O_a$.

Assume that $a\sqrt{2}/2 \in [\frac{\sqrt{2}}{2}, \frac{1}{2}]$ for some $1 \leq j \leq N_1$. Then choose $O'_j \in O_j$ such that $O'_j \subseteq O'_{a\sqrt{2}/2}$, and consider $g_1 P \in \mathcal{T}$ associated to $O'_k$, and any norm-one positive continuous function $g_2$ on $\mathbb{T}$ with support in $J_{a\sqrt{2}/2}$. Note that

$\varphi(g_1) \approx_{\epsilon''} a + b$

for some $b$ commutes with $B$ and the traces of $a$ are at most $\epsilon''$.

Consider the function $g(x, t) := g_1(x) P \cdot g_2(t)$. Then, for any $a \geq \sqrt{2}\eta > \eta$,

$$\mu_{\tau \circ L_2}(B_a) \geq \tau(L_2(g)) > \tau(b g_2(w)) - \epsilon' = \tau(b) \cdot t'(g_2(w)) - \epsilon'$$

$$\geq \tau(\varphi(g_1)) \cdot t(g_2(w)) - \epsilon' - \epsilon''$$

$$> \Delta(\frac{1}{2k+3}) \cdot t(g_2(w)) - \epsilon' - \epsilon''$$

Since this holds for any $g_2$, one has

$$\mu_{\tau \circ L_2}(B_a) \geq \Delta(\frac{1}{2k+3}) \cdot \mu_t(J_{a\sqrt{2}/2})/2 - \Delta(\frac{1}{2N_1+3})/2^{N_1+5} = \Delta(\frac{1}{2k+3}) a\sqrt{2}/8 > \Delta_0(a\sqrt{2}/2) a\sqrt{2}/8,$$

where $\mu_t$ is the spectral measure of $w$.

Note that $|\tau \circ L_1(g) - \tau \circ L_2(g)| < \epsilon < \epsilon'$, $\forall g \in \tilde{\mathcal{H}} \subseteq \mathcal{H}$,

by Lemma 3.4 of [13], one has

$$\mu_{\tau \circ L_1}(B_a) > \frac{\sqrt{2}}{48} (a\sqrt{2}/6) a = \Delta_0(a)$$

for any $a \geq 3\sqrt{2}\eta$.

\[ \square \]

**Definition 3.8.** Let $A$ be a unital C*-algebra. In the following, for any invertible element $x \in A$, let $\langle x \rangle$ denote the unitary $x(x^* x)^{-\frac{1}{2}}$, and let $\overline{x}$ denote the element $\overline{\langle x \rangle}$ in $U(A)/CU(A)$.

Consider a subgroup $\mathbb{Z}^k \subseteq K_1(A)$, and write the unitaries $\{ u_1, ..., u_k \} \subseteq U_c(A)$ corresponding to the standard generators $\{ e_1, e_2, ..., e_k \}$ of $\mathbb{Z}^k$. Suppose that $\{ u_1, u_2, ..., u_k \} \subset M_n(A)$ for some integer $n \geq 1$. Let $\Phi : A \to B$ be a unital positive linear map such that $\Phi \otimes \text{id}_{M_n}$ is
at least \{u_1, ..., u_k\} are \(1/4\)-multiplicative (hence each \(\Phi \otimes \text{id}_{M_n}(u_i)\) is invertible), then the map
\[
\Phi^t|_{s_1(\mathbb{Z}^k)} : \mathbb{Z}^k \to U(B)/CU(B)
\]
is defined by
\[
\Phi^t|_{s_1(\mathbb{Z}^k)}(e_i) = (\Phi \otimes \text{id}_{M_n}(u_i)), \quad 1 \leq i \leq k.
\]
Thus, for any finitely generated subgroup \(G \subset U_c(A)\), there exists \(\delta > 0\) and a finite subset \(\mathcal{G} \subset A\) such that, for any unital \(\delta\)-\(G\)-multiplicative contractive completely positive linear map \(L : A \to B\) (for any unital \(C^\ast\)-algebra \(B\)), the map \(L^t\) is well defined on \(s_1(G)\). (Please see 2.4 for \(U_c(A)\) and \(s_1\).)

**Theorem 3.9.** Let \(C = C(X)\) with \(X\) a compact metric space and let \(\Delta : (0, 1) \to (0, 1)\) be a non-decreasing map. For any \(\epsilon > 0\) and any finite subset \(\mathcal{F} \subseteq C\), there exists \(\delta > 0\), \(\eta > 0\), \(\gamma > 0\), a finite subsets \(\mathcal{G} \subseteq C\), \(\mathcal{P} \subseteq K(C)\), a finite subset \(\mathcal{Q} = \{x_1, x_2, ..., x_m\} \subseteq K_0(C)\) which generates a free subgroup and \(x_i = [p_i] - [q_i]\), where \(p_i, q_i \in M_n(C)\) (for some integer \(n \geq 1\)) are projections, satisfying the following:

Suppose that \(A\) is a unital simple \(C^\ast\)-algebra with \(TR(A) \leq 1\), \(\varphi : C \to A\) is a unital homomorphism and \(u \in A\) is a unitary, and suppose that
\[
|||\varphi(c), u||| < \delta, \quad \forall c \in \mathcal{G} \quad \text{and} \quad \text{Bott}(\varphi, u)\big|_{\mathcal{P}} = 0,
\]
and
\[
\mu_{\tau \circ \varphi}(O_a) \geq \Delta(a) \quad \forall \tau \in T(A),
\]
where \(O_a\) is any open ball in \(X\) with radius \(a < 1\) and \(\mu_{\tau \circ \varphi}\) is the Borel probability measure defined by \(\tau \circ \varphi\). Moreover, for each \(1 \leq i \leq m\), there is \(v_i \in CU(M_n(A))\) such that
\[
||((1_n - \varphi(p_i) + \varphi(p_i)1_n \otimes u)(1_n - \varphi(q_i) + \varphi(q_i)(1_n \otimes u^*) - v_i)\| < \gamma.
\]
Then there is a continuous path of unitaries \(\{u(t) : t \in [0, 1]\}\) in \(A\) such that
\[
u(0) = u, \quad u(1) = 1, \quad \text{and} \quad |||\varphi(c), u(t)||| < \epsilon
\]
for any \(c \in \mathcal{F}\) and for any \(t \in [0, 1]\).

**Proof.** Since \(A\) is a simple \(C^\ast\)-algebra with \(TR(A) \leq 1\), it is tracially approximately divisible (see 5.4 of [8]). Therefore 3.7 applies. Without loss of generality, one may assume that \(\mathcal{F}\) is in the unit ball of \(C\). Let \(e_0\) be the universal constant such that, for any unitaries \(u_1\) and \(u_2\) with \(\|u_1 - u_2\| < e_0\), there is a path of unitaries connecting \(u_1\) and \(u_2\) with length at most \(\epsilon/2\).

Let \(\eta' > 0\), \(\delta' > 0\), \(\mathcal{G}' \subseteq C \otimes C(T), \mathcal{H} \subseteq C \otimes C(T), \mathcal{P}' \subseteq K(C \otimes C(T), \mathcal{U}' \subseteq U_c(K_1(C \otimes C(T))), \gamma_1, \gamma_2\) be the finite subsets and constants of Theorem 5.3 of [11] with respect to \(X \times T, \Delta_0, \mathcal{F} \otimes \{1, z\}\), and \(\min\{\epsilon/2, e_0\}\). Without loss of generality, we may assume that \(\mathcal{P}' = \mathcal{P} \cup \beta(\mathcal{P})\), where \(\mathcal{P}\) is a finite subset of \(K(C)\), and
\[
\mathcal{G}' = \mathcal{G}'_1 \cup \{1_{C(T)}, z\},
\]
where \(\mathcal{G}'_1\) is a finite subset of \(C\). Moreover, we may assume
\[
[L']|_{\mathcal{P}} = [L''']|_{\mathcal{P}} \quad (e.3.18)
\]
for any unital \(\mathcal{G}'_1\)-\(\delta'\)-multiplicative contractive completely positive linear maps \(L', L'' : C \to A\) with
\[
||L'(g) - L''(g)|| < \gamma_2 \quad \text{for all} \quad g \in \mathcal{G}'_1.
\]
By choosing larger \(\mathcal{G}'_1\) and smaller \(\delta'\), we may assume further that \((L')^t\) is well defined on \(\overline{U'}\).
Since \( K_1(C \otimes C(\mathbb{T})) = K_1(C) \oplus K_2(C) \), without loss of generality, the set \( \mathcal{U}' \) may be chosen as \( \mathcal{U}'_1 \cup \mathcal{U}'_0 \), where \( \mathcal{U}'_1 = \{v_1 \otimes 1_{C(\mathbb{T})}, \ldots, v_{n'} \otimes 1_{C(\mathbb{T})}\} \) with each \( v_i \) a unitary \( M_n(C) \), and any element in \( \mathcal{U}_0 \) has the form

\[
(p \otimes z + (1_n - p) \otimes 1_{C(\mathbb{T})})(q \otimes z + (1_n - q) \otimes 1_{C(\mathbb{T})})^*
\]

for some projections \( p \) and \( q \) in \( M_n(C) \) for some integer \( n \geq 1 \). Without loss of generality, one may assume that \( \mathcal{U}'_0 \) exactly generates a free group \( \mathbb{Z}^m \) in \( K_1(C \otimes C(T)) \) as standard generators, and hence one may write

\[
\mathcal{U}_0 = \{(p_i \otimes z + (1_n - p_i) \otimes 1_{C(\mathbb{T})})(q_i \otimes z + (1_n - q_i) \otimes 1_{C(\mathbb{T})})^*; \ i = 1, \ldots, m\},
\]

where \( p_i \) and \( q_i \) are projections in \( M_n(C) \). Denote by \( x_i = [p_i] - [q_i] \) for \( 1 \leq i \leq m \), and put \( Q = \{x_1, \ldots, x_m\} \).

We may assume that \( \mathcal{F}_1 \subset C \) is a finite subset such that

\[ p_i, q_i \in \{(c_{j,k}) \in M_n(C) : c_{j,k} \in \mathcal{F}_1\}. \]

Put \( \mathcal{F}_2 = \{1_C\} \cup \mathcal{F} \cup \mathcal{F}_1 \). Let \( \delta > 0 \) and \( \mathcal{G} \subset C \) be the constant and finite subset of Lemma 3.7 with respect to \( \min\{\epsilon/8\sqrt{n}, \delta'/n^2, \gamma_1/2n^2, \gamma_2/16n^2\} \) (in place of \( \epsilon \), \( \mathcal{F}_2 \) (in place of \( \mathcal{F} \), \( \mathcal{G}' \) and \( \mathcal{H} \).

Without loss of generality, one may assume that \( \delta \) is sufficiently small and \( \mathcal{G} \) is sufficiently large such that \( \text{Bott}(\varphi, u_1 u_2)|_\mathcal{P} \) is well defined and

\[ \text{Bott}(\varphi, u_1 u_2)|_\mathcal{P} = \text{Bott}(\varphi, u_1)|_\mathcal{P} + \text{Bott}(\varphi, u_2)|_\mathcal{P} \]

for any unital homomorphisms \( \varphi : C \to B \) for some unital C*-algebra \( B \) and unitaries \( u_1, u_2 \in B \)

with

\[ ||(\varphi(a), u_i)|| < \delta, \ \forall a \in \mathcal{G}, i = 1, 2. \]

One asserts that \( \delta, \eta = \frac{7}{6} \eta', \gamma = \gamma_2/4, \mathcal{P}, \mathcal{G} \) and \( \mathcal{Q} \) satisfy the theorem.

Let \( (\varphi, u) \) be a pair which satisfies the condition of the theorem. By Lemma 3.7 there are unitary \( w = w_1 w_2 w_3^* w_4^* \) with \( w_1, w_2 \) unitaries in \( A \), a path of unitaries \( \{w'(t); t \in [0, 1]\} \) with \( w'(1) = 1 \) and \( w'(0) = w \), and unital \( \mathcal{G}'/\delta \)-multiplicative completely positive linear maps \( L_1, L_2 : C \otimes C(\mathbb{T}) \to A \) such that for any \( f \in \mathcal{F} \),

1. \( ||w_i, \varphi(a)|| < \min\{\epsilon/8n^2, \gamma_2/16n^2\}, \forall a \in \mathcal{F}_2, i=1, 2, \)
2. \( ||w'(t), \varphi(a)|| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}, \forall a \in \mathcal{F}_2 \cup \{a\}, \forall t \in [0, 1], \)
3. \( ||L_1(f \otimes a) - (\varphi(f) \otimes a)|| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}, \|L_1(f \otimes 1) - \varphi(f)\| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}, \)
4. \( ||L_2(f \otimes a) - (\varphi(f) \otimes a)|| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}, \|L_2(f \otimes 1) - \varphi(f)\| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}, \)
5. \( ||\tau \circ L_1(g) - \tau \circ L_2(g)|| < \gamma_1/2n^2, \forall g \in \mathcal{H}, \forall \tau \in T(A), \)
6. \( \mu_{\tau_0(L_1(O_a))} > \Delta_0(a), i = 0, 1 \) for any open ball \( O_a \) of \( X \times \mathbb{T} \) with radius \( a > 3\sqrt{2} \eta = \eta' \).

It follows from (2) that \( \text{Bott}(\varphi, w) = 0 \). Therefore

\[ \text{Bott}(\varphi, uw)|_\mathcal{P} = \text{Bott}(\varphi, u)|_\mathcal{P} + \text{Bott}(\varphi, w)|_\mathcal{P} = \text{Bott}(\varphi, u)|_\mathcal{P} = 0. \]

We also have, by (e3.18),

\[ [L_1]|_\mathcal{P} = [\varphi]|_\mathcal{P} = [L_2]|_\mathcal{P}. \]
Note that, by \((1)\),
\[w = w_1 w_2 w_3^* w_4^*
\]
with \[\|[w_1, \varphi(a)]\| < \min\{\varepsilon/8n^2, \gamma_2/16n^2\}, \forall a \in \mathcal{F}_2, i = 1, 2.\] Then for any projection \(p_i\) (or \(q_i\)), one estimates that
\[
\text{dist}((\mathbb{1}_n - \varphi(p_i)) + \varphi(p_i)w, CU(M_n(A))) < \gamma_2/16 \text{ and } (e 3.20)
\]
\[
\text{dist}((\mathbb{1}_n - \varphi(q_i)) + \varphi(q_i)w, CU(M_n(A))) < \gamma_2/16, \quad (e 3.21)
\]
\[1 \leq i \leq m.\] Therefore, for any \(1 \leq i \leq m,\)
\[
\text{dist}(L_2((p_i \otimes z + \mathbb{1}_n - p_i)(q_i \otimes z + \mathbb{1}_n - q_i)^*), \mathbb{1}) \approx_{\gamma_2/4} 0, \text{ and } (e 3.22)
\]
\[
\text{dist}(L_1((p_i \otimes z + \mathbb{1}_n - p_i)(q_i \otimes z + \mathbb{1}_n - q_i)^*), \mathbb{1}_n) \approx_{\gamma_2/8} \text{dist}((\mathbb{1}_n - \varphi(q_i)) + \varphi(q_i)uw((\mathbb{1}_n - q_i) + \varphi(q_i)uw)^*, \mathbb{1}_n) \approx_{\gamma_2/8} \text{dist}((\mathbb{1}_n - \varphi(p_i)) + \varphi(p_i)u((\mathbb{1}_n - q_i) + \varphi(q_i)u)^*, \mathbb{1}_n) \quad (e 3.23)
\]
\[
\approx \gamma = \gamma_2/4. \quad (e 3.26)
\]
Also note that for any \(v_i \otimes 1 \in U_1',\) one computes that
\[
\text{dist}(L_1(v_i \otimes 1), L_2(v_i \otimes 1)) \approx_{\gamma_2} \text{dist}(\varphi(v_i), \varphi(v_i)) = 0.
\]

Since \(U_0(A)/CU(A)\) is torsion free (Theorem 6.11 of [8]), one has that
\[
\text{dist}(L_1(u), L_2(u)) < \gamma_2, \quad \forall u \in U'. \quad (e 3.27)
\]

By \((e 3.19)\) \((e 3.27)\) and \((6)\), it follows from Theorem 5.3 of [14] that there is a unitary \(U \in A\) such that
\[
\|L_1(f) - U^*L_2(f)U\| < \min\{\varepsilon/2, \epsilon_0\}, \quad \forall f \in \mathcal{F} \otimes \{1, z\}.
\]

Consider the path of unitaries \(w(t) : t \mapsto U^*w'(2t - 1)U, t \in [1/2, 1].\) Then
\[
\|[\varphi(f), w(t)]\| < \epsilon, \quad \forall f \in \mathcal{F}, t \in [1/2, 1] \text{ and } \|w(1/2) - uw\| < \epsilon_0, \quad w(1) = 1. \quad (e 3.28)
\]

By the choice of \(\epsilon_0,\) there is a path of unitaries \(\{w''(t); t \in [1/4, 1/2]\}\) such that
\[
\|[\varphi(f), w(t)]\| < \epsilon, \quad \forall f \in \mathcal{F}, t \in [1/4, 1/2], \text{ and } (e 3.29)
\]
\[
w''(1/4) = uw \quad \text{and} \quad w''(1/2) = w(1/2). \quad (e 3.30)
\]

Also consider the path of unitaries \(w'''(t) : t \mapsto uw'(4t), t \in [0, 1/4].\) Then one has that \(w'''(0) = u, w'''(1/4) = uw\) and
\[
\|[w'''(t), \varphi(f)]\| < \epsilon, \quad \forall f \in C. \quad (e 3.31)
\]

Define the path
\[
w(t) = \begin{cases} 
    w'''(t), & \text{if } t \in [0, 1/4], \\
    w''(t), & \text{if } t \in [1/4, 1/2], \\
    w(t), & \text{if } t \in [1/2, 1].
\end{cases}
\]

Then it is clear that
\[
\|[\varphi(f), w(t)]\| < \epsilon, \quad \forall f \in \mathcal{F}, t \in [0, 1],
\]
\[
w(0) = u \quad \text{and} \quad w(1) = 1,
\]
as desired. \qed
Corollary 3.10. Let $X$ be a compact subset of finite CW-complex and let $C = PM_n(C(X))P$ for some integer $n \geq 1$ and $P$ a projection in $M_n(C(X))$. Then the statement of Theorem 3.9 still holds for the C*-algebra $C$ and using the measure define in 3.4.

Proof. If $C = M_n(C(X))$, it is clear that the corollary follows from Theorem 3.9 ($X$ is even not required to have finite covering space in this case). In what follows we will use this case of the corollary to prove the general case.

Assume that $C = PM_n(C(X))P$. Since $X$ is compact, the rank of $P$ has only finitely many values. It follows that, without loss of generality, we may assume that $P(x) \neq 0$ for all $x \in X$. Since $X$ is a compact subset of finite CW-complex, there is an integer $d$ and a projection $Q \in M_d(PM_n(C(X))P)$ such that

$$QM_d(PM_n(C(X))P)Q \cong M_r(C(X))$$

for some integer $r$. Note that $Q(x) \neq 0$ for all $x \in X$. Without loss of generality, one may assume that $P \preceq Q$, that is, there is also a partial isometry $V \in M_d(PM_n(C(X))P)$ such that $VV^* \preceq Q$ and $V^*V = \{P, 0, \ldots, 0\}$.

There is an integer $l \geq 1$ such that $X = X_1 \sqcup \cdots \sqcup X_l$ such that the ranks of the restrictions of $P$ and $Q$ to each $X_i$, $1 \leq j \leq l$, are constant. Denote by $P_j$ and $Q_j$ the restriction of $P$ and $Q$ to $X_j$ respectively. Let $R_1 = \max_{1 \leq j \leq l}\{\text{rank}P_j\}$ and $R_2 = \min_{1 \leq j \leq l}\{\text{rank}Q_j\}$.

Fix $d$, $Q$, and $V$. Let $\Delta : (0, 1) \rightarrow (0, 1)$ be a non-decreasing map, let $\varepsilon > 0$ and $F \subseteq PM_n(C(X))P$ be a finite subset of elements with norm one.

Pick $\frac{\varepsilon}{4} > \epsilon' > 0$ such that for any unitaries $u, v$ in a C*-algebra with $\|u - v\| < \epsilon'$, there is a path of unitaries $u(t)$ such that $u(0) = u$, $u(1) = v$, and $\|u(t) - v\| < \frac{\varepsilon}{2}$, $\forall t \in [0, 1]$.

Pick $\frac{\varepsilon}{4} > \epsilon'' > 0$ such that if there are a projection $p$ and a unitary $U$ in a C*-algebra $A$ with $\|[p, U]\| < \epsilon''$, then

$$\|(pu)p - pUp\| < \epsilon'/4.$$ (Recall that $\langle pu, p\rangle = pu(pu'^*pu) - \frac{1}{2}$.)

Denote by $\delta'$ (in place $\delta$), $\eta$, $\gamma'$ (in place of $\gamma$), $G' \subseteq QM_d(PM_n(C(X))P)Q \cong M_r(C(X))$ (in place of $G$), $P \subseteq K(C(X))$, and $Q \subseteq K_0(C(X))$ the constants and finite subsets of the corollary required for $M_r(C(X))$ with $\epsilon''$, $V^*FV^*$, and $\Delta$.

We may assume that $\gamma' < 1$. For each $x_i \in Q$, write $x_i = [p_i] - [q_i]$ with $p_i, q_i \in M_k(QM_d(C)Q)$ for some integer $k$. Choose an integer $k''$ such that

$$M_k(QM_d(C)Q) \subseteq M_{k''}(C).$$

Without loss of generality, one also assumes that any element of $G'$ has norm one, and $VV^* \in G'$. Choose a finite subset $G_1 \subseteq C$ and $\delta_1 > 0$ such that if there is a C*-algebra $A$ and a unitary $u \in A$ satisfies

$$\|[\varphi(c), u]\| < \delta_1, \quad \forall c \in G_1$$

for some homomorphism $\varphi$ to $A$, then

$$\|[\varphi \otimes \text{id}_{M_d}(c), u \otimes 1_{M_d}]\| < \delta'/2$$

for any $c \in G' \subseteq M_d(C)$, and

$$\|[\varphi \otimes \text{id}_{M_d}(Q), u \otimes 1_{M_d}]\| < \min\{\epsilon'', \delta'/2\}.$$

Let $B = QM_d(C)Q \otimes C(T)$. It is a full hereditary C*-subalgebra of $M_d(C) \otimes C(T)$. Choose a large finite subset $G_2 \subseteq C$ and a sufficiently small $\delta_2 > 0$ such that, if $L : M_d(C) \otimes C(T) \rightarrow M_d(A)$
is a unital $G_2 \times \{1, z\}$-multiplicative contractive completely positive linear map and $[L]|_p$ is well defined, then


Note that if we assume that

$$\text{Bott}(\varphi, u)|_p = 0,$$  

then

$$\text{Bott}(\varphi \otimes \text{id}_{M_d}, u \otimes 1_{M_d})|_p = 0. \tag{e.3.32}$$ 

It then follows that we can choose a larger $G_2$ and smaller $\delta_2$ so that if

$$||[\varphi(c), u]|| < \delta_2, \quad \forall c \in G_2 \text{ and } \text{Bott}(\varphi, u)|_p = 0, \tag{e.3.33}$$ 

we still have

$$\text{Bott}(\varphi|_{QM_d(C)Q}, \langle q(u \otimes 1_{M_d})q \rangle)|_p = 0,$$  

where $q = (\varphi \otimes \text{id}_{M_d})(Q)$.

Note that $p_i, q_i \in M_k(QM_d(C)Q) \subseteq M_{k'}(C)$. Define $\bar{q} = q \otimes 1_{M_k}$. Then there is a finite subset $G_3 \subseteq C$, and $\delta_3 > 0$ such that if

$$||[\varphi(c), u]|| < \delta_3, \quad \forall c \in G_3 \text{ and }$$

$$\|((1_{M_{k'}} - \varphi(p_i) + \varphi(p_i)1_{M_{k'}} \otimes u)(1_{M_{k'}} - \varphi(q_i) + \varphi(q_i)1_{M_{k'}} \otimes u^*)) - v_i\| < \gamma'/ (8(k'R_1 + \frac{1}{8})),$$

for some $v_i \in CU(M_{k'}(A))$, then

$$\|g_i - ((1_{M_{k'}} - \bar{q} + \langle \bar{q} \varphi i \bar{q} \rangle)) < \gamma'/(4(k'R_1 + \frac{1}{8}))$$  

and

$$\|g'_i - \langle \bar{q} \varphi i \bar{q} \rangle\| < \gamma'/(4(k'R_1 + \frac{1}{8})),$$  

where

$$g_i := ((1_{M_{k'}} - \varphi(p_i) + \varphi(p_i)u \otimes 1_{M_{k'}})(1_{M_{k'}} - \varphi(q_i) + \varphi(q_i)u^* \otimes 1_{M_{k'}}))$$

and

$$g'_i := ((\bar{q} - \varphi(p_i) + \varphi(p_i)\langle \bar{q}(u \otimes 1_{M_{k'}})\bar{q} - \varphi(q_i) + \varphi(q_i)\langle \bar{q}(u^* \otimes 1_{M_{k'}})\bar{q} \rangle)).$$

Note that, in particular, one has

$$\|g_i - ((1_{M_{k'}} - \bar{q}) + g'_i)\| < \gamma'/(2(k'R_1 + \frac{1}{8})). \tag{e.3.38}$$

Then

$$\text{dist}((1_{M_{k'}} - \bar{q}) + g'_i, CU(M_k'(A))) < \gamma'/(k'R_1 + \frac{1}{8}) \tag{e.3.39}.$$ 

Since $TR(A) \leq 1$, it follows from Lemma 6.9 of [3] that $CU(M_k'(A)) \subseteq U_0(M_{k'}(A))$. It follows from the fact that $\gamma' < 1$ and (e.3.39) that $(1_{M_{k'}} - \bar{q}) + g'_i \in U_0(M_{k'}(A))$. Since $A$ is a unital simple C*-algebra with $TR(A) \leq 1$, one has that $g'_i \in U_0(M_k(qM_d(A)q))$ (see 2.10 of [19]). Note that for any $\tau \in T(M_{k'}(A))$, one has

$$\tau(\bar{q}) \geq \frac{kR_2}{k'R_1} > \frac{1}{k'R_1},$$

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and hence
\[ k'R_1[\bar{q}] > [1_{M_k'} - \bar{q}] \cdot \]

Then by Lemma 3.3 of [II], one has
\[ \text{dist}(g'_t, CU(M_k(qM_d(A)q))) < (k'R_1 + \frac{1}{8}) \frac{\gamma'}{(k'R_1 + \frac{1}{8})} = \gamma'. \]

That is,
\[ ||(\bar{q} - \varphi(p_t) + \varphi(p_t)q(u \otimes 1_{M_d})\bar{q} - \varphi(q_t))q(u^* \otimes 1_{M_d})\bar{q})|| - v'_t|| < \gamma', \quad (e\, 3.40) \]

for some \( v'_t \in CU(M_k(qM_d(A)q)) \).

Put \( \gamma = \gamma'/(8(k'R_1 + \frac{1}{8})) \). Then, one asserts that \( \delta = \min\{\delta_1, \delta_2, \delta_3\}, \eta, \gamma, G_1 \cup G_2 \cup G_3, \mathcal{P}, \) and \( Q \) satisfy the corollary.

Let \( \varphi : PM_d(C(X)) \rightarrow A \) be a unital homomorphism satisfies the conditions of the corollary for some unitary \( u \in A \), where \( A \) is a simple C*-algebra with \( TR(A) \leq 1 \).

Put \( v = \varphi \otimes 1_{M_d}(V) \in M_d(A) \). The restriction of \( \varphi \otimes 1_{M_d} \) to \( QM_d(C)Q \) (which is isomorphic to \( M_d(C(X)) \)) is a unital homomorphism to \( qM_d(A)q \), which has \( TR(qM_d(A)q) \leq 1 \), and one also has that \( vv^* \leq q \) and \( v^*v = 1_A \).

Since \( ||(\varphi(c), u)|| < \delta_1, \forall c \in G_1 \), one has
\[ ||(\varphi \otimes 1_{M_d})(c), u \otimes 1_{M_d}|| < \delta'/2, \quad \forall c \in G'. \]

In particular,
\[ ||(\varphi \otimes 1_{M_d})|_{QM_d(C)Q}(c), (q(u \otimes 1_{M_d})q)|| < \delta', \quad \forall c \in G'. \]

Since \( \varphi \) also satisfies (e\, 3.33) and (e\, 3.35), Equations (e\, 3.34) and (e\, 3.40) are also satisfied.

Since \( \mu_{\sigma \varphi}(O_a) \geq \Delta(a) \) for any open ball \( O_a \) on \( X \) with radius \( 1 > a > \eta \) and any \( \tau \in T(A) \), one then also has that
\[ \mu_{\tau \varphi}(\varphi \otimes 1_{M_d}|_{QM_d(C)Q})(O_a) \geq \Delta(a) \]

for any open ball \( O_a \) on \( X \) with radius \( 1 > a > \eta \) and any tracial state \( \tau \) on \( qM_d(A)q \).

Then, applying the corollary to \( QM_d(C)Q \) and \( qM_d(A)q \), there is a path of unitaries \( \{U(t); \ t \in [0, 1]\} \subseteq qM_d(A)q \) such that
\[ U(0) = 1_{qM_d(A)q}, \quad U(1) = (q(u \otimes 1_{M_d})q), \]

and
\[ ||(\varphi \otimes 1_{M_d})(VfV^*), U(t)|| < \epsilon'', \quad \forall f \in \mathcal{F}. \]

Denote by \( e = vv^* \in qM_d(A)q \). Note that \( ||[e, U(t)]|| < \epsilon'' < \frac{1}{4} \). One considers the path of unitaries
\[ w(t) = \langle eU(t)c \rangle \otimes \in \mathcal{M}_d(A)e, \quad t \in [0, 1]. \]

Then
\[ w(0) = r, \quad ||w(1) - e(u \otimes 1_{M_d})e|| < \epsilon'/2, \]
\[ ||v(\varphi \otimes 1_{M_d})(f)v^*, w(t)|| < 2\epsilon' + 2\epsilon'', \quad \forall f \in \mathcal{F}. \]

Consider the path of unitaries \( u(t) := v^*w(t)v \in A \). One then has that
\[ u(0) = 1_A, \quad ||u(1) - u|| < \epsilon'/2 + \epsilon'' < \epsilon', \]

and
\[ ||(\varphi(f), u(t)|| < \epsilon, \quad \forall f \in \mathcal{F}. \]
Remark 3.11. In fact, the corollary above holds for the case that $X$ is a general compact metric space. One can use a standard argument reducing the general case to the case that $X$ is a compact subset of a finite CW-complex.

The following lemma is due to N.C. Phillips. (See the proof of 3.8 of [18].)

Lemma 3.12. Let $A$ be a unital $C^*$-algebra and $2 > d > 0$. Let $u_0, u_1, ..., u_n$ be $n + 1$ unitaries in $A$ such that
\[ u_n = 1 \quad \text{and} \quad \| u_i - u_{i+1} \| \leq d \quad i = 0, 1, ..., n - 1. \]
Then there exists a unitary $v \in CU(M_{2n+1}(A))$ with exponential length no more than $2\pi$ such that
\[ \|(u_0 \oplus 1_{M_{2n}(A)}) - v\| \leq d. \]

In the rest of the paper, unless otherwise specified, $z$ will be the identity function on the unit circle.

Theorem 3.13. Let $C = C(X)$ with $X$ a compact metric space, let $G \subset K_0(C)$ be a finitely generated subgroup. Write $G = \mathbb{Z}^k \oplus \text{Tor}(G)$ with $\mathbb{Z}^k$ generated by
\[ \{x_1 = [p_1] - [q_1], x_2 = [p_2] - [q_2], ..., x_k = [p_k] - [q_k]\}, \]
where $p_i, q_i \in M_n(C(X))$ (for some integer $n \geq 1$) are projections, $i = 1, ..., k$.
Let $A$ be a simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Suppose that $\varphi : C \to A$ is a monomorphism. Then, for any finite subsets $\mathcal{F} \subseteq C$ and $\mathcal{P} \subseteq K(C)$, any $\epsilon > 0$ and $\gamma > 0$, any homomorphism $\Gamma : \mathbb{Z}^k \to U_0(A)/CU(A)$, there is a unitary $w \in A$ such that
\[ \| \varphi(f), w \| < \epsilon \quad \forall f \in \mathcal{F}, \quad \text{Bott}(\varphi, w)|_{\mathcal{P}} = 0, \quad \text{and} \quad \text{(3.41)} \]
\[ \text{dist}((1_n - \varphi(p_i) + \varphi(q_i))w(1_n - \varphi(q_i) + \varphi(q_i)w^*), \Gamma(x_i)) < \gamma, \quad \forall 1 \leq i \leq k, \]
where $U_0(A)/CU(A)$ is identified as $U_0(M_m(A))/CU(M_m(A))$, and the distance above is understood as the distance in $U_0(M_m(A))/CU(M_m(A))$.

Proof. Without loss of generality, we may assume that $\epsilon < 1/2$. Denote by
\[ \Delta(a) = \inf\{\mu_{\tau \circ \varphi}(O_a); \quad \tau \in T(A), O_a \text{ is an open ball of } X \text{ with radius } a\}. \]
Since $A$ is simple and $T(A)$ is compact, $\Delta(a)$ is a nondecreasing function from $(0,1)$ to $(0,1)$.
Let $\eta' > 0$, $\delta' > 0$, $\mathcal{G}' \subseteq C$, $\mathcal{H}' \subseteq C_{s.a.}$, $\mathcal{P}' \subseteq K(C)$, $\mathcal{U}' \subseteq U_c(K_1(C))$, $\gamma_1 > 0$, $\gamma_2 > 0$ be the finite subsets and constants of Theorem 5.3 of [14] with respect to $X$, $\Delta(r/3)/2$, $F$, and $\epsilon/2$. Without loss of generality, one may assume that $\mathcal{F} \subseteq \mathcal{G}'$ and $\delta' < \epsilon/4$.
Let $\delta''$ and $\mathcal{H}'' \subseteq C$ be the constant and finite subset of lemma 3.4 of [14] with respect to $X$, $\Delta$, and $\eta'/3$.
Since $X$ is an inverse limit of finite CW-complexes, there is a $C^*$-algebra $C' \cong C(X')$ for a finite CW-complex $X'$ and a homomorphism $\iota : C' \to C$ such that
\[ G \subseteq \iota_{s0}(K_0(C')), \quad \{p_i, q_i; \quad i = 1, ..., k\} \subseteq \iota(M_n(C')), \quad \text{and} \quad \mathcal{P}' \subseteq [\iota](\mathcal{P}'_1), \quad \text{(3.42)} \]
where $\mathcal{P}'_1 \subset K(C')$ is a finite subset.
Furthermore, one may choose $X'$ such that there is a completely positive linear map $\pi : C \to C'$ so that if $\psi : C' \to A$ is $(G'',\delta')/2$-multiplicative (for some finite subset $G'' \subset C'$) then $\psi \circ \pi$ is $G'\delta'$-multiplicative, and moreover,

$$\|l \circ \pi(f) - f\| < \min\{\epsilon/8, \gamma_1/4\}, \quad \forall f \in \mathcal{F} \cup \mathcal{H}' \cup \mathcal{H}'',$$

and $[\pi](\mathcal{P}') \subseteq \mathcal{P}'_1$ is well defined.

Denote by $\mathcal{P}'' = \mathcal{P}'_1 \cup \beta(\mathcal{P}'_1) \subseteq K(C' \otimes C(\mathbb{T}))$, and then denote by $N_1$ the integers of Lemma 9.6 of [15] with respect to $C' \otimes C(\mathbb{T})$, $\pi(G') \otimes \{1, z\}$ (in the place of $G$), $\delta'/2$ (in the place of $\delta$), and $\mathcal{P}''$ (in the place of $\mathcal{P}$), where $z$ denotes the identity function on $\mathbb{T}$.

Let $M$ (in place of $N$) be the constant of Theorem 2.1 of [5] with respect to $X'$, $\mathcal{H}' \cup \mathcal{H}''$ and $\gamma_1/2$. Without loss of generality, one may assume that $M > 8/(N_1 \gamma_1)$.

Set

$$u_i = ((1_n - p_i) + (p_i \otimes z))(1_n - q_i) + (q_i \otimes z))^* \quad i = 1, 2, ..., k. \quad (e.3.44)$$

We may assume that there are unitaries $u'_i \in M_n(C')$ such that $\pi(u'_i) = u_i$, $i = 1, 2, ..., k$.

Choose unitaries $v_i \in U_0(A)$ such that $\Gamma(x_i) = \pi$ for each $1 \leq i \leq k$, and choose $T > 0$ such that $\ker(v_i) < T$, $i = 1, ..., k.$

Also write $K_1(C') = \mathbb{Z}^I \oplus \text{Tor}(K_1(C'))$ and $K_0(C') = \mathbb{Z}^{k'} \oplus \text{Tor}(K_0(C'))$, and let

$$\{y_1 = [e_1], ..., y_{r'} = [e_{r'}], y_{r'+1}, ..., y_{k'}\}$$

be the standard generators of $\mathbb{Z}^{k'}$ with $y_i \in \ker \rho_{C'}$, $i = r' + 1, ..., k'$, and $e_i, i = 1, ..., r'$, projections.

By choosing a larger $G''$ and a smaller $\delta'$, we may assume that, for any unital $G'' \cup \{1, z\}$-$\delta'$-multiplicative contractive completely positive linear map $L'$ from $C'$ to an arbitrary C*-algebra induces a well-defined homomorphism on $s_1(K_1(C' \otimes C(\mathbb{T})))$.

Since $TR(A) \leq 1$, there is an interval algebra $I \subset A$ with $p = 1_I$ and $G''\delta'/4$-multiplicative completely positive linear maps $L_0 : C' \to (1 - p)A(1 - p)$ and $L_1 : C' \to I$ such that

1. $\|(L_0(\pi(f)) + L_1(\pi(f))) - \varphi(f)\| < \min\{\epsilon/8, \delta'/16, \gamma_1/4\}$, for any $f \in \mathcal{F} \cup \mathcal{H}' \cup \mathcal{H}'$,
2. $[\varphi]|_{\mathcal{P}'} = [L_0 \circ \pi]|_{\mathcal{P}'} + [L_1 \circ \pi]|_{\mathcal{P}'}$,
3. $I = \bigoplus_i M_{n_i}(C([0, 1]))$ with $n_i > \max\{16(\dim(X) + 1)N_1/\gamma_1, 2M - 2N_1(\dim(X) + 1)\}$,
4. there are unitaries $v'_i \in (1 - p)A(1 - p)$ and $v''_i \in I$ such that $\ker(v'_i + p) < \gamma/4$ in $A$ (by Lemma 3.12) and $\|v_i - (v'_i + v''_i)\| < \gamma/4$, $i = 1, ..., k$,
5. moreover, by applying 2.21 of [15], one may assume that for any $r' + 1 \leq i \leq k'$

$$|\tau(L_1(y_i))| < \gamma_1/8N_2, \quad \forall \tau \in T(I).$$

There is a subgroup $G_0 \subset \mathbb{Z}^{k'} \subset K_0(C')$ such that $G_0 \cong \mathbb{Z}^k$ and generators $\{g_1, g_2, ..., g_k\} \subset G_0$ such that $s_0(g_i) = x_i$, $i = 1, 2, ..., k$. Without loss of generality, we may assume that $u'_i = g_i, i = 1, 2, ..., k$. Define a homomorphism $\Gamma_1 : K_0(C') \to U_0(I)/CU(I)$ as follows: First define $\Gamma_1(g_i) = v''_i, i = 1, 2, ..., k$. This gives a homomorphism from $G_0 \to U_0(I)/CU(I)$. Since $U_0(I)/CU(I)$ is divisible, it extends to a homomorphism $\Gamma_1$ from $K_0(C')$ to $U_0(I)/CU(I)$. Note that since $U_0(I)/CU(I)$ is also torsion free, $\Gamma_1|_{\text{Tor}(K_0(C'))} = 0$.

Denote by $m_i = n_i\gamma_1/8 + 2N_1(\dim(X) + 1)$. Note that

$$n_i - m_i > M \quad \text{and} \quad m_i/n_i < \gamma_1/4.$$
By Theorem 2.1 of [5], there is a homomorphism

$$\Psi : C' \to \bigoplus_i M_{n_i-m_i}(C([0,1])) \subseteq I$$

such that

$$|τ \circ \Psi(h) - τ \circ L_1(π(h))| < γ_1/2, \quad ∀h ∈ H∪H', \; ∀τ ∈ T(I).$$

Define

$$κ = ([L_1] - [Ψ]) ⊕ 0 ∈ \text{Hom}_A(K(C' ⊗ C(T)), K(A)),$$

where $K(C' ⊗ C(T))$ is identified as $K(C') ⊕ β(K(C'))$.

Note that $K_1(C' ⊗ C(T)) ≅ K_1(C') ⊕ K_0(C')$. It may also be written as $Z^l ⊕ Z^k' ⊕ \text{Tor}(K_1(C' ⊗ C(T)))$, where $k'$ is the rank of of $K_0(C')$.

Define a map $λ : Z^l ⊕ Z^k' → U_0(I)/CU(I)$ as follows:

$$λ(x) = L^t \circ s_1(x)(Ψ^t(x^*)) \text{ for all } x ∈ K_1(C') \quad \text{and} \quad \lambda|_{Z^k'} = Γ_1|_{Z^k'}.$$ (e3.45)

Note that for any $τ ∈ T(I)$ and any $i = r' + 1, ..., k'$, one has that

$$|τ(κ(y_i))| = |τ(L_1(y_i)) - τ(Ψ(y_i))| = |τ(L_1(y_i))| < δ.$$ (e3.46)

By Lemma 9.6 of [15], there is a $G'' ⊗ \{1, z\}-δ'/4$-multiplicative map

$$Φ : C' ⊗ C(T) → \bigoplus_i M_{n_i}(C([0,1]))$$

such that

$$[Φ] = κ \quad \text{and} \quad Φ^t|_{s_1(Z^l ⊕ Z^k')} = λ.$$ 

Denote by

$$w' = (1 - p) ⊕ (Φ(1 ⊕ z)) ⊕ \bigoplus_i 1_{M_{n_i-m_i}}$$

and $ψ : C' → A$ by

$$ψ = L_0 ⊕ Φ|_{C' ⊗ 1} ⊕ Ψ.$$ 

Since $Φ$ is $G'' ⊗ \{1, z\}-δ'/4$-multiplicative, it is clear that

$$||ψ(π(f)), w'|| < ε/4$$

and

$$\text{Bott}(ψ \circ π, w') = κ \circ β \circ π = 0.$$ 

Moreover,

\[
\text{dist}(ψ(u_i'), Γ(x_i)) \approx γ/4 = \text{dist}((1 - p) ⊕ (Φ(u_i')) ⊕ \bigoplus_i 1_{M_{n_i-m_i}}, v_i' ⊕ v_i'')
\]

\[
= \text{dist}(1 - p) ⊕ Γ_1([u_i']) ⊕ \bigoplus_i 1_{M_{n_i-m_i}}, v_i' ⊕ v_i''
\]

\[
= \text{dist}(1 - p ⊕ v_i', v_i' ⊕ v_i'')
\]

\[
= \text{dist}(1 - p, v_i'') \approx γ/4 0.
\]
On the other hand, the map $\psi \circ \pi$ is $G' \cdot \delta'$-multiplicative, and

$$[\psi \circ \pi]|_{P'} = [L_0 \circ \pi]|_{P'} + [\Psi|_{C \cdot \delta' \cdot \pi}]|_{P'} + [\Phi \circ \pi]|_{P'} = [L_0 \circ \pi]|_{P'} + [L_1 \circ \pi]|_{P'} = [\varphi]|_{P'}.$$ 

One also has that, for any $u \in U'$,

$$\text{dist}(\varphi(u), (\psi(\pi(u)))) 
\approx_{\gamma_2} \text{dist}(L_0(\pi(u)) + L_1(\pi(u))), (L_0(\pi(u)) + \Psi(\pi(u) \Psi(\pi(u^*)))) + \Psi(\pi(u))) 
= 0,$$

and for any $h \in H' \cup H''$,

$$|\tau(\varphi(h)) - \tau(\psi(\pi(h)))| 
\approx_{\gamma_1/4} |\tau(L_0(\pi(h)) + L_1(\pi(h))) - \tau(L_0(\pi(h)) + \Psi(\pi(h) \otimes 1) + \Phi(\pi(h)))| 
\approx_{\gamma_1/2} \tau(\Psi(\pi(h) \otimes 1)) \approx_{\gamma_1/4} 0.$$

It then follows from Lemma 3.4 of [14] that

$$\mu_{\tau \psi \pi}(O_r) > \Delta(r/3)/2$$

for any $r > \eta'$. By Theorem 5.3 of [14], there is a unitary $v$ such that

$$\|\varphi(f) - v^* \psi(\pi(f))v\| < \epsilon/2, \quad \forall f \in F.$$

Then the unitary $w := v^*w' v$ satisfies the lemma. \qed

**Corollary 3.14.** The statement of Theorem 3.13 still holds if $C(X)$ is replaced by $PM_n(C(Y))P$ for a compact subset $Y$ of a finite CW-complex and a projection $P$ in $M_n(C(Y))$.

**Proof.** The corollary clearly holds for $C = M_n(C(X))$ (in this case, $X$ is even not required to be finite dimensional). In what follows we will use this case of the corollary to prove the general case.

Assume that $C = PM_n(C(X))P$. As in the proof of [3.10], without loss of generality, we may assume that $P(x) \neq 0$ for all $x \in X$. Since $X$ is a compact subset of a finite CW-complex, there is an integer $d$ and a projection $Q \in M_d(PM_n(C(X))P)$ such that

$$QM_d(PM_n(C(X))P)Q \cong M_{r}(C(X))$$

for some integer $r$. Note that $Q(x) \neq 0$ for all $x \in X$.

Without loss of generality, one may assume that $P \leq Q$, that is, there is also a partial isometry $V \in M_d(PM_n(C(X))P)$ such that $VV^* \leq Q$ and $V^*V = \{P, 0, \ldots, 0\}$. In particular, $V$ induces an isomorphism between $PM_n(C(X))P$ and the unital hereditary subalgebra of $QM_d(PM_n(C(X))P)Q$ generated by $VV^*$.

Fix $d$, $Q$, and $V$.

Since $X$ is compact, there is an integer $l \geq 1$ such that $X = X_1 \sqcup \cdots \sqcup X_l$ such that the ranks of the restrictions of $P$ and $Q$ to each $X_i$, $1 \leq j \leq l$, are constant. Denote by $P_j$ and $Q_j$ the restriction of $P$ and $Q$ to $X_j$ respectively. Let $R = \max_{1 \leq j \leq l} \{\text{rank} Q_j\}$.

Let $G \subseteq K_0(C)$ be a finitely generated group with a fixed decomposition $G = \mathbb{Z}^k \oplus \text{Tor}(G)$ with $\mathbb{Z}^k$ generated by

$$\{x_1 = [p_1] - [q_1], x_2 = [p_2] - [q_2], \ldots, x_k = [p_k] - [q_k]\},$$

where $p_i, q_i \in M_m(C)$ (for some integer $m \geq 1$) are projections, $i = 1, \ldots, k$. 

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Let $A$ be a unital simple C*-algebra with $TR(A) \leq 1$, and let $\varphi : C \to A$ be a monomorphism. Let $\mathcal{F} \subseteq C$, $\mathcal{M} \subseteq K(C)$, $\epsilon > 0$, $\gamma > 0$, and $\Gamma : \mathbb{Z}^k \to U_0(A)/CU(A)$ be a homomorphism.

Denote by $q = (\varphi \otimes 1_{M_d})(Q)$, $e = \varphi \otimes 1_{M_d}(VV^*) \in qM_d(A)q$ and $v = \varphi \otimes 1_{M_d}(V) \in qM_d(A)q$.

By Remark 2.3 one may choose unitaries $v_1, ..., v_k \in U(M_n(A))$ such that

$$\Gamma(x_i) = \overline{\pi} \in U_0(M_n(A))/CU(M_n(A)), \quad i = 1, ..., k.$$  

Then the elements $(v \otimes 1_m)v_i(v \otimes 1_m)^*$, $i = 1, ..., k$, are unitaries in $M_n(eM_d(A)e) = (e \otimes 1_m)(M_n(qM_d(A)q))(e \otimes 1_m)$.

Choose $\epsilon_1 > 0$ and a finite subset $\mathcal{G}_1 \subseteq QM_d(C)Q$ such that $VV^* \in \mathcal{G}_1$ and if there is a unitary $u \in qM_d(A)q$ with

$$\|[(\varphi \otimes 1_{M_d})|_{QM_d(C)Q}(c), u]\| < \epsilon_1, \quad \forall c \in \mathcal{G}_1,$$

then

$$\|[(\varphi \otimes 1_{M_d})|_{VCV^*}(VcV^*), u]\| < \epsilon, \quad \forall c \in \mathcal{G}.$$  

By choosing $\epsilon_1$ sufficiently small (note that $VV^* \in \mathcal{G}_1$), the element $v^*uv$ can be assumed to be invertible in $A$ and

$$\|[(\varphi(c), \langle v^*uv \rangle)]\| < \epsilon, \quad \forall c \in \mathcal{G}. \quad (3.47)$$

Using the same argument as that of Corollary 3.10 one may choose a finite subset $\mathcal{G}_2 \subseteq QM_d(C)Q$ and $\epsilon_2 > 0$ such that if

$$\|[(\varphi \otimes 1_{M_d})|_{QM_d(C)Q}(c), u]\| < \epsilon_2, \quad \forall c \in \mathcal{G}_2,$$

and

$$Bott(\varphi \otimes 1_{M_d}|_{QM_d(C)Q}, u)|_{VPV^*} = 0,$$

then

$$Bott(\varphi \otimes 1_{M_d}|_{VCV^*}, \langle eue \rangle)|_{VPV^*} = 0.$$  

Then one may assume further that $\epsilon_2$ is sufficiently small so that $\|v^*\langle eue \rangle v - \langle v^*uv \rangle\|$ is small enough so that

$$Bott(\varphi, \langle v^*uv \rangle)|_P = 0. \quad (3.48)$$

Denote by $\tilde{V} = V \otimes 1_m$ and $\tilde{v} = v \otimes 1_m$. Note that $\tilde{V}p_i\tilde{V}^*, \tilde{V}q_i\tilde{V}^* \in M_n(QM_d(C)Q)$.

Define

$$\Gamma' : \mathbb{Z}^k \to U_0(M_n(qM_d(A)q))/CU(M_n(qM_d(A)q))$$

by

$$\Gamma'(x_i) = \overline{vv_i\tilde{v}^* + (q \otimes 1_m - e \otimes 1_m)}, \quad i = 1, ..., k.$$  

One may choose a finite subset $\mathcal{G}_3 \subseteq QM_d(C)Q$ and $\epsilon_3 > 0$ such that if there is a unitary $u \in qM_d(A)q$ such that

$$\|[(\varphi \otimes 1_d)|_{QM_d(C)Q}(c), u]\| < \epsilon_3, \quad \forall c \in \mathcal{G}_3,$$

and if

$$\text{dist}((\varphi \otimes 1_m - \varphi(Vp_iV^*) + \varphi(Vp_iV^*)u \otimes 1_m)(q \otimes 1_m - \varphi(Vq_iV^*) + \varphi(Vq_iV^*)u^* \otimes 1_m), \Gamma'(x_i)) < \frac{\gamma}{(R + \frac{1}{8})}$$

for any $1 \leq i \leq k$, then

$$\text{dist}(g_i, CU(M_n(qM_d(A)q))) < \frac{\gamma}{2(R + \frac{1}{8})}, \quad i = 1, ..., k. \quad (3.49)$$

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where
\[ g_i = ((q \otimes 1_m - \varphi(Vp_iV^*) + \varphi(Vp_iV^*)u \otimes 1_m)(q \otimes 1_m - \varphi(Vq_iV^*) + \varphi(Vq_iV^*)u^* \otimes 1_m)(\bar{v}v_i^* \bar{v}^* + (q \otimes 1_m - e \otimes 1_m)). \]

One may assume that \( \epsilon_3 \) is sufficiently small so that
\[
\|g_i - (g'_i + (q \otimes 1_m - e \otimes 1_m))\| < \frac{\gamma}{2(R + \frac{1}{8})}, \quad i = 1, \ldots, k, \tag{e3.50}
\]
where
\[
g'_i = ((e \otimes 1_m - \varphi(Vp_iV^*) + \varphi(Vp_iV^*)(eue) \otimes 1_m)(e \otimes 1_m - \varphi(Vq_iV^*) + \varphi(Vq_iV^*)(eue)^* \otimes 1_m)(\bar{v}v_i^* \bar{v}^*).
\]

As in the proof of [3.10], since \( A \) is a unital simple C*-algebra with \( TR(A) \leq 1 \), one has that \( g' \in U_0(M_m(eM_d(A)e)) \). Note that for any \( \tau \in T(M_m(qM_d(A)q)) \), one has
\[
\tau(e \otimes 1_m) \geq \frac{1}{R},
\]
and therefore \( R[e \otimes 1_m] \geq [q \otimes 1_m - e \otimes 1_m] \). By Lemma 3.3 of [11], one has that
\[
\text{dist}(g'_i, CU(M_m(eM_d(A)e))) < (R + \frac{1}{8}) \frac{\gamma}{(R + \frac{1}{8})} = \gamma, \quad i = 1, \ldots, k.
\]

Then one may also assume further that \( \epsilon_3 \) is sufficiently small so that
\[
\text{dist}((1_m - \varphi(p_i) + \varphi(p_i)(v^*uv) \otimes 1_m)(1_m - \varphi(q_i) + \varphi(q_i)(v^*uv)^* \otimes 1_m)v_i^*, CU(M_m(A)) < \gamma, \tag{e3.51}
\]
for any \( 1 \leq i \leq k \). That is,
\[
\text{dist}((1_m - \varphi(p_i) + \varphi(p_i)(v^*uv) \otimes 1_m)(1_m - \varphi(q_i) + \varphi(q_i)(v^*uv)^* \otimes 1_m), \Gamma(x_i)) < \gamma, \tag{e3.52}
\]
for any \( 1 \leq i \leq k \).

Now, since \( Q(M_d(PM_d(C(X)))P))Q \cong M_r(C(X)) \), applying the corollary to \( M_r(C(X)) \), one obtains a unitary \( u \in qM_d(A)q \) such that
\[
\|[\varphi \otimes 1_{M_d}]_Q |Q_{M_d(C)}(c), u]\| < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}, \quad \forall c \in G_1 \cup G_2 \cup G_3,
\]
and
\[
\text{Bott}((\varphi \otimes 1_{M_d})|Q_{M_d(C)}(c), u)|vPv^* = 0,
\]
and
\[
\text{dist}((q \otimes 1_m - \varphi(Vp_iV^*) + \varphi(Vp_iV^*)u \otimes 1_m)(q \otimes 1_m - \varphi(Vq_iV^*) + \varphi(Vq_iV^*)u^* \otimes 1_m), \Gamma'(x_i))
\]
\[
< \frac{\gamma}{(R + \frac{1}{8})}
\]
for any \( 1 \leq i \leq k \).

By (e3.47), (e3.48), and (e3.52), the unitary \( w = (v^*uv) \in A \) satisfies the corollary. \( \square \)

**Lemma 3.15.** Let \( C = C(X) \) with \( X \) a compact metric space, and let \( A \) be a simple C*-algebra with \( TR(A) \leq 1 \). Suppose that \( h : C \rightarrow A \) is a unital homomorphism, and \( \varphi : C \rightarrow A \) is a non-unital homomorphism with
\[
h_{*1} = \varphi_{*1}.
\]

Denote by \( p = \varphi(1_C) \). For any \( \epsilon > 0 \), any finite subset \( F \subseteq C \), any finite subset \( P \subseteq K(C) \), and any finite subset \( U \subseteq U_c(C) \), there is a \( \epsilon \)-multiplicative map \( \Phi : C \rightarrow (1 - p)A(1 - p) \) such that
\[
[\Phi]|_P = [H]|_P,
\]
where \( H : C \rightarrow pAp \) is the direct sum of finitely many point-evaluations and
\[
\text{dist}(h^+(\mathcal{P}^{-1}((\Phi \oplus \varphi)(u)), \mathcal{I}) < \epsilon, \quad \forall u \in U.
\]
Proof. Since $C$ can be written as an inductive limit of the $C^*$-algebras of continuous functions on finite CW-complexes, without loss of generality, one may assume that $X$ is a finite CW-complex.

Denote by $N_1$ and $N_2$ the constants of Lemma 9.6 of [15] with respect to $X$, $\mathcal{F}$ (in the place of $G$), $\epsilon$ (in the place of $\delta$).

Choose $G \subseteq C$ and $\delta > 0$ such that for any $C^*$-algebra $B$ and any $G$-$\delta$-multiplicative map $L : C \to B$, the element $L(u)$ is invertible for all $u \in U$. Since $h_{a_1} = \varphi_{a_1}$ and $A$ has stable rank one, there is $T > 0$ such that
\[ \text{cel}(h(u)((1 - p) \oplus \varphi(u^*))) < T, \quad \forall u \in U. \]

Since $TR(A) \leq 1$, there is an interval algebra $I \subseteq A$ with $q = 1_I$ and $G$-$\delta$-multiplicative maps $h_0, \varphi_0 : C \to (1 - q)A(1 - q)$, $h_1, \varphi_1 : C \to I$ such that
(1) $\|h(u) - (h_0(u) \oplus h_1(u))\| < \epsilon/4$ and $\|\varphi(u) - (\varphi_0(u) \oplus \varphi_1(u))\| < \epsilon/4$ for any $u \in U$.

(2) The element $p_0 := h_0(1_C) - \varphi_0(1_C)$ is a projection in $(1 - q)A(1 - q)$, and the element $p_1 := h_1(1_C) - \varphi_1(1_C)$ is a projection in $I$; moreover, $p_0 + p_1 = p$.

(3) $\text{dist}(\overline{\{h_0(u) \oplus q\}, I_A}) < \epsilon/4$ and $\text{dist}(\overline{\{\varphi_0(u) \oplus p_0 \oplus q\}, I_A}) < \epsilon/4$ for any $u \in U$.

(4) The rank of $p_1$ is at least $N_1(\dim(X) + 1)$.

Then it follows from Lemma 9.6 of [15] that there is a $\mathcal{F}$-$\epsilon$-multiplicative map $\Phi' : C \to p_1 Ip_1$ such that
\[ [\Phi']_p = [H']_p, \]
where $H' : C \to p_1 Ip_1$ is the direct sum of finitely many point evaluation maps and
\[ (\Phi')^\perp(u) = h_1(u)(\varphi_1(u^*)^\perp), \quad \forall u \in U. \]

Let $\Phi_0 : C \to (1 - q)A(1 - q)$ be the map $f \to f(\xi)(1 - p_0)$ for some $\xi \in X$, and define
\[ \Phi = \Phi_0 \oplus \Phi' : C \to (p_0 + p_1)A(p_0 + p_1) = pAp. \]

It is clear that
\[ [\Phi]_p = [H]_p, \]
where $H : C \to pAp$ is a direct sum of finitely many point-valuations. Furthermore, for any $u \in U$, one has
\[
\text{dist}(\overline{\{(\Phi(u) \oplus \varphi(u), h(u))\}}) < \text{dist}(\overline{\{(\Phi_0(u) \oplus \varphi_0(u) \oplus (\Phi'(u) \oplus \varphi_1(u)), (h_0(u) \oplus h_1(u))\}}) + \epsilon/4
\]
\[
= \text{dist}(\overline{\{p_0 \oplus \varphi_0(u) \oplus q, I_A\}}) + \text{dist}(\overline{\{h_0(u) \oplus q\}}) + \text{dist}(\overline{\{(1 - q) \oplus \Phi'(u) \oplus \varphi_1(u)), (1 - q) \oplus h_1(u)\}}) + \epsilon/4
\]
\[
\leq \text{dist}(\overline{\{(\Phi'(u) \oplus \varphi_1(u)), (h_1(u))\}}) + 3\epsilon/4
\]
\[
= 3\epsilon/4 < \epsilon.
\]

This proves the lemma.

\[ \square \]

**Theorem 3.16.** Let $C$ be an AH-algebra, and let $A$ be a simple $C^*$-algebra with $TR(A) \leq 1$. Suppose that $h : C \to A$ is a monomorphism. Then, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subseteq C$ and any finite subset $\mathcal{P} \subseteq K(C)$, there is a $C^*$-algebra $C' \cong PM_n(C(X'))\mathcal{P}$ for some finite CW-complex $X'$ with $K_1(C') = \mathbb{Z}^k \oplus \text{Tor}(K_1(C'))$ and a homomorphism $\iota : C' \to C$ with
\(\mathcal{P} \subseteq [i](K(C'))\), a finite subset \(\mathcal{Q} \subseteq \mathbb{Z}^k \subset K_1(C')\) and \(\delta > 0\) satisfying the following: Suppose that \(\kappa \in \text{Hom}_A(K(C' \otimes \mathbb{C}(T)), K(A))\) with
\[|\rho_A \circ \kappa(\beta(x))(\tau)| < \delta, \quad \forall x \in \mathcal{Q}, \forall \tau \in T(A).\]
Then there exists a unitary \(u \in A\) such that
\[\|h(c, u)\| < \epsilon, \quad \forall c \in \mathcal{F}\text{ and } \text{Bott}(h \circ \iota, u) = \kappa \circ \beta.\]
Moreover, there is a sequence of \(C^\ast\)-algebras \(C_n\) with the form \(C_n = P_nM_{r(n)}(C(X_n))P_n\), where each \(X_n\) is a finite CW-complex and \(P_n \in M_{r(n)}(C(X_n))\) a projection, such that \(C = \lim\langle C_n, \varphi_n \rangle\) for a sequence of unital homomorphisms \(\varphi_n : C_n \to C_{n+1}\) and one may choose \(C' = C_n\) and \(\iota = \varphi_n\) for some integer \(n \geq 1\).

**Proof.** The proof is similar to that of Theorem 6.3 of \cite{12}. Without loss of generality, one may assume that \(C = C(X)\) for some compact metric space \(X\). Denote by
\[\Delta(a) = \inf \{\mu(x)(O, \tau) ; \tau \in T(A), O, a\text{ is an open ball of } X\text{ with radius } a\}.
\]
Since \(A\) is simple and \(T(A)\) is compact, \(\Delta(a)\) is a nondecreasing function from \((0, 1)\) to \((0, 1)\).

Let \(B\) be a unital separable simple amenable \(C^\ast\)-algebra with \(TR(B) = 0\) satisfying the UCT and
\[(K_0(B), K_0^+(B), [1_B], K_1(B)) \cong (K_0(A), K_0^+(A), [1_A], K_1(A)).\]
Then there is an embedding \(\iota' : B \to A\) such that \([\iota']\) induces an identification of the above. In the following, we identify \(B\) as a \(C^\ast\)-subalgebra of \(A\).

Let \(\epsilon_1 > 0\) with \(\epsilon_1 < \epsilon\), and let \(\mathcal{F}_1 \supseteq \mathcal{F}\) be a finite subset such that for any unital homomorphism \(H : C \to A\) and unitary \(u' \in A\) satisfying
\[\|H(c) - H'(c)\| < \epsilon_1, \quad \forall c \in \mathcal{F}_1,
\]the map \(\text{Bott}(H, u')|_{\mathcal{P}}\) is well defined; moreover, if \(H' : C \to A\) is any other unital monomorphism satisfying
\[\|H(c) - H'(c)\| < \epsilon_1, \quad \forall c \in \mathcal{F}_1,
\]then
\[\text{Bott}(H, u')|_{\mathcal{P}} = \text{Bott}(H', u')|_{\mathcal{P}}.
\]
Let \(\eta, \delta_1\) (in the place of \(\delta\)), \(\mathcal{G}_1 \subseteq C\) (in the place of \(\mathcal{H}\)), \(\mathcal{P}' \subseteq K(C)\) (in the place of \(\mathcal{P}\)), and \(\mathcal{U} \subseteq U_c(K_1(C))\), \(\gamma_1, \gamma_2\) be the constants and finite subsets of Theorem 5.3 of \cite{14} with respect to \(\epsilon_1/2, \mathcal{F}_1\), and \(\Delta(\cdot/3)/2\).

Let \(\delta_2\) and \(\mathcal{G}'_2 \subseteq C\) be the constant and finite subset required by Lemma 3.4 of \cite{14} with respect to \(\Delta, \eta, \lambda_1, \lambda_2 = 1/2\). Denote by \(\mathcal{G}_2 = \mathcal{G}_1 \cup \mathcal{G}'_2\). Without loss of generality, one may assume that \(\delta_2 < \gamma_1\).

By Lemma 6.2 of \cite{12}, there is a \(\mathcal{G}_1\)-\(\delta_1\)-multiplicative map \(h_0 : C \to p_0BP_0\) with \(\tau(p_0) < \delta_2/4\) and a unital homomorphism \(h'_1 : C \to F\), where \(F\) is a finite dimensional \(C^\ast\)-subalgebra of \(B\) with \(1_F = 1 - p_0\) such that
\[|h_0 + h'_1|_{\mathcal{P}'} = |h|_{\mathcal{P}'} \in KL(C, A).
\]
Let \(C' \subseteq C, 1 > \delta_3 > 0\) and \(Q' \subseteq K_1(C')\) (in place of \(Q\)) be the constant and finite subset required by Lemma 6.1 of \cite{12} with respect to \(\mathcal{F}, \mathcal{P}\), and \(p_0BP_0\). We may write \(K_1(C') = \mathbb{Z}^k \oplus \text{Tor}(K_1(C'))\). Let \(Q \subseteq \mathbb{Z}^k\) be a finite subset such that
\[Q' = \{x + y : x \in Q \text{ and } y \in \text{Tor}(K_1(C'))\}.
\]
Let \( \delta = \min\{\delta_3 \delta_1/16\pi, \delta_3 \delta_2/4\} \). Now let \( \kappa \in \text{Hom}_A(\overline{K(C') \otimes C(T)}), \overline{K(A)} \) with
\[
|\rho_A \circ \kappa(\beta(x))| < \delta \quad \text{for all } x \in Q \quad \text{and for all } \tau \in T(A).
\]
Note that this implies
\[
|\rho_A \circ \kappa(\beta(x))| < \delta \quad \text{for all } x \in Q'
\]
and for all \( \tau \in T(A) \). By Lemma 6.1 of [12], there is a unitary \( u_0 \in p_0 Bp_0 \) such that
\[
||[h_0(c), u_0]|| < \epsilon_1/2, \quad \forall c \in F,
\]
and
\[
\text{Bott}(h_0 \circ \iota, u_0) = \kappa \circ \beta.
\]
Put \( u = u_0 + (1 - p_0) \). Then there is a nonzero projection \( q_0 \in (1 - p_0)A(1 - p_0) \) such that
\[
q_0 f = f q_0, \quad \forall f \in F, \quad \text{and } \tau(q_0) < \delta, \quad \forall \tau \in T(A).
\]
Define \( \psi_0(c) = q_0 h_1'(c) \) and \( \psi_0'(c) = (1 - p_0 - q_0)h_1' \) for all \( c \in C \). By Lemma 9.5 of [8], there is \( C^* \)-subalgebra \( B_0 \in (1 - p_0 - q_0)A(1 - p_0 - q_0) \) with \( B_0 \) an interval algebra and a unital homomorphism \( h_1 : C \rightarrow B_0 \) such that \( (h_1)_* \psi_0 = (\psi_0)_0 \) and
\[
|\tau(h_1(f)) - \tau(1 - p_0 - q_0)\tau(h(f))| < \delta_2/4, \quad \forall f \in G_2.
\]
Define \( \psi_1 = h_0 + h_1 \). By Lemma 3.15, there is \( \mathcal{G}'-\delta_1 \)-multiplicative map \( \Phi : C \rightarrow q_0 Aq_0 \) with \( \Phi_* \psi_1 = (\psi_0)_0, \quad \Phi|_{\mathcal{P}'[\mathcal{P}]} = [H]|_{\mathcal{P}'} \) in \( KL(C, A) \) for some point evaluation map \( H : C \rightarrow p_0 Ap \), and
\[
\text{dist}(\overline{h^+ \tau^{-1}(\Phi \oplus \psi_1)(u)}), \overline{I}) < \gamma_2, \quad \forall u \in \mathcal{U}.
\]
Define \( h_2 = \Phi \oplus \psi_1 \). Then \( |h_2|_{\mathcal{P}'} = |h|_{\mathcal{P}'} \) in \( KL(C, A) \) and for any \( u \in \mathcal{U} \),
\[
\text{dist}(\overline{(h_2(u))}, \overline{h(u)}) = \text{dist}(\overline{(\Phi(u) \oplus \psi_1(u))}, \overline{h(u)}) \approx \gamma_2 0.
\]
Moreover, for any \( f \in G_2 \) and any \( \tau \in T(A) \),
\[
|\tau(h(f)) - \tau(h_2(f))| < \delta_2 / 4 + |\tau(h(f)) - \tau(h_1(f))| \leq 3\delta_2 / 4 + |\tau(1 - p_0 - q_0)\tau(h(f)) - \tau(h_1(f))| < \delta_2 < \gamma_1.
\]
Note that \( \mu_{\text{roh}}(O_a) \geq \Delta(a) \) for any \( a \), by Lemma 3.4 of [14], one has
\[
\mu_{\text{roh}^2}(O_a) \geq \frac{1}{2} \Delta(a/3)
\]
for any \( a \geq \eta \). Then, by Theorem 5.3 of [14], there is a unitary \( U \in A \) such that
\[
\text{ad}U \circ h_2 \approx_{\epsilon_1/2} h, \quad \text{on } \mathcal{F}_1.
\]
Define \( u = U^*(u_0 + (1 - p_0))U \). Then
\[
||[h(c), u]|| < \epsilon_1, \quad \forall c \in \mathcal{F}_1.
\]
Moreover, by the choice of \( \epsilon_1 \), one has
\[
\text{Bott}(h \circ \iota, u) = \text{Bott}(h_2 \circ \iota, u_0 + (1 - p_0)) = \text{Bott}(h_0 \circ \iota, u_0) = \kappa \circ \beta,
\]
as desired.
4 Asymptotic unitary equivalence

**Lemma 4.1.** Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Suppose that

1. $[\varphi_1] = [\varphi_2]$ in $KL(C, A)$, $\varphi_1^\dagger = \varphi_2^\dagger$, $(\varphi_1)_z = (\varphi_2)_z$,
2. $R_{\varphi_1,\varphi_2}(K_1(M_{\varphi_1,\varphi_2})) \subseteq \rho_A(K_0(A))$.

Then, for any increasing sequence of finite subsets $(F_n)$ of $C$ whose union is dense in $C$, any increasing sequence of finite subsets $(P_n)$ of $K_1(C)$ with $\bigcup_{n=1}^\infty P_n = K_1(C)$ and any decreasing sequence of positive number $(\delta_n)$ with $\sum_{n=1}^\infty \delta_n < \infty$, there exists a sequence of unitaries $(u_n)$ in $U(A)$ such that

$$\text{ad}(u_n) \circ \varphi_1 \approx_{\delta_n} \varphi_2 \quad \text{on } F_n,$$

and

$$\rho_A(\text{bott}_1(\varphi_2, u_n^*u_{n+1})(x)) = 0,$$

for all $x \in P_n$ for all sufficiently large $n$.

**Proof.** The proof is a simple modification of the proof of Lemma 7.1 of [12]. In the place of Theorem 6.3 of [12] being used, one uses the second part of Theorem 3.16 instead. $\square$

**Theorem 4.2.** Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Then there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\} \subseteq A$ such that

$$\lim_{t \to \infty} \text{ad}(U(t)) \circ \varphi_1(c) = \varphi_2(c) \quad \text{for all } c \in C$$

if and only if

$$[\varphi_1] = [\varphi_2] \quad \text{in } KK(C, A), \quad (\varphi_1)^\dagger = (\varphi_2)^\dagger, \quad (\varphi_1)_z = (\varphi_2)_z$$

and

$$\overline{R}_{\varphi_1,\varphi_2} = 0.$$

**Proof.** We only have to show the “if” part.

Let $C = \lim\!(C_n, \psi_n)$, where $C_n$ is a $C^*$-algebra in the form of $P_nM_{\psi_n}(C(X_n))P_n$ with $X_n$ having a finite covering dimension, and $\psi_n : C_n \to C_{n+1}$ is a unital monomorphism. Let $(F_n)$ be an increasing sequence of finite subsets of $C$ such that $\bigcup_{n=1}^\infty F_n$ is dense in $C$.

For each $n$ and $0 < a < 1$, define

$$\Delta_n(a) = \inf\{\mu_{r_0\varphi_1}(O_a) : O_a \text{ an open ball of } X_n \text{ with radius } a\}.$$

Since $A$ is simple, one has that $\Delta_n(a) \in (0, 1)$ for any $a \in (0, 1)$.

Consider the mapping torus

$$M_{\varphi_1,\varphi_2} = \{f \in C([0,1], A) : f(0) = \varphi_1(a) \text{ and } f(1) = \varphi_2(a) \text{ for some } a \in C\}.$$

Since $C$ satisfies the Universal Coefficient Theorem, the assumption of $[\varphi_1] = [\varphi_2]$ in $KK(C, A)$ implies the following short exact sequence splits:

$$0 \to K(SA) \to K(M_{\varphi_1,\varphi_2}) \to \pi_0 K(C) \to 0.$$

Denote by $\theta : K(C) \to K(M_{\varphi_1,\varphi_2})$ the splitting map.
Since \( \tau \circ \varphi_1 = \tau \circ \varphi_2 \) for all \( \tau \in T(A) \) and \( R_{\varphi_1, \varphi_2}(\theta(x)) = 0 \), we may also assume that
\[
R_{\varphi_1, \varphi_2}(\theta(x)) = 0,
\]
for all \( x \in K_1(C) \).

In what follows, for any C*-algebras \( C'' \) and \( A \) and a homomorphism \( \varphi : C'' \to A \), for any \( x = [p] - [q] \in K_0(C) \) with projections \( p, q \in M_n(A) \) (for some integer \( n \geq 1 \)) and a unitary \( u \in A \) with \( \| ([\varphi(p), \bar{u}] ) \| < 1/4 \) and \( \| ([\varphi(q), \bar{u}] ) \| < 1/4 \), where \( \bar{u} = \text{diag}(u, \ldots, u) \), define
\[
g_{x,u}^\varphi := \langle (1 - \varphi(p) + \varphi(q)\bar{u})(1 - \varphi(q) + \varphi(q)\bar{u}) > \rangle \in U_n(A)/CU_n(A). \tag{e 4.53}
\]

Let \( \delta_n''' > 0 \) (in place of \( \delta \)), \( \eta_n''' \) (in place of \( \eta \)), \( \gamma_n''' \) (in place of \( \gamma \)), \( G_n' \subseteq C_n \) (in place of \( G \)), \( P_n' \subseteq K_0(C_n) \) and \( Q_n' = \{ x_n,1, \ldots, x_n,m(n) \} \subseteq K_0(C_n) \) (in place of \( Q \)) be the constants and finite subsets corresponding to \( 1/2^{n+1} \), \( \mathcal{F}_n \) and \( \Delta_n \) required by Theorem 3.4. Without loss of generality, one may assume that \( [\psi_{n+1}'] (P_n') \subseteq P_{n+1}' \) for all \( n \). Note that \( \{ x_{n,1}, \ldots, x_{n,m(n)} \} \) are free (hence generate a group \( \mathbb{Z}^{m(n)} \subseteq K_0(C_n) \)), and write \( x_{n,i} = [p_{n,i}] - [q_{n,i}] \) for some projections \( p_{n,i}, q_{n,i} \in M_{l(n)}(C_n) \).

Consider the image \( [\psi_{n+1}'] (\mathbb{Z}^{m(n)}) \), and fix a decomposition
\[
[\psi_{n+1}'] (\mathbb{Z}^{m(n)}) = \mathbb{Z}^{k(n)} \oplus \text{Tor}([\psi_{n+1}'] (\mathbb{Z}^{m(n)}))
\]
for some integer \( k(n) \). One also fixes a lifting of \( \mathbb{Z}^{k(n)} \) in \( \mathbb{Z}^{m(n)} \). Write \( \{ y_{n,1}, y_{n,2}, \ldots, y_{n,k(n)} \} \) a set of generators of \( \mathbb{Z}^{k(n)} \), and \( \{ y_{n,1}', y_{n,2}', \ldots, y_{n,k(n)'} \} \) the corresponding elements in \( \mathbb{Z}^{m(n)} \). Note that there are integers \( c_{i,j}^{(n)} \) such that
\[
x_{n,i} = \sum_{j=1}^{k(n)} c_{i,j}^{(n)} y_{n,j} + r_i, \quad i = 1, \ldots, m(n)
\]
with \( [\psi_{n+1}'] (r_i) \) a torsion element in \( K_0(C_{n+1}) \).

Therefore, without loss of generality, one may assume that \( \delta_n''' \) is sufficiently small and \( G_n' \) is sufficiently large such that if \( h' : C \to A \) is a homomorphism and \( u' \in A \) a unitary with \( \| (h'(a), u') \| < \delta_n''' \) for all \( a \in G_n' \), then
\[
\text{dist}(g_{x_{n,i}, u'}^h, \sum_{j=1}^{k(n)} c_{i,j}^{(n)} g_{y_{n,j}, u'}^h) < \gamma_n''' /8, \quad i = 1, \ldots, m(n). \tag{e 4.54}
\]

We also assumes that \( \text{Bott}(h', u')|_{\mathcal{P}_n} = \text{Bott}(h', u)|_{\mathcal{P}_n} \) is well defined whenever \( \| (h'(a), u') \| < \delta_n''' \) for any homomorphism \( h' \) and unitary \( u' \), and moreover, if \( h \approx_{\delta_n''} h' \) on \( G_n' \), then
\[
\text{Bott}(h, u)|_{\mathcal{P}_n} = \text{Bott}(h', u)|_{\mathcal{P}_n}.
\]

Let \( C_n' \) (in place of \( C' \)) with \( K_1(C_n') = \mathbb{Z}^{r(n)} \oplus \text{Tor}(K_1(C_n')) \), \( \iota_n : C_n' \to C_n \), \( Q_n'' \subseteq \mathbb{Z}^{r(n)} \) (in place of \( Q \), and \( \eta_n \) (in place of \( \delta \)) be required by Theorem 3.16 for \( G_n' \) (in place of \( G \)), \( P_n' \) (in place of \( P \)) and \( \delta_n'' /4 \) (in place of \( \epsilon \)). One also fixes a finite set of generators of \( K_1(C_n') \) for each \( n \). Without loss of generality, one may assume that \( Q_n'' \) is the set of standard generators of \( \mathbb{Z}^{r(n)} \).

Put \( \delta_n = \min \{ \eta_n, \delta_n'' /2 \} \).

By Lemma 4.31, there are unitaries \( v_n \in U(A) \) such that
\[
\text{ad}(v_n) \circ \varphi_1 \approx_{\delta_n+1/4} \varphi_2 \quad \text{on} \quad \psi_{n+1,\infty}(G_{n+1}'),
\]
\[
\rho_A(\text{bott}(\varphi_2 \circ \iota_n, v_n v_n^*(x))) (x) = 0 \quad \text{for all} \quad x \in \psi_{n+1,\infty}(K_1(C_{n+1})),
\]

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and
\[ ||[\varphi_2(a), v_n^*v_{n+1}]|| < \delta_{n+1}/2 \quad \text{for all } a \in \psi_{n+1,\infty}(G_{n+1}'). \]

Then we have that
\[ \text{Bott}(\varphi_1 \circ \iota_{n+1}, v_n v_{n+1}^*) = \text{Bott}(v_n^*(\varphi_1 \circ \iota_{n+1})v_n, v_{n+1}^*(v_n v_{n+1}^*)v_n) = \text{Bott}(\varphi_2 \circ \iota_{n+1}, v_n^*v_{n+1}). \]

In particular, for any \( x \in (\psi_{n+1,\infty} \circ \iota_{n+1})_1(K_1(C_{n+1}')) \), one has
\[ \text{bott}_1(v_n^*\varphi_1 v_n, v_n^*v_{n+1})(x) = \text{bott}_1(\varphi_2, v_n^*v_{n+1})(x). \]

By applying 10.4 and 10.5 of [9], without loss of generality, we may assume that \( \varphi_1 \circ \psi_{n+1,\infty} \circ \iota_{n+1} \) and \( v_n \) define an element \( \gamma_n \in \text{Hom}_A(K_1(C_{n+1}'), K(M_{\varphi_1 \circ \iota_{n+1}, \varphi_2 \circ \iota_{n+1}})) \) and \( [\pi_0] \circ \gamma_n = [\iota_{n+1}] \).

Moreover, \( \gamma_n \) factors through \( H_n := [\psi_{n+1,\infty} \circ \iota_{n+1}]_1(K_1(C_{n+1}')) \). Thus, one may also regard \( \gamma_n \) being defined on \( H_n \).

Furthermore, by 10.4 and 10.5 of [9], without loss of generality, we may assume that
\[ \tau(\log((\varphi_2 \circ \psi_{n+1,\infty} \circ \iota_{n+1}(z_j^*))\overline{\tau}_n(\varphi_1 \circ \psi_{n+1,\infty} \circ \iota_{n+1}(z_j))\overline{\tau}_n)) < \delta_{n+1} \quad (e.4.55) \]

where \( \{z_1, ..., z_{r(n)}\} \subseteq U(M_k(C_{n+1}')) \) induces a set of standard generators of \( \mathbb{Z}^{r(n)} \subseteq K_1(C_{n+1}') \) and \( \overline{\tau}_n = \text{diag}(v_n, ..., v_n) \).

Since \( \bigcup_{n=1}^\infty [\psi_{n+1,\infty} \circ \iota_{n+1}]_1(K_1(C_{n+1}')) = K_1(C) \) and \( [\pi_0] \circ \gamma_n = [\iota_{n+1}] \), one concludes
\[ K(M_{\varphi_1, \varphi_2}) = K(SA) + \bigcup_{n=1}^\infty \gamma_n(H_n). \quad (e.4.56) \]

By passing to a subsequence, one may assume that
\[ \gamma_n(H_n) \subseteq K(SA) + \gamma_{n+1}(H_{n+1}), \quad n = 1, 2, ... \]

By 10.6 of [9], \( \Gamma(\text{Bott}(\varphi_1, v_n v_{n+1}^*))|_{H_n} = (\gamma_n - \gamma_{n+1} \circ [\psi_{n+1}])|_{H_n} \) defines a homomorphism \( \xi_n : H_n \to K(SA) \). Then define a map \( j_n : K(SA) \oplus H_n \to K(SA) \oplus H_{n+1} \) by
\[ (x, y) \mapsto (x + \xi_n(y), [\psi_{n+1}](y)). \]

By (e.4.56), the limit is \( K(M_{\varphi_1, \varphi_2}) \). One has the following diagram
\[
\begin{array}{cccccccc}
0 & \to & K(SA) & \rightarrow & K(SA) & \rightarrow & \bigoplus H_n & \rightarrow & H_n & \rightarrow & 0 \\
0 & \rightarrow & K(SA) & \rightarrow & K(SA) & \rightarrow & \bigoplus H_{n+1} & \rightarrow & H_{n+1} & \rightarrow & 0.
\end{array}
\]

By the assumption that \( R_{\varphi_1, \varphi_2} = 0 \), the map \( \theta \) also induces the following
\[ \ker R_{\varphi_1, \varphi_2} = \ker \rho_A \oplus K_1(C). \]

Define \( \zeta_n = \gamma_{n+1}|_{H_n}, \) \( \theta_n = \theta \circ [\psi_{n+1,\infty}] \), and \( \kappa_n = \zeta_n - \theta_n \). Note that
\[ \theta_n = \theta_{n+1} \circ [\psi_{n+2}] \]

and
\[ \zeta_n - \zeta_{n+1} \circ [\psi_{n+2}] = \xi_n. \]
Since $[\pi_0] \circ (\zeta_n - \theta_n) = 0$, $\kappa_n$ maps $H_n$ into $K(SA)$. It follows that
\[
\begin{align*}
\kappa_n - \kappa_{n+1} &= \zeta_n - \theta_n - \zeta_{n+1} \circ [\psi_{n+2}] + \theta_{n+1} \circ [\psi_{n+2}] \\
&= \zeta_n - \zeta_{n+1} \circ [\psi_{n+2}] = \xi_n.
\end{align*}
\]

It follows from 10.3 of [9] that there are integers $N_1 \geq 1$, a $\delta_{n+1}\psi_{n+1}(C'_{n+1})$-multiplicative map
\[
L_n : \psi_{n+1,\infty} \circ \iota_{n+1}(C'_{n+1}) \to M_{1+N_1}(M_{\psi_1,\psi_2}),
\]
a unitary homomorphism $h_\theta : \psi_{n+1,\infty} \circ \iota_{n+1}(C'_{n+1}) \to M_{N_1}(C)$, and a continuous path of unitaries \{\(V_n(t) : t \in [0,3/4]\)\} of $M_{1+N_1}(A)$ such that $[L_n]|_{p'_{n+1}}$ is well defined, $V_n(0) = 1_{M_{1+N_1}(A)}$,
\[
[L_n \circ \psi]|_{p'_{n}} = (\theta \circ [\psi_{n+1,\infty}] + [h_0 \circ \psi_{n+1,\infty}])|_{p'_{n}},
\]
\[
\pi_t \circ L_n \circ \psi_{n+1,\infty} \approx_{\delta_{n+1}/4} \text{ad}V_n(t) \circ ((\varphi_1 \circ \psi_{n+1,\infty}) \oplus (h_0 \circ \psi_{n+1,\infty}))
\]
on $G_{n+1}$ and $t \in [0,3/4]$, and
\[
\pi_t \circ L_n \circ \psi_{n+1,\infty} \approx_{\delta_{n+1}/4} \text{ad}V_n(3/4) \circ ((\varphi_1 \circ \psi_{n+1,\infty}) \oplus (h_0 \circ \psi_{n+1,\infty}))
\]
on $G_{n+1}$ and $t \in (3/4,1)$, and
\[
\pi_1 \circ L_n \circ \psi_{n+1,\infty} \approx_{\delta_{n+1}/4} (\varphi_1 \circ \psi_{n+1,\infty}) \oplus (h_0 \circ \psi_{n+1,\infty})
\]
on $G_{n+1}$. Note that $R_{\varphi_1,\varphi_2}(\theta(x)) = 0$ for all $x \in (\psi_{n+1,\infty})_{**1}(K_1(C_{n+1}))$. As computed in 10.4 of [9],
\[
\tau(\log((\varphi_2(z) \oplus h_0(z))^*V_n^*(3/4)(\varphi_1(z) \oplus h_0(z))V_n(3/4))) = 0
\]
for $z = (\psi_{n+1,\infty} \circ \iota_{n+1})_{**1}(y)$, where $y$ in the fixed set of generators of $K_1(C'_{n+1})$ and for all $\tau \in T(A)$.

Define $W'_n = \text{diag}(v_n, 1) \in M_{1+N_1}(A)$. Then
\[
\text{Bott}((\varphi_1 \oplus h_0) \circ \psi_{n+1,\infty} \circ \iota_{n+1}, W'_n(V_n(3/4))^*)
\]
defines a homomorphism $\tilde{\kappa}_n \in \text{Hom}_A(K(C'_{n+1}), K(SA))$.

By (e 4.59), one has
\[
\tau(\log((\varphi_2 \oplus h_0) \circ \psi_{n+1,\infty} \circ \iota_{n+1}(z_j))^*\tilde{V}_n^*(\varphi_1 \oplus h_0) \circ \psi_{n+1,\infty} \circ \iota_{n+1}(z_j)\tilde{V}_n)) < \delta_{n+1}
\]
for $j = 1, 2, ..., r(n)$, where $\tilde{V}_n = \text{diag}(\pi_n, 1)$. Then, by (e 4.59), one has
\[
\rho_A(\tilde{\kappa}_n(z_j))(\tau) < \delta_{n+1}, \quad j = 1, 2, ..., r(n).
\]
It then follows from Theorem [3.16] that there is a unitary $u'_n \in U(A)$ such that
\[
\|\varphi_1(a), w'_n\| < \delta_{n+1}/4, \quad \forall a \in \psi_{n+1,\infty}(G_{n+1}),
\]
and
\[
\text{Bott}(\varphi_1 \circ \psi_{n+1,\infty} \circ \iota_{n+1}, u'_n)|_{K(C'_{n+1})} = -\tilde{\kappa}_n.
\]
Put $w_n = v_n^* w'_n v_n$. One has
\[
\text{Bott}(\varphi_2 \circ \psi_{n+1,\infty} \circ \iota_{n+1}, w_n)|_{K(C'_{n+1})} = -\tilde{\kappa}_n, \quad \text{Bott}(\varphi_1 \circ \psi_{n+2,\infty} \circ \iota_{n+1}, w'_n)|_{K(C'_{n+1})} = -\kappa_{n+1}.
\]
It follows from 10.6 of [9] that
\[
\Gamma(\text{Bott}(\varphi_1 \circ \psi_{n+1,\infty}, w'_n)) = -\kappa_n \quad \text{and} \quad \Gamma(\text{Bott}(\varphi_1 \circ \psi_{n+2,\infty}, w'_n)) = -\kappa_{n+1},
\]
\[29\]
where $\Gamma$ is defined in 10.6 of [9]. One also has

$$\Gamma(\text{Bott}(\varphi_1 \circ \psi_{n+1,\infty}, v_n v_{n+1}^*))|_{H_n} = \zeta_n - \zeta_{n+1} \circ [\psi_{n+2}] = \xi_n.$$  

Then, by (e 4.57), one has

$-\text{Bott}(\varphi_1 \circ \psi_{n+1,\infty}, w_n') + \text{Bott}(\varphi_1 \circ \psi_{n+1,\infty}, v_n v_{n+1}^*) + \text{Bott}(\varphi_1 \circ \psi_{n+1,\infty}, w_{n+1}') = 0.$

Define $u_n' = v_n v_n^*, n = 1, 2, \ldots$ Then,

$$\text{ad}(u_n') \circ \varphi_1 \approx \delta_{n+1}/2 \varphi_2, \quad \forall a \in \psi_{n+1,\infty}(G'_{n+1}),$$

and

$$\text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, (u_n')^* u_{n+1}')
= \text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, w_n v_n v_{n+1}^* u_{n+1}')
= \text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, w_n) + \text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, v_n v_{n+1}^*) + \text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, w_{n+1}')
= \text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, w_n') - \text{Bott}(\varphi_1 \circ \psi_{n+1,\infty}, v_n v_{n+1}^*) - \text{Bott}(\varphi_1 \circ \psi_{n+1,\infty}, w_{n+1}')
= 0.$$

In what follows, we shall construct unitaries $\{s_n\} \subseteq A$ such that

$$||[\varphi_2 \circ \psi_{n+1,\infty}(a), s_n]|| < \delta_{n+1}/2, \quad \forall a \in G'_{n+1}$$

$$\text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, s_n)|_{p_{n+1}'} = 0,$$

and

$$\text{dist}(g_{x_{n+1,j}, s_n u_n'}, g_{x_{n+1,j}, (u_n')^* u_{n+1}'}) < \zeta_{n+1}/2.\quad (e 4.62)$$

(Recall that $x_{n,j} = [p_{n,j}] - [q_{n,j}], j = 1, \ldots, m(n)$ and $n = 1, \ldots$, and $\{x_{1,n}, \ldots, x_{n,m(n)}\}$ is free.)

Put $s_1 = 1$, and assume that $s_1, \ldots, s_n$ has been constructed. Let us construct $s_{n+1}$. Define the map $\Xi_n : \mathbb{Z}^{m(n+1)} \to U(A)/CU(A)$ by

$$\Xi_n(x_{n+1,j}) = g_{x_{n+1,j}, s_n u_n'}, \quad j = 1, \ldots, m(n+1)$$

with the map $\varphi_2 \circ \psi_{n+1,\infty}$ in the place of $\varphi$ in (e 4.53).

Recall that there are fixed decomposition $[\psi_{n+1,\infty}(x_{n+2})](\mathbb{Z}^{m(n+1)}) = \mathbb{Z}^{k(n+1)} \oplus \text{Tor}([\psi_{n+1,\infty}(x_{n+2})](\mathbb{Z}^{m(n+1)}))$ (for some integer $k(n+1)$) and a fixed lifting of $\mathbb{Z}^{k(n+1)}$ in $\mathbb{Z}^{m(n+1)}$ for each $n$. Also recall that $\{y_{n+1,1}, y_{n+1,2}, \ldots, y_{n+1,k(n+1)}\}$ is a fixed set of generators for $\mathbb{Z}^{k(n+1)}$, and $\{y_{n+1,1}, y_{n+1,2}, \ldots, y_{n+1,k(n+1)}\}$ are their liftings in $\mathbb{Z}^{m(n+1)}$. Then define the map $\Xi_n : \mathbb{Z}^{k(n+1)} \to U(A)/CU(A)$ by

$$\Xi_n(y_j) = \Xi_n(y_j'), \quad j = 1, \ldots, k(n+1).$$

Let $\epsilon'' > 0$ be arbitrary (which will be fixed later). Applying Theorem 3.13 to $C_{n+2}$ (in place of $C$), $A$, $G'_{n+2}$ (in place of $F$, $P'_{n+2}$ (in place of $P$, $\epsilon''$ (in place of $\epsilon$ and in place of $\gamma$), and $\Xi_n$ (in place of $\Gamma$), there is a unitary $s_{n+1} \in A$ such that

$$||[\varphi_2 \circ \psi_{n+1,\infty}(a), s_{n+1}]|| < \epsilon''_{n+1}, \quad \forall a \in G'_{n+1},$$

$$\text{Bott}(\varphi_2 \circ \psi_{n+2,\infty}, s_{n+1})|_{p_{n+2}'} = 0,$$

and

$$\text{dist}(g_{y_{n+1,j}, s_{n+1}}, \Xi_n(y_{n+1,j})) < \epsilon''_{n}, \quad j = 1, \ldots, k(n+1).$$

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By choosing \( \epsilon_n'' < \delta_{n+1}'/2 \) sufficiently small, it follows from (e 4.63) that the unitary \( s_{n+1} \) satisfies (e 4.60). Since \( \pi([\psi_{n+1,n+2}(x_{n+1,j})]) \) is in the subgroup generated by \( \{y_{n+1,1}, \ldots, y_{n+1,k(n+1)}\} \), where \( \pi \) is the projection map from \( [\psi_{n+1,n+2}]'(Z^{m(n+1)}) \) to \( Z^{k(n+1)} \), by choosing \( \epsilon_n'' \) smaller, it follows from (e 4.65) and (e 4.54) that
\[
\text{dist}(g_{x_{n+1,j},s_{n+1}+1}^2, \Xi'(x_{n+1,j})) < \gamma_{n+1}'/4, \quad j = 1, \ldots, m(n+1),
\]
which is
\[
\text{dist}(g_{x_{n+1,j},s_{n+1}+1}^2, g_{x_{n+1,j},s_{n+1}+1}^2 u_n s_{n+1}^* w_{n+1}) < \gamma_{n+1}'/4, \quad j = 1, \ldots, m(n+1).
\]
Hence
\[
\text{dist}(g_{x_{n+1,j},s_{n+1}+1}^2, g_{x_{n+1,j},s_{n+1}+1}^2 u_n s_{n+1}^* w_{n+1}) < \gamma_{n+1}'/2, \quad j = 1, \ldots, m(n+1),
\]
which verifies (e 4.62).

Therefore, one obtains the sequence of unitaries \( (s_n) \) satisfying (e 4.60), (e 4.61) and (e 4.62). Define \( u_n = u_n s_n^*, n = 1, 2, \ldots \) Then it follows from (e 4.60) and (e 4.61) that
\[
\| [\varphi_2 \circ \psi_{n+1,\infty}, u_n^* u_{n+1}] \| < \delta_n', \quad (e 4.66)
\]
and
\[
\text{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, u_n^* u_{n+1}) |_{\mathcal{F}_{n+1}} = 0. \quad (e 4.67)
\]

It also follows from (e 4.62) that
\[
\text{dist}(g_{x_{n+1,j},s_{n+1}+1}^2, A_n) < \gamma_{n+1}' j = 1, \ldots, m(n+1),
\]
which is
\[
\text{dist}(g_{x_{n+1,j},s_{n+1}+1}^2, A_n) < \gamma_{n+1}' j = 1, \ldots, m(n+1). \quad (e 4.68)
\]
Moreover, it also follows from the definition of \( \Delta_n \) such that
\[
\mu_{\tau \varphi_2 \circ \psi_{n,\infty}} (O_a) \geq \Delta_n(a), \quad \forall \tau \in T(A), \quad (e 4.69)
\]
where \( O_a \) is any open ball in \( X_n \) with radius \( a \geq \eta_n' \).

With (e 4.66), (e 4.67), (e 4.68) and (e 4.69), one applies Theorem 3.9 to obtain a path of unitaries \( \{z(t) : t \in [0, 1]\} \) in \( A \) such that
\[
z(0) = 0, \quad z(1) = u_n^* u_{n+1},
\]
and
\[
\| z(t), \varphi_2 \circ \psi_{n+1,\infty} \| < 1/2^{n+1}, \quad \forall t \in [0, 1].
\]

Define
\[
u(t + n - 1) = u_n z_{n+1}(t), \quad t \in (0, 1),
\]
and then \( \{z(t) : t \in [0, \infty)\} \) is a continuous path of unitary in \( A \).

Note that
\[
ad u(t + n - 1) \circ \varphi_1 \approx \delta_{n+1}' \quad \text{ad}(z_{n+1}(t)) \circ \varphi_2 \approx 1/2^{n+1} \qquad \varphi_2
\]
on \( \mathcal{F}_{n+1} \) for all \( t \in (0, 1) \). It then follows that
\[
\lim_{t \to \infty} u_n^* u(t) \varphi_1(a) u(t) = \varphi_2(a)
\]
for all \( a \in C \), as desired.
Let $C$ and $A$ be two unital $C^*$-algebras. Recall that (see 10.2 of [9])

$$H_1(K_0(C), K_1(A)) := \{x \in K_1(A) : h([1_C]) = x \text{ for some } h \in \text{Hom}(K_0(C), K_1(A))\}.$$

**Lemma 4.3.** Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Suppose that $\{F_n\}$ is an increasing sequence of finite subsets of $C$ such that $\cup_{n=1}^{\infty} F_n$ is dense in $C$, and suppose that $\{P_n\}$ is an increasing sequence of finite subsets of $K_1(C)$ such that its union is $K_1(C)$. Suppose also that there is a sequence of decreasing positive numbers $\delta_n > 0$ with $\sum_{n=1}^{\infty} \delta_n < \infty$ and a sequence of unitaries $\{u_n\} \subset A$ such that

$$\text{Ad} u_n \circ \varphi \approx_{\delta_n} \psi \text{ on } F_n \quad \text{and} \quad \rho_A(\text{bott}_1(\psi, u_n^* u_{n+1})(x) = 0 \text{ for all } x \in P_n).$$

Then we may further require that $u_n \in U_0(A)$ if $H_1(K_0(C), K_1(A)) = K_1(A)$.

**Proof.** The proof is exactly the same as that of Lemma 10.4 of [12]. Note that, we will apply the second part of 4.10 instead of 6.3 of [12].

**Theorem 4.4.** Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Suppose that $H_1(K_0(C), K_1(A)) = K_1(A)$ and suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms which are asymptotically unitarily equivalent. Then they are strongly asymptotically unitarily equivalent, i.e., there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\} \subset U(A)$ such that

$$u(0) = 1_A \quad \text{and} \quad \lim_{t \to \infty} u(t)^* \varphi(c) u(t) = \psi(c) \text{ for all } c \in C.$$

**Proof.** The proof is exactly the same as that of Theorem 10.5 in [12]. However, we apply 4.3 instead of Lemma 10.4 of [12] as needed in the proof of [12].

**Corollary 4.5.** Let $X$ be a compact metric space and let $A$ be a unital separable simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Then $\varphi$ and $\psi$ are strongly asymptotically unitarily equivalent if and only if

$$[\varphi] = [\psi] \text{ in } KK(C(X), A), \quad \varphi^t = \psi^t, \quad \text{and} \quad \tau \circ \varphi = \tau \circ \psi \quad \text{for all } \tau \in T(A) \quad \text{and} \quad \overline{R_{\varphi, \psi}(K_1(M_{\varphi, \psi}))} \subset R_A(K_0(A)).$$

**Proof.** Note that $K_0(C(X)) = (\mathbb{Z} \cdot [1_{C(X)}]) \oplus G$ for some abelian subgroup $G$ of $K_0(C(X))$. For each $x \in K_1(A)$, define a homomorphism $h : K_0(C(X)) \to K_1(A)$ by $h([1_{C(X)}]) = x$ and $h|_G = 0$. In other words, one has that $H_1(K_0(C), K_1(A)) = K_1(A)$.

**Proposition 4.6.** Let $C$ be a unital amenable $C^*$-algebra satisfying the UCT and let $A$ be a unital separable simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Suppose that $\varphi$ and $\psi$ are two unital monomorphisms. Suppose also that

$$[\varphi] = [\psi] \text{ in } KL(C, A), \quad \tau \circ \varphi = \tau \circ \psi \quad \text{for all } \tau \in T(A) \quad \text{and} \quad R_{\varphi, \psi}(K_1(M_{\varphi, \psi})) \subset R_A(K_0(A)).$$

Then

$$\varphi^t = \psi^t.$$
Proof. Let $u \in M_l(C)$ be a unitary, where $l \geq 1$ is an integer. Let $z \in M_l(M_{\varphi,\psi})$ be a unitary which is piecewise smooth on $[0, 1]$ such that $\pi_0 \circ z = \varphi(u)$ and $\pi_1 \circ z = \psi(u)$. Let $G$ be a finitely generated subgroup of $K_1(C)$ which contains $[u]$. By the assumption, there is an injective homomorphism $\theta_G : G \to K_1(M_{\varphi,\psi})$ such that
\[(\pi_0)_{x_1} \circ \theta_G = \text{id}_G \text{ and } R_{\varphi,\psi} \circ \theta_G \in \text{Hom}(G, \rho_A(K_0(A))).\] (e 4.78)

It follows that there exists projections $p, q \in M_l(A)$ such that
\[\theta([u]) = [zv] \in K_1(M_{\varphi,\psi}),\] (e 4.79)
where $v(t) = (e^{2\pi it}p + (1-p)(e^{-2\pi it}p + (1-p))) \in M_l(M_{\varphi,\psi})$. To simplify the notation, without loss of generality, we may assume that $l = l'$. By (e 4.78),
\[R_{\varphi,\psi}([zv]) \in \rho_A(K_0(A)).\] (e 4.80)

Since $R_{\varphi,\psi}([v]) \in \rho_A(K_0(A))$, one computes that
\[R_{\varphi,\psi}([z]) \in \rho_A(K_0(A)).\] (e 4.81)

Now let $w(t) \in C([0, 1], A)$ be a unitary which is piecewise smooth such that $w(0) = \psi(u)^*\varphi(u)$ and $w(1) = 1_{M_l(A)}$. Then
\[\psi(u)w(t) \in M_l(M_{\varphi,\psi}).\] (e 4.82)

Moreover $[z] = [\psi(u)w]$ in $K_1(M_{\varphi,\psi})$. It follows that, for any $\tau \in T(A)$,
\[\frac{1}{2\pi i} \int_0^1 \tau \left( \frac{d(w(t))}{dt} \frac{w^*(t)}{dt} \right) dt = \frac{1}{2\pi i} \int_0^1 \tau (\psi(u) \frac{d(w(t))}{dt} w^*(t)) \psi(u)^* dt \]
\[= \frac{1}{2\pi i} \int_0^1 \tau \left( \frac{d(\psi(u)w(t))}{dt} \right) \psi(u)w^*(t) dt \]
\[= R_{\varphi,\psi}([z])(\tau).\] (e 4.85)

Thus, by (e 4.81), there exists $x \in K_0(A)$ such that
\[\text{Det}([w]) = \rho_A(x)(\tau)\] (e 4.86)
for all $\tau \in T(A)$. It follows from a result of P. Ng (16) that
\[\psi(u)^*\varphi(u) \in DU(M_l(A)).\]

Since this holds for all unitaries $u \in M_l(C)$, it follows that
\[\varphi^\dagger = \psi^\dagger.\]

**Corollary 4.7.** Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Then $\varphi$ and $\psi$ are asymptotically unitarily equivalent if and only if
\[\varphi_1 = \varphi_2 \text{ in } KK(C, A),\] (e 4.87)
\[\tau \circ \varphi = \tau \circ \psi \text{ for all } \tau \in T(A) \text{ and} \]
\[\overline{R}_{\varphi_1,\varphi_2} = 0.\] (e 4.89)
Proof. We only need to show the “if part” of the statement. It follows from \[\text{4.6}\] that, in addition, one has
\[
\varphi^\dagger = \psi^\dagger. \tag{e 4.90}
\]
This of course implies that \(\varphi^\dagger = \psi^\dagger\). Then \[\text{4.2}\] applies.

Theorem 4.8. Let \(C\) be a unital AH-algebra and let \(A\) be a unital simple \(C^*\)-algebra with \(\text{TR}(A) \leq 1\). Suppose that \(\varphi, \psi : C \to A\) are two unital monomorphisms such that
\[
[\varphi] = [\psi] \text{ in } KK(C, A), \tag{e 4.91}
\]
\[
\tau \circ \varphi = \tau \circ \psi, \text{ for all } \tau \in T(A), \text{ and} \tag{e 4.92}
\]
\[
\varphi^\dagger = \psi^\dagger, \tag{e 4.93}
\]
then \(\varphi\) and \(\psi\) are asymptotically unitarily equivalent, provided that one of the following holds:

1. \(K_1(C)\) is finitely generated, or
2. \(K_0(A)\) is finitely generated, or
3. the short exact sequence
\[
0 \to \text{kre} \rho_A \to K_0(A) \to \rho_A(K_0(A)) \to 0
\]
splits.

Proof. Let \(C\) and \(A\) be as in the statement. Suppose that \(\varphi, \psi : C \to A\) are two unital monomorphisms which satisfy the assumptions \(\text{[e 4.91]}, \text{[e 4.92]}\) and \(\text{[e 4.93]}\). In particular, \(\text{[e 4.93]}\) implies that
\[
\varphi^\dagger = \psi^\dagger. \tag{e 4.94}
\]
Since \([\varphi] = [\psi]\), there exists a splitting map \(\theta : K(C) \to K(M_{\varphi, \psi})\) such that
\[
\theta \circ [\pi_0] = [\text{id}_C]. \tag{e 4.95}
\]
Let \(u \in M_l(C)\) be a unitary for some integer \(l \geq 1\). Let \(z \in M_l(M_{\varphi, \psi})\) be a unitary such that \(z(0) = \varphi(u)\) and \(z(1) = \psi(u)\). Moreover, we may assume that \(z\) is piecewise smooth. Define \(z_1(t) = \psi(u)^* z(t)\) for \(t \in [0, 1]\). Then \(z_1\) is a piecewise smooth and continuous path of unitaries in \(A\) such that \(z_1(0) = \psi(u)^* \varphi(u)\) and \(z_1(1) = 1_{\text{M}_l}\). It follows from \(\text{[e 4.93]}\) that
\[
\frac{1}{2\pi i} \int_0^1 \tau \left( \frac{dz_1(t)}{dt} z_1(t)^* dt \right) \rho_A(K_0(A)), \tag{e 4.96}
\]
where \(\tau \in T(A)\). One then easily computes that
\[
R_{\varphi, \psi}([z]) \in \rho_A(K_0(A)). \tag{e 4.97}
\]
On the other hand, there is a projection \(p \in M_{l'}(A)\) such that the following holds:
\[
\theta([u]) = [zv], \tag{e 4.98}
\]
where \(v(t) = e^{2\pi it} p + (1_{M_{l'}} - p)\) for all \(t \in [0, 1]\). To simplify the notation, without loss of generality, we may assume that \(l' = l\). It follows that
\[
R_{\varphi, \psi}([zv]) = R_{\varphi, \psi}([z]) + R_{\varphi, \psi}([v]) \in \rho_A(K_0(A)). \tag{e 4.99}
\]
It follows that
\[ R_{\varphi,\psi} \circ \theta \in \text{Hom}(K_1(C), \rho_A(K_0(A))). \] (e 4.100)

In all three cases (1), (2) and (3), there exists a homomorphism \( \lambda_0 : R_{\varphi,\psi} \circ \theta(K_1(C)) \to K_0(A) \) such that
\[ \rho_A \circ \lambda_0 = \text{id}_{R_{\varphi,\psi} \circ \theta(K_1(C))}. \] (e 4.101)

Define \( \lambda = \lambda_0 \circ R_{\varphi,\psi} \circ \theta \). So \( \lambda \) is a homomorphism from \( K_1(C) \) into \( K_0(A) \).

\[ \theta_1 = \theta - \lambda. \] (e 4.102)

Then
\[ R_{\varphi,\psi} \circ \theta_1 = 0. \] (e 4.103)

It follows that
\[ R_{\varphi,\psi} = 0. \] (e 4.104)

The theorem then follows from 4.2.

Corollary 4.9. Let \( X \) be a finite CW-complex and let \( A \) be a unital simple \( C^* \)-algebra with finite tracial rank. Suppose that \( \varphi, \psi : C(X) \to A \) are two unital monomorphisms. Then \( \varphi \) and \( \psi \) are asymptotically unitarily equivalent if and only if
\[ [\varphi] = [\psi] \text{ in } KK(C,A), \] (e 4.105)
\[ \tau \circ \varphi = \tau \circ \psi \text{ for all } \tau \in T(A) \] and
\[ \varphi^\dagger = \psi^\dagger. \] (e 4.106)

Remark 4.10. We would point out that the assumptions in 4.2 is more sensitive than those in (e 4.91), (e 4.92) and (e 4.93), in general.

Let \( A \) be a unital simple AF-algebra with \( K_0(A) \) given by a non-splitting short exact sequence
\[ 0 \to G \to K_0(A) \to \mathbb{Q} \to 0, \] (e 4.108)

where \( G \) is a countable abelian group and where the order of an element is determined by its image in \( \mathbb{Q} \). In particular, \( A \) has a unique tracial state \( \tau \) and \( \rho_A(K_0(A)) = \mathbb{Q} \). Let \( C \) be a unital simple \( C^* \)-algebra of tracial rank zero with \( K_1(C) = \mathbb{Q} \oplus \text{Tor}(K_1(C)) \) which also satisfies the UCT. Let \( \kappa \in KK(C,A)^{++} \) such that \( \kappa([1_C]) = [1_A] \). Then there exists a unital monomorphism \( \varphi : C \to A \) such that \( [\varphi] = \kappa \). Let \( \lambda = \varphi_T : T(A) \to T(C) \) be the affine continuous map induced by \( \varphi \). Let \( \gamma : K_1(C) \to \rho_A(K_0(A)) \) be an isomorphism as abelian group. It follows from 4.8 of [13] that there exists a unital monomorphism \( \psi : C \to A \) such that \( [\psi] = \kappa = [\varphi] \), \( \psi_T = \lambda = \varphi_T \) and there exists a splitting map \( \theta : K_1(C) \to K_1(M_{\varphi,\psi}) \) such that
\[ R_{\varphi,\psi} \circ \theta = \gamma + \gamma_0, \] (e 4.109)

where \( \gamma_0 \in R_0 \). We may write \( \gamma_0 = \rho_A \circ f \), where \( f : K_1(C) \to K_0(A) \) is a homomorphism. It follows from 4.10 that
\[ \varphi^\dagger = \psi^\dagger. \] (e 4.110)
However, there is no homomorphism $\lambda_1 : K_1(C) \to K_0(A)$ such that

$$R_{\phi,\psi} \circ \theta = \rho_A \circ \lambda_1.$$  

Otherwise, $\eta = (\lambda_1 - f) \circ \gamma^{-1}$ would split the short exact sequence (e 4.108), since

$$\rho_A \circ \eta = \rho_A \circ (\lambda_1 - f) \circ \gamma_1^{-1} = (R_{\phi,\psi} \circ \theta - \rho_A \circ f) \circ \gamma_1^{-1} = (\gamma + \gamma_0 - \gamma_0) \circ \gamma^{-1} = \text{id}_{\rho_A(K_0(A))}.$$  

In other words,

$$R_{\phi,\psi} \neq 0.$$

References


