CONJUGACY RELATION ON COXETER ELEMENTS

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Abstract. Let (W,S,Γ) be an irreducible finitely presented Coxeter system. The present paper is mainly concerned with conjugacy relation on Coxeter elements in the case where Γ containing just one circle, in particular when Γ is itself a circle. In the cases where Γ is either a three multiple circle or a circle with three nodes, we show that the ss-equivalence relation on Coxeter elements of W is the same as the W-conjugacy relation. An explicit formula is given for the characteristic polynomial of a Coxeter element in the natural reflection representation of W when Γ is a circle. We also give the answers to some questions raised by Coleman and extend some results of Geck and Pfeiffer concerning conjugacy relation in Coxeter groups.

Let (W,S,Γ) be a Coxeter system, where W is a Coxeter group, S a distinguished generator set, and Γ the corresponding Coxeter graph. In the present paper, we always assume S finite (write $S=\{s_1,s_2,\cdots,s_r\}$) and Γ connected unless otherwise specified.

By a Coxeter element $w\in W$, we mean a product $s_{i_1}s_{i_2}\cdots s_{i_r}$ with $i_1,i_2,\cdots,i_r$ a permutation of $1,2,\cdots,r$. Let $C(W)$ be the set of all the Coxeter elements in W. The spectral classes in $C(W)$ have been studied extensively by a number of people (see [2, 3, 8, 9, 12, 13] and §5.). But the conjugacy relation in $C(W)$ is relatively less known except for the case where Γ is a tree, in the latter case, $C(W)$ is wholly contained in a single W-conjugacy class (see, for example, [7, 3.16]). In the present paper, we mainly consider the case where Γ contains just one circle. We first introduce the concept of ss-equivalence

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in $C(W)$ (see 1.3). Each ss-class of $C(W)$ is contained in some W-conjugacy class. The study of ss-classes in $C(W)$ with Γ containing just one circle can be reduced to the case where Γ is itself a circle (Proposition 2.3). Then we describe all the ss-classes of $C(W)$ when Γ is a circle (Theorem 1.6). Furthermore we show that the ss-equivalence relation in $C(W)$ is actually the same as W-conjugacy relation in the following two special cases: one is when Γ is a three multiple circle (Theorem 3.4), and the other is when Γ is a circle with three nodes (Theorem 4.8). Apply Coleman's ν-function on $C(W)$ for Γ a circle (see [3] and 1.6), we show that $w, y \in C(W)$ satisfy $\nu(w) = \nu(y)$ if and only if w, y are ss-equivalent. Since ss-equivalence implies W-conjugacy, this verifies a conjecture of Coleman in [3, IX, Question 1]. In the cases where Γ is either a three multiple circle, or a circle with three nodes, we also show the converse: if $w, y \in C(W)$ are W-conjugate then $\nu(w) = \nu(y)$. This implies that the ss-equivalence relation in $C(W)$ is the same as W-conjugacy relation in these two special cases. We conjecture that it should still hold for any Coxeter system (W, S, Γ) with Γ containing at most one circle. In the direction of verifying the conjecture, we give a formula for the characteristic polynomial of any $w \in C(W)$ in the natural reflection representation of W (in the sense of [7, §5.3]) where Γ is a circle. The formula shows that two Coxeter elements w, y are not W-conjugate if $\nu(w) \neq \pm\nu(y)$. This phenomenon was also observed by Coleman in [3, Theorem 4.6]. Then the proof of the conjecture is reduced to showing that if two Coxeter elements w, y satisfy $\nu(w) = -\nu(y) \neq 0$ then w and y are not W-conjugate. On the other hand, by computing the characteristic polynomials of Coxeter elements in some hyperbolic Coxeter systems, we give a negative answer to another conjecture of Coleman (see [3, IX, Question 2]) which proposed a formula for the number of spectral classes in $C(W)$ in the case when Γ contains more than one circles. In the case where Γ is a circle with three nodes, we deal with the conjugacy problem not just for $C(W)$, but also for the whole group W. Let (W, S, Γ) be such a Coxeter system, C a conjugacy class in W, and $C_{\min}$ the set of all elements of C of minimal length. We show that for any $w \in C$, there exist an nc-sequence $x_0 = w, x_1, \cdots, x_t$ (see 4.3 for definition) in C such that $\ell(x_i) \leqslant \ell(x_{i-1})$ for every i, $1 \leqslant i \leqslant t$, and $x_t \in C_{\min}$. We show that for any $w, y \in C_{\min}$ with w cuspidal (see 4.4 for definition), there exist an nc-sequence inside $C_{\min}$ to connect w, y. We also show that for a non-cuspidal conjugacy class C, $C_{\min}$ consists of non-cuspidal elements, which are either of length 1 or all contained in some dihedral standard parabolic subgroup of W (see Remark 4.9 (2)). This extends the corresponding results of Geck and Pfeiffer in [4].

The content is organized as follows. In Section 1, we introduce the concept of ss-equivalence relation in $C(W)$, and discuss some properties of ss-equivalence in the case where Γ is either a tree or a circle. In Section 2, we reduce the study of ss-classes of $C(W)$ from the case where Γ contains exactly one circle to the case where Γ is itself a circle. In Sections 3 and 4, we describe the W-conjugacy classes of $C(W)$ for Γ being a three multiple circle and a circle with three nodes respectively. Finally, in Section 5, we give a formula for the characteristic polynomial of a Coxeter element when Γ is a circle. We also give a negative answer to a conjecture of Coleman on the number of spectral classes of $C(W)$ by providing some counter-examples.

§1. ss-equivalence.

Fix a Coxeter system (W, S, Γ). In this section, we introduce the concept of ss-equivalence relation in the set $C(W)$ of Coxeter elements of W. We deduce some properties of ss-classes of $C(W)$ in the case where Γ is either a tree or a circle. The main results of this section are Lemma 1.5 and Theorem 1.6, which will be used later. In particular, Theorem 1.6 gives an affirmative answer to a question of Coleman in the paper [3, IX, Question 1].

1.1. As a node of the graph Γ, we sometimes denote j for $s_j \in S$. Let $O(\Gamma)$ be the set of all the orientations of Γ such that the corresponding digraph contains no directed circle. It is known that there exists a natural bijection between the sets $C(W)$ and $O(\Gamma)$ as follows (see [10]). To

$$w = s_{i_1} s_{i_2} \cdots s_{i_r} \tag{1.1.1}$$

in $C(W)$, we associate an orientation $\Gamma_w \in O(\Gamma)$ such that an arrow connecting two nodes j, k is from j to k if and only if the factor s_j is on the left of s_k in the expression (1.1.1).

Here and later, for any concepts and terminologies in graph theory not explained in the paper, we refer the reader to any graph theory textbook, e.g., [1].

1.2. To each $w \in W$, we associate two subsets in S:

$$\mathcal{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S \mid ws < w\},$$

where $\leqslant$ is the Bruhat order in (W, S). A node s in $\Gamma_0 \in O(\Gamma)$ is called a *source* (resp. a *sink*), if no arrow of Γ_0 is incident to s at its head (resp. tail). Then the following properties of the correspondence $w \longrightarrow \Gamma_w$ can be seen easily.

(1) $s \in \mathcal{L}(w)$ if and only if s is a source of Γ_w.

(2) $s \in \mathcal{R}(w)$ if and only if s is a sink of Γ_w.

1.3. Suppose that $w, y \in C(W)$ and $s \in S$ satisfy $w = sys$. Then the node s is a source (resp. a sink) in exactly one of Γ_w and Γ_y. The digraphs Γ_w and Γ_y can be obtained from one to another by a source-sink exchanging operation at s (or an *ss-operation at* s in short), i.e., by reversing the directions of all arrows incident to s.

More generally, suppose that there exists a sequence

$$\Gamma_0, \Gamma_1, \cdots, \Gamma_t \tag{1.3.1}$$

in $O(\Gamma)$ such that for every j, $1 \leqslant j \leqslant t$, Γ_j is obtained from Γ_{j-1} by an ss-operation at some $i_j \in \{1, 2, \cdots, r\}$. Then we say that Γ_t is obtained from Γ_0 by ss-operations. In this case, if all s_{i_j}'s are in a subset S' of S, then we also say that Γ_t is obtained from Γ_0 by *ss-operations at* S'. This defines an equivalence relation in $O(\Gamma)$, called S'*-ss-equivalence*. The corresponding equivalence classes in $O(\Gamma)$ are called S'*-ss-classes*. In particular, S-ss-equivalence (resp. an S-ss-class) is simply called *ss-equivalence* (resp. an *ss-class*).

1.4. For any sets I, J with $I \subset J$, denote by J_I the set difference $J - I$. In particular, when $I = \{x\}$ or $\{x, y\}$, we simply denote $J_x := J_{\{x\}}$ and $J_{xy} := J_{\{x,y\}}$.

A node in a digraph is *extreme* if it is either a source or a sink. A node in an unoriented graph is *terminal*, if it has at most one neighboring node, i.e., there is at most one edge incident to this node.

For any $S' \subset S$, a subset H of $O(\Gamma)$ is said S'*-ss-transitive*, if any two elements of H can be transformed from one to another by ss-operations at S'.

Lemma. *Assume that Γ is a tree. Let t be a terminal node of Γ. Then $O(\Gamma)$ is S_t-ss-transitive.*

Proof. There exists a unique $\Gamma_0 \in O(\Gamma)$ with t a unique source. To show our result, we must show that any $\Gamma_1 \in O(\Gamma)$ can be transformed to Γ_0 by ss-operations at S_t. We

may assume $m := |S| \geqslant 2$ since otherwise the result is trivial. It is obvious when $m = 2$. Now assume $m > 2$. Given $\Gamma_1 \in O(\Gamma)$.
(1) First assume that t is a source of Γ_1. Let s be the neighboring node of t in Γ. Let $t_1, \cdots, t_k$ be the list of neighboring nodes of s in Γ differing from t. Let $\Gamma^{(i)}$, $1 \leqslant i \leqslant k$, be the subgraphs of Γ with vertex set $S^{(i)}$ which corresponds to the branch containing the node t_i. Thus we have $\cap_{i=1}^k S^{(i)} = \{s\}$ and $\cup_{i=1}^k S^{(i)} = S_t$. In particular, s is a common terminal node of $\Gamma^{(i)}$'s. Let $\Gamma_j^{(i)}$, $j = 0, 1$, be the one in $O(\Gamma^{(i)})$ corresponding to Γ_j. By applying induction on $|S| \geqslant 2$, we see that $\Gamma_1^{(i)}$ can be transformed to $\Gamma_0^{(i)}$ by ss-operations at $S_s^{(i)}$. This implies that Γ_1 can be transformed to Γ_0 by ss-operations at S_{ts} and hence also by ss-operations at S_t.
(2) Next assume that t is a sink of Γ_1. Let $s \in S$ be the unique neighboring node of t in Γ. If s is a source of Γ_1, then let Γ_2 be obtained from Γ_1 by an ss-operation at s. Now assume that s is not a source of Γ_1. Let t_i, $\Gamma^{(i)}$, $S^{(i)}$, $1 \leqslant i \leqslant k$, be as in (1). Let $\Gamma_1^{(i)}$ be the orientation of $\Gamma^{(i)}$ corresponding to Γ_1. By inductive hypothesis, $\Gamma_1^{(i)}$ can be transformed to some $\Gamma_2^{(i)} \in O(\Gamma^{(i)})$ by ss-operations at $S_s^{(i)}$ such that s is a source in $\Gamma_2^{(i)}$. This implies that Γ_1 can be transformed to some $\Gamma_h \in O(\Gamma)$ by ss-operations at S_{st} such that s is a source of Γ_h. Let Γ_2 be obtained from Γ_h by an ss-operation at s. Then in either case, we go back to the case (1) with Γ_2 in the place of Γ_1. This implies our result. □

1.5. Let Γ be a circle. Then in any $\Gamma_0 \in O(\Gamma)$, the number of sources is equal to the number of sinks, denote by $c(\Gamma_0)$ this common number. This implies that $|\mathcal{L}(w)| = |\mathcal{R}(w)|$ for any $w \in C(W)$. It is clear that for any $w \in C(W)$, we have $1 \leqslant c(\Gamma_w) \leqslant \left[\frac{n}{2}\right]$, where n is the number of nodes in Γ, and $[a]$ is the largest integer not greater than a.

Lemma. *Let Γ be a circle. For any $w \in C(W)$, there exists some $y \in C(W)$ such that y is ss-equivalent to w with $c(\Gamma_y) = 1$. In particular, we have $w \underset{\mathrm{W}}{\sim} y$, where the notation $y \underset{\mathrm{W}}{\sim} w$ means that y, w are conjugate in W.*

Proof. Let $\Gamma_0 \in O(\Gamma)$. To show our result, we need only show that if $c(\Gamma_0) > 1$ then we can transform Γ_0 by ss-operations such that the resulting $\Gamma_1 \in O(\Gamma)$ satisfies $c(\Gamma_1) < c(\Gamma_0)$. Now assume $c(\Gamma_0) > 1$. Labeling the nodes of Γ by $0, 1, 2, \cdots, n-1$ in an anti-clockwise ordering such that 0 is a source of Γ_0. Clearly, the sources and sinks of Γ_0 occur alternately in the sequence $0, 1, \cdots, n-1, 0$. By our assumption, we can take the nodes i, j, k, $1 \leqslant i < j < k \leqslant n-1$, such that i, k are sinks and j is a source in Γ_0 and that i, j, k are minimal with this property . We can define a sequence $\Lambda_j = \Gamma_0, \Lambda_{j-1}, \cdots, \Lambda_i$ in $O(\Gamma)$ such that for every v, $i < v \leqslant j$, Λ_{v-1} is obtained from Λ_v by changing the source v into a sink. Let us observe the orientation Λ_i. If $k = j+1$, then there is only one extreme node among $1, \cdots, k$ in Λ_i, i.e., the sink $i+1$. If $k > j+1$, then there are three extreme nodes among $1, \cdots, k$ in Λ_i, i.e., two sinks $i+1$, k and one source $j+1$. The orientations of the edges incident to the nodes $k+1, \cdots, n-1, 0, 1, \cdots, i-1$ in Λ_i are all the same as those in Γ_0. Applying reversing induction on i, j, $1 \leqslant i < j < n-1$, we can eventually get some $\Gamma_1 \in O(\Gamma)$ from Γ_0 by a sequence of ss-operations such that $c(\Gamma_1) < c(\Gamma_0)$. This proves our result. $\square$

1.6. Let Γ be a circle. Label the nodes of Γ in an anti-clockwise order by $0, 1, \cdots, n-1$. For $w \in C(W)$ and $0 \leqslant i \leqslant n-1$, an arrow in Γ_w connecting the nodes i and $i+1$ is called a *rise* (resp. a *fall*) if it directs from i to $i+1$ (resp. from $i+1$ to i), where we stipulate $n := 0$. Let $\nu_r(w)$ (resp. $\nu_f(w)$) be the number of the rises (resp. falls) in Γ_w. Define $\nu(w) := \nu_r(w) - \nu_f(w)$ (this notation follows from [3]). It is easily seen that an ss-operation on Γ_w preserves the number $\nu(w)$.

Let $C(W)_1$ be the set of all the elements $w \in C(W)$ with $c(\Gamma_w) = 1$ (see 1.5). Given $w \in C(W)_1$, let s, t be the source and the sink in Γ_w, respectively. Starting with s and counting the nodes anti-clockwisely in Γ_w. If the node t occurs in the mth place after s then we denote $d(w) = m$. It is easily seen that

$$(1.6.1) \qquad 2d(w) = n + \nu(w) \qquad \text{for any} \quad w \in C(W)_1.$$

Theorem. *Let Γ be a circle with n nodes. Then $w, y \in C(W)$ are ss-equivalent if and only if $\nu(w) = \nu(y)$. So $C(W)$ contains exactly $n-1$ ss-classes.*

Proof. Since ss-operations on $C(W)$ preserve the function ν, it implies that $\nu(w) = \nu(y)$ if $w, y \in C(W)$ are ss-equivalent. Now assume that $w, y \in C(W)$ satisfy $\nu(w) = \nu(y)$. By Lemma 1.5, there exist some $w', y' \in C(W)_1$ which are ss-equivalent to w, y, respectively. Since ss-operations on $C(W)$ preserve the function ν, we have $\nu(w') = \nu(y')$ and hence $d(w') = d(y')$ by (1.6.1). Denote $m := d(w')$.

It remains to show that w' and y' are ss-equivalent. Labeling the nodes of Γ in an anti-clockwise ordering by $0, 1, \cdots, n-1$ such that $0, j$ are the sources of $\Gamma_{w'}, \Gamma_{y'}$ respectively.

We may assume $j = 1$ without loss of generality. Then $w' = s_0 s_{n-1} s_{n-2} \cdots s_{m+1} s_1 s_2 \cdots s_{m-1} s_m$ and $y' = s_1 s_2 \cdots s_m s_0 s_{n-1} s_{n-2} \cdots s_{m+1}$. Let $z = s_0 s_{n-1} s_{n-2} \cdots s_{m+1}$. Clearly, w' and y' are ss-equivalent with $y' = z^{-1} w' z$. This proves the first assertion of the theorem. Then the second assertion is an immediate consequence of the first one. □

Remark 1.7. In his paper [3, IX, Question 1], Coleman considered a Coxeter system (W, S, Γ) with Γ a circle. He asked if the equation $\nu(w) = \nu(y)$ could always imply $w \underset{W}{\sim} y$. Now Theorem 1.6 gives an affirmative answer to Coleman's question.

§2. A graph containing exactly one circle.

Let (W, S, Γ) be a Coxeter system. In this section, we consider the case where Γ contains exactly one circle Γ'. The result is Proposition 2.3, which tells us that the study of the ss-classes in $C(W)$ can be reduced to the case where Γ itself is a circle.

2.1. As above, let S' be the vertex set of Γ' which generates a parabolic subgroup W' of W. Then (W', S', Γ') is also a Coxeter system. Let $\overline{O(\Gamma)}$ be the set of S-ss-classes in $O(\Gamma)$. Also, let $\overline{O(\Gamma')}$ be the set of S'-ss-classes in $O(\Gamma')$.

2.2. Let H be a subgraph of a graph K. Given $K_0 \in O(K)$, let $H_0 \in O(H)$ be obtained from K_0 by removing all the nodes not in H. A natural map $\tau_{KH} : O(K) \longrightarrow O(H)$ is defined by setting $\tau_{KH}(K_0) = H_0$. Now let $K = \Gamma$ and $H = \Gamma'$ be as in 2.1. Then it is clear that $\tau_{\Gamma\Gamma'}$ is surjective.

Proposition 2.3. *The map $\tau_{\Gamma\Gamma'}$ induces a bijection from the set $\overline{O(\Gamma)}$ to $\overline{O(\Gamma')}$.*

Proof. Consider the ss-operations on $O(\Gamma)$ and on $O(\Gamma')$. Given any $\Gamma'_0 \in O(\Gamma')$, we see by Lemma 1.4 that the inverse image $\tau_{\Gamma\Gamma'}^{-1}(\Gamma'_0)$ is contained in some $S_{S'}$-ss-class of $O(\Gamma)$ (and hence in some S-ss-class of $O(\Gamma)$). On the other hand, let $\Gamma'_1 \in O(\Gamma')$ be obtained from Γ'_0 by an ss-operation at some $s \in S'$. Note that in that case, s is an extreme node of Γ'_0 (see 1.4.). Take any $\Gamma_0 \in \tau_{\Gamma\Gamma'}^{-1}(\Gamma'_0)$. If s is an extreme node of Γ_0, then we can perform on Γ_0 an ss-operation at s to obtain some $\Gamma_1 \in \tau_{\Gamma\Gamma'}^{-1}(\Gamma'_1)$. Now assume that s is not an extreme node of Γ_0. Then s is a branch node of Γ. By Lemma 1.4, we can perform ss-operations at $S_{S'}$ on Γ_0 to obtain some $\Gamma_{01} \in \tau_{\Gamma\Gamma'}^{-1}(\Gamma'_0)$ with s an extreme node. Hence an ss-operation at s transforms Γ_{01} to some $\Gamma_1 \in \tau_{\Gamma\Gamma'}^{-1}(\Gamma'_1)$. This implies that the inverse image of an S'-ss-class of $O(\Gamma')$ under $\tau_{\Gamma\Gamma'}$ is contained in some S-ss-class of $O(\Gamma)$. On the other hand, let $\Gamma_0, \Gamma_1 \in O(\Gamma)$ be such that Γ_1 is obtained from Γ_0 by an ss-operation at t . Let $\Gamma'_i = \tau_{\Gamma\Gamma'}(\Gamma_i)$, $i = 0, 1$. Then $\Gamma'_1 = \Gamma'_0$ if

$t \in S - S'$, and Γ'_1 is obtained from Γ'_0 by an ss-operation at t if $t \in S'$. This implies that the image of an S-ss-class of $O(\Gamma)$ under $\tau_{\Gamma\Gamma'}$ is contained in some S'-ss-class of $O(\Gamma')$. Therefore $\tau_{\Gamma\Gamma'}$ induces a bijective map from $\overline{O(\Gamma)}$ to $\overline{O(\Gamma')}$. $\square$

From Proposition 2.3, we see that in the study of ss-classes in $C(W)$, the case of Γ containing exactly one circle can be reduced to the case of Γ itself being a circle.

§3. The case of a three multiple circle.

Let (W, S, Γ) be a Coxeter system with Γ a circle. Γ is *three multiple*, if for any neighboring nodes s, t in Γ, the order $o(st)$ of the product st is either a multiple of three or infinite. The simplest case for a three multiple circle is when W is the affine Weyl group of type $\widetilde{A}_{n-1}$, $n > 2$, where Γ is a circle with $o(sr) = 3$ for any neighboring nodes $s, t \in S$. In 3.1-3.3, we assume W to be such an affine Weyl group. By a matrix presentation of this group, we show that two elements in $C(W)$ are W-conjugate if and only if their ν-values are the same. Then we extend this result to the case of Γ being an arbitrary three multiple circle.

3.1. Recall in [11] that the affine Weyl group of type $\widetilde{A}_{n-1}$ has a matrix presentation $G_0(x, n)$, where $G_0(x, n)$ is the group consisting of all the $n \times n$ monomial matrix $\sum_{i=1}^{n} x^{k_i} E_{i,r_i}$ with $\sum_{i=1}^{n} k_i = 0$ and $r_1, \cdots, r_n$ a permutation of $1, 2, \cdots, n$, where E_{hk} is the $n \times n$ unit matrix whose entries are all zero except for the (h, k)-entry which is 1, and x is an indeterminate. Define the following $n \times n$ matrices

$$s_i = \sum_{j \neq i, i+1} E_{jj} + E_{i,i+1} + E_{i+1,i}, \qquad 1 \leqslant i < n$$
$$s_0 = \sum_{1<j<n} E_{jj} + x^{-1} E_{1n} + x E_{n1}.$$

Then $S = \{s_0, s_1, \cdots, s_{n-1}\}$ forms a distinguished generator set of $G_0(x, n)$ as a Coxeter group.

Theorem 3.2. *Let $W = G_0(x, n)$ and S be as above. Then $w, y \in C(W)$ are conjugate in W if and only if $\nu(w) = \nu(y)$ (see 1.6).*

Proof. The implication “ $\Longleftarrow$ ” follows by Theorem 1.6. So it remains to show the implication “ $\Longrightarrow$ ”. By Lemma 1.5, we may assume $w, y \in C(W)_1$. We have that $d(w) = d(y)$ if and only if $\nu(w) = \nu(y)$ by (1.6.1). So we need only show that if $d(w) \neq$

$d(y)$ then w and y are not conjugate in W. Consider the characteristic polynomial $f_w(\lambda)$ of an element $w \in C(W)$. Suppose $w = s_0 s_1 \cdots s_{d(w)-1} s_{n-1} s_{n-2} \cdots s_{d(w)}$. Then

$$w = \sum_{i=2}^{d(w)} E_{i,i-1} + \sum_{j=d(w)+1}^{n-1} E_{j,j+1} + x^{-1} E_{1,d(w)} + x E_{n,d(w)+1}.$$

It is easily seen that the characteristic polynomial of w is

$$f_w(\lambda) = (\lambda^{n-d(w)} - x)(\lambda^{d(w)} - x^{-1}).$$

This implies that if $w, y \in C(W)_1$ satisfy $d(w) \neq d(y)$ with

(3.2.1) $\qquad s_0$ the common source of Γ_w and Γ_y,

then $f_w(\lambda) \neq f_y(\lambda)$ and hence w and y are not W-conjugate. By Theorem 1.6, we see that this result still holds when we drop the assumption (3.2.1). So our proof is completed. □

Remark 3.3. The reader may wonder why we do not use the characteristic polynomials of elements of W in the natural reflection representation for the description of conjugacy in $C(W)$. Our reason is as follows. Two elements $w, y \in C(W)$ have the same characteristic polynomials in the natural reflection representation if and only if $\nu(w) = \pm\nu(y)$ (see (5.4.1)). This result is not enough to solve the conjugacy problem in $C(W)$.

3.4. To extend Theorem 3.2 to a more general case, let us first point out a simple fact. Given two Coxeter systems (W', S', Γ') and (W'', S'', Γ''). Suppose that there exists a bijective map $\phi : S' \longrightarrow S''$ such that $o(\phi(s)\phi(r))$ divides $o(sr)$ for any $s, r \in S'$, with the convention that ∞ is divisible by any integer and by itself. Then ϕ can be extended to a group homomorphism from W' to W'', which is obviously surjective. In this case, ϕ also induces a surjective map from $C(W')$ to $C(W'')$. For any $w, y \in C(W')$, the relation $w \underset{W'}{\sim} y$ implies $\phi(w) \underset{W''}{\sim} \phi(y)$. This fact, together with Theorems 1.6 and 3.2, implies the following more general result immediately.

Theorem. *Let (W, S, Γ) be a Coxeter system with Γ a three multiple circle. Then $w, y \in C(W)$ are conjugate in W if and only if $\nu(w) = \nu(y)$.*

§4. A circle with three nodes.

In the present section, we assume (W, S, Γ) to be of rank 3 with Γ a circle. We shall show that in the set $C(W)$ the ss-classes are the same as W-conjugacy classes. In showing this result, we actually get more: we extend two results of Geck and Pfeiffer to our case concerning the conjugacy in W.

Let $\overline{S}$ be the set of all the two-elements subsets in S. Denote by $o(I)$ the order $o(sr)$ of the product sr if $I = \{s, r\}$. In order to simplify our statements, we always assume $2 < o(I) < \infty$ for any $I \in \overline{S}$ unless otherwise specified.

For any $I \subset S$, denote by w_I the longest element in the subgroup W_I of W generated by I.

Lemma 4.1. *For any $I \in \overline{S}$ and $t \in S - I$, let $w = t \cdot w_I \cdot x$ be with $x \in W$ and $\ell(w) = \ell(x) + \ell(w_I) + 1$. Then $|\mathcal{L}(w)| = 1$.*

Proof. Suppose not. Take a counter-example of $w \in W$ with $\ell(w)$ smallest possible. Let $I = \{s, r\}$. Then either s or r must be in $\mathcal{L}(w)$. Say $s \in \mathcal{L}(w)$ for the sake of definity. Then we have two reduced expressions of w as follows:

$$w = t \cdot \underbrace{srs \cdots}_{p \text{ factors}} \cdot x = \underbrace{tst \cdots}_{q \text{ factors}} \cdot y,$$

where $p = o(sr)$ and $q = o(ts)$, both $\geqslant 3$ by our assumption, and that $x, y \in W$ satisfy the conditions $\mathcal{L}(x) \subseteq \{t\}$ and $\mathcal{L}(y) \subseteq \{r\}$. Let

$$z = tw = \underbrace{srs \cdots}_{p \text{ factors}} \cdot x = \underbrace{st \cdots}_{q-1 \text{ factors}} \cdot y.$$

Then $\mathcal{L}(z) = \{s, r\}$ and $\mathcal{L}(sz) = \{t, r\}$. Since $\ell(z) = \ell(w) - 1$, the element z is another counter-example with shorter length, contradicting the minimum assumption on w. This implies our result. $\square$

We also have the right analogue of the above lemma.

Corollary 4.2. *Let $S = \{s, r, t\}$. Assume that $w \in W$ has a reduced expression $w = \underbrace{srs \cdots}_{p \text{ factors}} \cdot x$ with $\mathcal{L}(x) \subseteq \{t\}$, and $2 \leqslant p < o(sr)$. Then $\mathcal{L}(w) = \{s\}$.*

Proof. It is enough to show $t \notin \mathcal{L}(w)$. Suppose not. Then $\mathcal{L}(w) = \{t, s\}$ and $\mathcal{L}(sw) = \{r, t\}$. But this contradicts Lemma 4.1. □

4.3. Call a sequence

$$\xi : x_0, x_1, \cdots, x_m \tag{4.3.1}$$

in W a *normal conjugate sequence* (or an *nc-sequence* in short) if $x_h \neq x_k$ for any $h \neq k$, and $x_i = s_i x_{i-1} s_i$ with some $s_i \in S$ for every i, $1 \leqslant i \leqslant m$.

$$s_1, s_2, \cdots, s_m \tag{4.3.2}$$

is called the associated sequence of ξ in S.

4.4. An element $w \in W$ is *cuspidal*, if $s < w$ for any $s \in S$. A conjugacy class C of W is *cuspidal*, if all the elements in C are cuspidal.

For any $w \in W$ and $I \in \overline{S}$, there is a reduced expression of the form

$$w = w_1 \cdot z \cdot w_2. \tag{4.4.1}$$

with $\mathcal{L}(z), \mathcal{R}(z) \subseteq S - I$ and $w_1, w_2 \in W_I$. Call (4.4.1) an *I-form* of w. The sum $\ell(w_1) + \ell(w_2)$ only depends on w, I, but not on the choice of an I-form. Denote this sum by $c_I(w)$.

Let $y \in W$ be with $\mathcal{L}(y), \mathcal{R}(y) \subseteq S - I$. Call y an *I-insulator* if $ry \neq yt$ for any $r, t \in I$; an *I-conductor*, if for any $r \in I$, there exists some $t \in I$ with $ry = yt$; and an *I-semiconductor*, if otherwise. Note that when w is cuspidal, the element z in (4.4.1) satisfies $\mathcal{L}(z) = \mathcal{R}(z) = S - I$, and is either I-insulator or I-semiconductor.

4.5. Suppose that an nc-sequence ξ in (4.3.1) satisfies an additional condition

$$\ell(x_0) = \cdots = \ell(x_m) \tag{4.5.1}$$

with $m \geqslant 1$. Then $\mathcal{L}(x_j) \neq \mathcal{R}(x_j)$ for $0 \leqslant j \leqslant m$. The following facts can be checked easily by Lemma 4.1 and Corollary 4.2.

(a) If $c_I(x_i) > o(I) + 1$ for some i, $0 \leqslant i \leqslant m$, and some $I \in \overline{S}$, then $s_k \in I$ for all k, $0 \leqslant k \leqslant m$ (see (4.3.2)).

(b) If $c_{I_i}(x_i) = o(I_i) + 1$ for some i, $0 \leqslant i \leqslant m$, and some $I_i \in \overline{S}$,, then any x_k, $0 \leqslant k \leqslant m$, satisfies $c_{I_k}(x_k) = o(I_k) + 1$ with some $I_k \in \overline{S}$.

(c) If $c_I(x_i) \leqslant o(I)$ for some i, $0 \leqslant i \leqslant m$, and any $I \in \overline{S}$, then $c_I(x_k) \leqslant o(I)$ for any k, $0 \leqslant k \leqslant m$, and any $I \in \overline{S}$. In particular, we have $\mathcal{L}(x_k) \cap \mathcal{R}(x_k) = \emptyset$.

These facts will be used later.

Lemma 4.6. *Let $\xi : x_0, x_1, \cdots, x_m$ be an nc-sequence in W with $m > 1$, $\ell(x_0) + 2 = \ell(x_1) = \cdots = \ell(x_{m-1}) = \ell(x_m) + 2$, and x_1 cuspidal. Then we can find another nc-sequence $\zeta : y_0 = x_0, y_1, \cdots, y_n = x_m$ such that $\ell(y_k) < \ell(x_1)$ for all k, $0 \leqslant k \leqslant n$.*

Proof. Let $s_1, s_2, \cdots, s_m$ be the associated sequence of ξ in S. Set $s = s_1$ and $r = s_2$. Then $s \neq r$. The element $x_1 = sx_0s$ has an $\{s, r\}$-form

$$\underbrace{srs\cdots}_{q \text{ factors}} \cdot x \cdot \underbrace{\cdots srs}_{p \text{ factors}}, \tag{4.6.1}$$

where either $1 \leqslant p \leqslant q = o(sr)$, or $1 \leqslant q \leqslant p = o(sr)$. Without loss of generality, we may assume $1 \leqslant p \leqslant q = o(sr)$. If $p > 1$, then we take $y_0 = x_0$, and

$$y_k = \begin{cases} ry_{k-1}r & \text{if } k \text{ is odd,} \\ sy_{k-1}s & \text{if } k \text{ is even.} \end{cases}$$

for $k = 1, 2, \cdots\cdots$. We see by the fact 4.5 (a) that $y_0, y_1, \cdots, y_n$ is an nc-sequence with $y_n = x_m$ and $\ell(y_k) \leqslant \ell(x_0)$ for all k, $1 \leqslant k < n$, and some $n \in \{q + p - 2, q + p - 1\}$ ($n = q + p - 1$ only if x is an $\{s, r\}$-semiconductor). If $p = 1$, then by Lemma 4.1, Corollary 4.2 and the fact 4.5 (b), we see that x_0 must have a reduced expression of the form

$$x_0 = w'_{I_1} w'_{I_2} w'_{I_3} \cdots w'_{I_l} \cdot z \tag{4.6.2}$$

satisfying:

(1) $I_j \in \overline{S}$ for $1 \leqslant j \leqslant l$.

(2) sw'_{I_1} is the longest element in W_{I_1} with $I_1 = \{s, r\}$.

(3) Let $sw'_{I_1} = w'_{I_1} a_2$. Then $a_2 \in I_2$ and $a_2 w'_{I_2}$ is the longest element in W_{I_2}. In general, if we have known that for some $i < l$, $a_i w'_{I_i} = w'_{I_i} a_{i+1}$ is the longest element in W_{I_i} with $a_i, a_{i+1} \in I_i$, then $a_{i+1} \in I_{i+1}$ and $a_{i+1} w'_{I_{i+1}}$ is the longest element in $W_{I_{i+1}}$.

(4) $\mathcal{L}(z) \subseteq S - I_l$ and $\mathcal{R}(z) =\subseteq S - I_1$.

(5) (i) If $a_{l+1}z \neq zr$ then $x_1 = sx_0s = w'_{I_1}w'_{I_2}\cdots w'_{I_l}a_{l+1}\cdot z\cdot s$, $x_{m-1} = a_{l+1}\cdot z\cdot s$ $\cdot w'_{I_1}w'_{I_2}\cdots w'_{I_{l-1}}w'_{I_l}$ and $x_m = a_{l+1}\cdot x_{m-1}\cdot a_{l+1} = z\cdot w'_{I_1}w'_{I_2}\cdots w'_{I_{l-1}}w'_{I_l}$.

(ii) If $a_{l+1}z = zr$, let $y = w'_{I_2}\cdots w'_{I_l}\cdot z$, then y is an I_1-semiconductor with $a_2y = yr$. We have $x_0 = w'_{I_1}y = \underbrace{rsr\cdots}_{p-1}\cdot y$, $x_1 = sw'_{I_1}ys = \underbrace{rsr\cdots}_{p-1}\cdot yrs$, $x_m = yrsw'_{I_1} = y\underbrace{srs\cdots}_{p-1}$.

In the case (5)(i), define an nc-sequence $y_0 = x_0, y_1, \cdots, y_{m-2}$ with the associated sequence $s_2, s_3, \cdots, s_{m-1}$ in S. Then $y_{m-2} = x_m$ and $\ell(y_0) = \ell(y_1) = \cdots = \ell(y_{m-2})$. In the case (5)(ii), the associated sequence of ξ in S is $s, r, s, \cdots$ ($p = o(sr)$ terms). We define an nc-sequence $y_0 = x_0, y_1, \cdots, y_p$ with the associated sequence in S being $r, s, r, \cdots$ (p terms). We again have $y_p = x_m$ and $\ell(y_0) = \ell(y_1) = \cdots = \ell(y_p)$. So in either case, we get a required nc-sequence $y_0 = x_0, y_1, y_2, \cdots, y_n = x_m$ for some $n \geqslant 1$. $\square$

Note that in Lemma 4.6, the condition of x_1 being cuspidal is necessary. The result may be false without it. For example, when (W, S, Γ) is of type $\widetilde{A}_2$ and $S = \{s, r, t\}$, the nc-sequence $x_0 = r, x_1 = srs, x_2 = s$ is a counter-example to the conclusion of Lemma 4.6.

Recall that in the present section we always assume the Coxeter graph Γ to be a circle with three nodes.

Theorem 4.7. *Let C be a cuspidal conjugacy class of W and let $w, y \in C$ be of the same length. Then there exist an nc-sequence*

$$\xi : x_0 = w, x_1, \cdots, x_m = y \tag{4.7.1}$$

in W, $m \geqslant 1$, with $s_1, s_2, \cdots, s_m$ the associated sequence in S, such that $\ell(x_k) \leqslant \ell(x_0)$ for all k, $1 \leqslant k < m$.

Proof. There exist an nc-sequence (4.7.1) in W with $s_1, s_2, \cdots, s_m$ the associated sequence in S for some $m \geqslant 1$. If there exists no i, $1 \leqslant i < m$, with $\ell(x_i) > \ell(w)$, then there is nothing to do. Otherwise, we take x_i such that $\ell(x_i) = \max\{\ell(x_j) \mid 1 \leqslant j < m\}$ and i the smallest possible positive integer with this property. Thus $\ell(x_i) = \ell(x_{i-1}) + 2$. Let j be the smallest integer greater than i with $\ell(x_j) < \ell(x_i)$. Then $\ell(x_j) = \ell(x_i) - 2$.

By Lemma 4.6, we can replace in (4.7.1) the subsequence $x_i, x_{i+1}, \cdots, x_{j-1}$ by some nc-sequence $x_{i1}, x_{i2}, \cdots, x_{i,n-1}$ with $\ell(x_{ij}) \leqslant \ell(x_{i-1})$, $1 \leqslant j < n$, such that the resulting is again an nc-sequence. By applying induction first on the number $m(\xi) := \max\{\ell(x_j) \mid 1 \leqslant j < m\} - \ell(x_0) \geqslant 0$ and then on the cardinality of the set $\{k \mid 1 \leqslant k < m, \ell(x_k) = m(\xi)\}$, we can eventually get a required nc-sequence (4.7.1). □

Theorem 4.8. *The Coxeter elements rst and tsr are not conjugate in W. So in $C(W)$, the ss-classes are the same as W-conjugacy classes.*

Proof. It is known that in the group W, a Coxeter element is minimal in length in the conjugacy class containing it. It is also known to be cuspidal for any conjugacy class of W containing a Coxeter element. By Theorem 4.7, we see that if x, y are two conjugate Coxeter elements in W, then there exists an nc-sequence $x_0 = x, x_1, \cdots, x_m = y$ such that $\ell(x_k) \leqslant \ell(x)$ for $1 \leqslant k < m$. This implies that $\ell(x_k) = \ell(x)$ for $1 \leqslant k < m$ and hence all the x_k's are in $C(W)$. That is, x and y are in the same ss-class of $C(W)$. But rst and tsr are not ss-equivalent. So they are not conjugate in W.

We know from [10, Lemma 3.2] that the set $C(W)$ contains exactly six elements which belong to two ss-classes in $C(W)$: $\{rst, str, trs\}$ and $\{tsr, srt, rts\}$. This tells us that these two sets are also W-conjugacy classes in $C(W)$. □

Remark 4.9. (1) From the above discussion and from the fact pointed out in 3.4, it is easily seen that all the results in this section remain valid if we drop the restriction of $o(I) < \infty$ for any $I \in \overline{S}$. The proof is similar to and even simpler than the above if there is some $I \in \overline{S}$ with $o(I) = \infty$.

(2) In [4], Geck and Pfeiffer considered the set $C_{\min}$ of all the shortest elements in a conjugacy class C of a Coxeter system (W, S, Γ). Given $w \in C$, they considered if there always exist an nc-sequence $w_0 = w, w_1, \cdots, w_r$ in C with $\ell(w_{i-1}) \geqslant \ell(w_i)$ for every i, $1 \leqslant i \leqslant r$, and $w_r \in C_{\min}$. They also considered if for any $w, y \in C_{\min}$, there always exist some sequence $w_0 = w, w_1, \cdots, w_r = y$ in $C_{\min}$ such that $w_i = x_i w_{i-1} x_i^{-1}$ with $\ell(w_i) = \ell(w_{i-1})$, and either $\ell(x_i w_{i-1}) = \ell(x_i) + \ell(w_{i-1})$ or $\ell(w_{i-1} x_i^{-1}) = \ell(w_{i-1}) + \ell(x_i^{-1})$ for all i and some $x_i \in W$. These two problems have been solved by Geck and Pfeiffer in the case where W is a Weyl group (see [4]), but not in general. Now let C be a conjugacy class of W containing a Coxeter element. Then the set $C_{\min}$ just consists of Coxeter elements. The above results (i.e., Lemma 1.4, Theorems 1.6, 3.4 and 4.8)

show that when the Coxeter graph Γ of W is either a tree, or a circle with three nodes, or a three multiple circle (i.e., the case assumed in Theorem 3.4), the second problem for a conjugacy class containing a Coxeter element has an affirmative answer. Now assume Γ a circle with three nodes. By Lemma 4.6, we see that the first problem has an affirmative answer for C a cuspidal conjugacy class of W. In this case, we see from Lemma 4.6 that a cuspidal element $w \in W$ is in the set $C_{\min}$ for some conjugacy class C of W if and only if w satisfies $c_I(w) \leqslant o(I)$ for any $I \in \overline{S}$ and $\mathcal{L}(w) \cap \mathcal{R}(w) = \emptyset$ (see 4.5 (c)). Now assume that w is a cuspidal element in a non-cuspidal conjugacy class of W. Again by Lemma 4.6 and the facts 4.5 (a)-(c), we see that there exist an nc-sequence $x_0 = w, x_1, \cdots, x_t$ such that $\ell(x_i) \leqslant \ell(x_{i-1})$ for every i, $1 \leqslant i \leqslant t$, and x_t is non-cuspidal. This implies that for a non-cuspidal conjugacy class C of W, $C_{\min}$ consists of non-cuspidal elements. Therefore by the knowledge of the conjugacy in the dihedral groups and by Lemma 4.6, we can show that for any $I \neq J$ in $\overline{S}$, two elements $w \in W_I$ and $y \in W_J$ are W-conjugate if and only if $w \underset{W_I}{\sim} s$, $y \underset{W_J}{\sim} t$ for some $s \in I$, $t \in J$ with either $s = t$ or $o(st)$ odd. These results imply that both problems considered by Geck and Pfeiffer have affirmative answers for Γ a circle with three nodes.

We conclude the section by proposing the following

Conjecture 4.10. *Let (W, S, Γ) be an irreducible finitely-presented Coxeter system with Γ containing at most one circle. Then two elements of $C(W)$ are W-conjugate if and only if they are ss-equivalent.*

Remark 4.11. (1) By the fact pointed out in 3.4, to verify the conjecture, it is enough to deal with the case where $o(sr)$ is either a prime or 4 for any neighboring nodes s, r in Γ.

(2) For a Coxeter system (W, S, Γ) with Γ a circle, we ask if one can always find a suitable matrix presentation of W (just as we did in the case of type $\widetilde{A}_{n-1}$) such that $w, y \in C(W)$ are W-conjugate if and only if they have the same characteristic polynomial.

§5. Characteristic polynomials of $w \in C(W)$ in one or more circles cases.

Beside the cases considered in §§3-4, the conjugacy problem in $C(W)$ still remains open for all other Coxeter systems (W, S, Γ) with Γ not a tree. Thus we may consider a weaker problem of spectral classes. The *spectral class* of a linear operator is its eigenvalues together with their multiplicity. When we talk about the spectral class of $w \in W$, we always identify w with the corresponding linear transformation in the natural reflection representation of W. Clearly, the spectral class of w is precisely determined by the characteristic polynomial of w, which gives some important (but not complete) information for the conjugacy in $C(W)$. In this section, we give a formula for the characteristic polynomials of $w \in C(W)$ in the case where Γ is a circle. We also compute the characteristic polynomials $w \in C(W)$ in some hyperbolic Coxeter systems (W, S, Γ) where Γ contains more than one circles. By enumerating the spectral classes for $C(W)$, we give a negative answer to a conjecture of Coleman in [3, IX, Question 2].

5.1. In 5.1-5.4, we assume that (W, S, Γ) is a Coxeter system with Γ a circle, where $S = \{s_1, s_2, \cdots, s_n\}$ satisfies that $2 < m_{i,i+1} := o(s_i s_{i+1}) < \infty$ for $1 \leqslant i \leqslant n$ with the subscripts regarded as the congruence classes modulo n. Let $a_i = 2\cos\dfrac{\pi}{m_{i,i+1}}$. Then the matrix form of the generator s_i is

$$I_n - 2E_{ii} + a_{i-1}E_{i,i-1} + a_iE_{i,i+1}, \tag{5.1.1}$$

here and later, when we talk about the matrix of $w \in W$, it always means that this is with respect to a fixed simple root basis in the natural reflection representation of W.

5.2. Let Γ° be the graph obtained from Γ by adding a loop at each node i. To each s_i, we associate a set consisting of three weighted paces in Γ°:

(i) a pace along the loop at the node i with weight b_{ii};

(ii) a pace along the edge from the node i to $i+1$ with weight $b_{i,i+1}$;

(iii) a pace along the edge from the node i to $i-1$ with weight $b_{i,i-1}$.

These weights satisfy the conditions:

(1) $b_{ii} = -1$ for $1 \leqslant i \leqslant n$;

(2) $b_{i,i+1} = b_{i+1,i} = a_i$ for $1 \leqslant i \leqslant n$.

To each $w = s_{i_1}s_{i_2}\cdots s_{i_n} \in C(W)$ and any h, k, $1 \leqslant h, k \leqslant n$, define by $A_{hk}(w)$ the set of all weighted paths $\xi : x_0 = h, x_1, \cdots, x_t = k$ on Γ°, where $x_0, x_1, \cdots, x_{t-1}$ is a subsequence of $i_1, i_2, \cdots, i_n$, $x_i \equiv x_{i-1} \pm 1$ modulo n for all i, $1 \leqslant i < t$, and $x_t \equiv x_{t-1}, x_{t-1} \pm 1$ modulo n. The weight $\mathrm{wt}(\xi)$ of ξ is defined to be $\prod_{i=1}^t b_{x_{i-1},x_i}$. Then define the weight of $A_{hk}(w)$ by $a_{hk} = \sum_{\xi \in A_{hk}(w)} \mathrm{wt}(\xi)$ with the convention that $a_{hk} = 0$ if $A_{hk}(w) = \emptyset$. We have the following result.

Proposition 5.3. *In the above setup, the matrix of $w \in C(W)$ in the natural reflection representation is $A(w) = (\, a_{ij} \,)$.*

By Proposition 5.3, we can show that

Theorem 5.4. *The characteristic polynomial $f_m(x)$ of $w_m = s_1 s_2 \cdots s_m s_n s_{n-1} \cdots s_{m+1} \in C_1(W)$ is*
(5.4.1)

$$f_m(x) = \sum_{k=0}^{n} \left(\sum_{p=0}^{k} (-1)^p \binom{n-2p}{k-p} \sum_{(i_1,\cdots,i_p)\in J_n^p} \prod_{j=1}^{p} a_{i_j}^2 - (\delta_{km} + \delta_{k,n-m}) \cdot \prod_{i=1}^{n} a_i \right) x^k,$$

where J_n^p is the set of all subsets $\{i_1, \cdots, i_p\}$ of $\{1, 2, \cdots, n\}$ such that $i_a \not\equiv i_b \pm 1$ modulo n for any $a \neq b$, and δ_{ij} is the Kronecker symbol which takes the value 1 or 0 according to i, j being equal or not. This polynomial is reciprocal in x, and symmetric in the parameters $a_1, a_2, \cdots, a_n$.

From (5.4.1), we see that $f_m(x) = f_{m'}(x)$ if and only if $m' = m, n-m$. This implies that if $m' \neq m, n-m$ then w_m and $w_{m'}$ are not conjugate in W.

Note that $d(w_m) = m$. By Theorem 1.6 and (1.6.1), we see that (5.4.1) is the characteristic polynomial for any $w \in C(W)$ with $\nu(w) = 2m-n, n-2m$.

Example 5.5. Let Γ be a circle with four nodes s_1, s_2, s_3, s_4 in an anti-clockwise ordering. Assume $o(s_i s_{i+1}) = m_{i,i+1} < \infty$ for $i = 1, 2, 3, 4$. Then $C(W)$ contains 14 elements by [10, Lemma 3.2]. They belong to 3 ss-classes: $\{\mathbf{1234}, \mathbf{4123}, \mathbf{3412}, \mathbf{2341}\}$, $\{\mathbf{1432}, \mathbf{2143}, \mathbf{3214}, \mathbf{4321}\}$ and $\{\mathbf{1243}, \mathbf{3124}, \mathbf{4312}, \mathbf{2431}, \mathbf{3241}, \mathbf{4312}\}$, here and later, sometimes we denote s_j by $\mathbf{j}$ (boldfaced) for simplifying the notation. Let $a = 2\cos\frac{\pi}{m_{12}}$, $b = 2\cos\frac{\pi}{m_{23}}$, $c = 2\cos\frac{\pi}{m_{34}}$ and $d = 2\cos\frac{\pi}{m_{41}}$. Then the matrices of s_1, s_2, s_3, s_4 are as follows.

$$s_1 = \begin{pmatrix} -1 & a & 0 & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & -1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & -1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d & 0 & c & -1 \end{pmatrix}$$

By Proposition 5.3, we get

$$\mathbf{1234}=\begin{pmatrix} a^2+d^2-1+abcd & ab^2-a & abc^2-ab+cd & -abc-d \\ a+bcd & b^2-1 & bc^2-b & -bc \\ cd & b & c^2-1 & -c \\ d & 0 & c & -1 \end{pmatrix}$$

$$\mathbf{1432}=\begin{pmatrix} a^2+d^2-1+abcd & -a-bcd & ab-cd+b^2cd & c^2d-d \\ a & -1 & b & 0 \\ ab & -b & b^2-1 & c \\ abc+d & -bc & b^2c-c & c^2-1 \end{pmatrix}$$

$$\mathbf{1243}=\begin{pmatrix} a^2+d^2-1 & -a+ab^2+bcd & -ab-cd & abc+c^2d-d \\ a & b^2-1 & -b & bc \\ 0 & b & -1 & c \\ d & cb & -c & c^2-1 \end{pmatrix}$$

Then by (5.4.1) we see that the characteristic polynomials of **1234** and **1432** are equal to

$$\begin{aligned} x^4+&(4-a^2-b^2-c^2-d^2-abcd)x^3+(6-2a^2-2b^2-2c^2-2d^2+a^2c^2+b^2d^2)x^2 \\ &+(4-a^2-b^2-c^2-d^2-abcd)x+1. \end{aligned}$$

The characteristic polynomial of **1243** is

$$\begin{aligned} x^4+&(4-a^2-b^2-c^2-d^2)x^3+(6-2a^2-2b^2-2c^2-2d^2+a^2c^2+b^2d^2-2abcd)x^2 \\ &+(4-a^2-b^2-c^2-d^2)x+1. \end{aligned}$$

So there are two spectral classes in $C(W)$.

Next we consider some cases of hyperbolic Coxeter groups (W,S,Γ) with Γ containing more than one circle.

5.6. Let $S=\{s_1,s_2,s_3,s_4\}$ be with $o(s_1s_3)=2$ and $o(s_1s_2)=o(s_1s_4)=o(s_2s_3)=o(s_2s_4)=o(s_3s_4)=3$. Then $|C(W)|=18$ by [10, Theorem 3.8]. The set $C(W)$ contains four ss-classes

$$\{\mathbf{1432},\mathbf{4321},\mathbf{3214},\mathbf{2143}\},\qquad \{\mathbf{1234},\mathbf{4123},\mathbf{3412},\mathbf{2341}\},$$

$$\{\mathbf{1423},\mathbf{3142},\mathbf{3421},\mathbf{4231},\mathbf{2314}\},\qquad \{\mathbf{1243},\mathbf{3124},\mathbf{3241},\mathbf{2431},\mathbf{4312}\}.$$

The characteristic polynomials of **1432** and **1234** are equal to $x^4-4x^3-6x^2-4x+1$, and those of **1423** and **1243** are $x^4-3x^3-8x^2-3x+1$. So there are two spectral classes in $C(W)$.

5.7. Let $S=\{s_1,s_2,s_3,s_4,s_5\}$ be with $o(s_1s_3)=o(s_1s_5)=o(s_3s_5)=o(s_2s_4)=2$ and $o(s_is_j)=3$ for all other pairs i,j, $i\neq j$. Then $|C(W)|=46$ by [10, Theorem 3.8]. The set $C(W)$ contains seven ss-classes:

$$\{\mathbf{12345},\mathbf{23451},\mathbf{52341},\mathbf{51234},\mathbf{45123},\mathbf{34512}\},$$
$$\{\mathbf{12543},\mathbf{25431},\mathbf{32541},\mathbf{31254},\mathbf{43125},\mathbf{54312}\},$$
$$\{\mathbf{32145},\mathbf{35214},\mathbf{52143},\mathbf{21435},\mathbf{14352},\mathbf{43521}\},$$
$$\{\mathbf{14523},\mathbf{45231},\mathbf{34521},\mathbf{31452},\mathbf{23145},\mathbf{52314}\},$$
$$\{\mathbf{14325},\mathbf{43251},\mathbf{54321},\mathbf{51432},\mathbf{25143},\mathbf{32514}\},$$
$$\{\mathbf{34125},\mathbf{41235},\mathbf{54123},\mathbf{35412},\mathbf{23541},\mathbf{12354}\},$$
$$\{\mathbf{24351},\mathbf{21354},\mathbf{13524},\mathbf{41352},\mathbf{52431},\mathbf{15243},\mathbf{12435},\mathbf{31245},\mathbf{32451},\mathbf{35241}\}.$$

The characteristic polynomials of the elements in the first six ss-classes are equal to $x^5-3x^4-6x^3-6x^2-3x+1$. The ones in the last ss-class are $x^5-x^4-8x^3-8x^2-x+1$. So there are two spectral classes in $C(W)$.

5.8. Given a connected graph K with n nodes and m edges. Fix a spanning tree of K. Call the edges which were removed *special edges*. If one of these edges is put back on the tree then the resulting graph will contain one circle which is called a *special circle*. Clearly, we have exactly $q=m-n+1$ special circles in K.

In [3, IX, Question 2], Coleman conjectured that in a Coxeter system (W,S,Γ), the number of spectral classes for $C(W)$ should be $\prod_\gamma[n_\gamma/2]$, where γ ranges over the special circles of Γ, n_γ is the number of nodes in the circle γ. According to this conjecture, the number of spectral classes of $C(W)$ should be 1 or 2 in the case (5.6) and 4 in the case (5.7). But now we have 2 spectral classes of $C(W)$ in both cases. So the case (5.7) is a counter-example for the Coleman's conjecture, and the case (5.6) tells us that the formula in the Coleman's conjecture is not well defined, which depends on the choice of spanning tree.

Adding a note. After this paper was completed, R. B. Howlett told me an unpublished result of him by an e-mail which asserted that a minimal length element in a non-cuspidal conjugacy class of a Coxeter group W is always non-cuspidal. This extends one of our results in Remark 4.9 (2). Later the referee told me that the result of Howlett had appeared as Corollary 3.1.11 in the newly-published paper [5] by Geck and Pfeiffer. I would like to express my sincere gratitude to Howlett and the referee. The proof of Howlett's result is as follows. Suppose that w is of minimal length in its conjugacy class, and suppose that $w = xvx^{-1}$ where v is in the proper parabolic subgroup W_J. Replacing v by a suitable W_J-conjugate of itself, we may assume that x is the minimal length element of the coset xW_J. Now let $N(v)$ be the set of positive roots a such that $v(a)$ is negative. Then $N(v)$ is contained in Φ_J, the root system of W_J. (i.e. the elements of $N(v)$ are linear combinations of the simple roots in the set J.) Thus $x(N(v))$ consists of positive roots (since x takes positive roots in Φ_J to positive roots). But $w(x(N(v))) = x(v(N(v))$ consists of negative roots, since $v(N(v))$ consists of negative roots in Φ_J. So $N(w)$ contains $x(N(v))$; so the cardinality of $N(w)$ is at least as great as that of $N(v)$. i.e. $\ell(w) \geq \ell(v)$. We can even say more in the case that $\ell(w) = \ell(v)$. Then $N(w) = x(N(v))$, and if a is a simple root in $N(w)$ then $x^{-1}(a)$ is in $N(v)$ and hence in Φ_J (and positive). Writing $x^{-1}(a) = \sum_{b\in J} \lambda_b b$ gives $a = \sum_{b\in J} \lambda_b x(b)$, a positive combination of positive roots. (Note that $x(b)$ is positive since b is in J and x is a minimal coset representative.) Since a is simple, this forces $x^{-1}(a)$ to be simple – i.e. there is only one term in the sum. So $x^{-1}a = b$, where b is a simple root in J. Now the elements wr_a and vr_b are conjugate, and we can apply the same argument to them, eventually getting that w and v are conjugate by an element that conjugates all the simple reflections appearing in a reduced expression for v to simple reflections appearing in a reduced expression for w. i.e. $x(J) = K \subset \Pi$.

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