SIGN TYPES CORRESPONDING TO AN AFFINE WEYL GROUP

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ABSTRACT

The sign types corresponding to an affine Weyl group \( W_a \) were first studied in [3]. In the present paper, I generalize all the results of [3] on sign types to the case when \( W_a \) is an indecomposable affine Weyl group of an arbitrary type. As a result, I verify Carter's conjecture on the cardinality of sign types of type \( \Phi \), where \( \Phi \) is the root system determined by \( W_a \).

In [3], we defined sign types of the Euclidean space \( E \) spanned by the root system \( \Phi \) of type \( A_{n-1} \). These sign types are the connected components of the complement of a certain set of hyperplanes in \( E \) and can be regarded as certain equivalence classes of \( W_a \), where \( W_a \) is the affine Weyl group of type \( A_{n-1} \) identified with the set of alcoves of \( E \) via its action on \( E \). The sign types play an important role in the study of the affine Weyl groups [3]. We described sign types of \( E \) and showed that the number of sign types of \( E \) is \((n+1)^{n-1}\) in [3].

I am very grateful to Professor R. W. Carter who told me that the formula \((n+1)^{n-1}\) can be rewritten \((h+1)^l\), where \( l = n-1 \) is the rank of \( \Phi \) and \( h \) is the Coxeter number of \( \Phi \). He then conjectured that this result can be generalized to the case when \( \Phi \) is an indecomposable root system of any other type.

In the present paper, I shall generalize all the results of [3] on sign types to the case when \( \Phi \) is an indecomposable root system of an arbitrary type. The main results are Theorems 2.1 and 8.1. We start with the definition of an admissible sign type in terms of a \( \Phi \)-tuple over \( \mathbb{Z} \). Then §§3–5 are reserved for the proof of Theorem 2.1. Theorem 2.1 asserts that the set \( \mathcal{S}(\Phi) \) of admissible sign types can be identified with the set of certain equivalence classes of \( W_a \). We also deduce in §6 that \( \mathcal{S}(\Phi) \) can be identified with the set of connected components of the complement of a certain set of hyperplanes in \( E \). Finally, we prove Theorem 8.1 in §§7–8 and thus verify the above conjecture of Carter.

1. Preliminary

Let \( \Phi \) be an indecomposable reduced root system. Choose a simple root system \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \) of \( \Phi \). Let \( \Phi^+, \Phi^- \) be the corresponding positive and negative root systems of \( \Phi \). Let \( E \) be the Euclidean space spanned by \( \Phi \) with positive definite inner product \( \langle , \rangle \) such that \( \alpha^2 = \langle \alpha, \alpha \rangle = 1 \) for any short root \( \alpha \) of \( \Phi \). For any \( \alpha \in \Phi \), \( \alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle \) is called the coroot of \( \alpha \). The set \( \Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\} \) is a root system such that the set \( \{\alpha_1^\vee, \ldots, \alpha_l^\vee\} \) affords a choice of simple root system in it. Let \( -\alpha_0 \) be the highest short root of \( \Phi \). Then \( (-\alpha_0)^\vee \) is the highest (co)root of \( \Phi^\vee \). Let \( h \) be the Coxeter number of \( \Phi \). Then \( h \) is also the Coxeter number of \( \Phi^\vee \).

Let \( W \) be the Weyl group of \( \Phi \) generated by the reflections \( s_\alpha \) on \( E \) for \( \alpha \in \Phi \), where

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\(s_\alpha\) sends \(x\) to \(x - \langle x, \alpha' \rangle \alpha\). Let \(Q\) denote the root lattice \(\mathbb{Z}\Phi\). Let \(N\) denote the group consisting of all translations \(T_\lambda\) operating on \(E\) for \(\lambda \in Q\), where \(T_\lambda\) sends \(x\) to \(x + \lambda\). We denote by \(W_a\) the group of affine transformations of \(E\) generated by \(N\) and \(W\).

It is well known that \(W_a\) is the semidirect extension of \(W\) by the normal subgroup \(N\) on which the action of \(W\) is known. Any \(w \in W_a\) has a unique decomposition \(w = wT_\lambda\) with \(w \in W\) and \(\lambda \in Q\).

For linear and affine transformations, we shall denote the operation on the right and compose them accordingly. With this convention, we define \(s_0 = s_{\alpha_0}T_{-\alpha_0}, s_i = s_{\alpha_i}, 1 \leq i \leq l\). It is known that \(W_a\) (respectively \(W\)) is a Coxeter group on generators \(s_0, s_1, \ldots, s_l\) (respectively \(s_1, \ldots, s_l\)). We denote \(\Delta = \{s_0, s_1, \ldots, s_l\}\). The group \(W_a\) will be called an affine Weyl group.

We define the length \(l(w)\) of an element \(w \in W_a\) to be the smallest number \(r\) such that there exists an expression \(w = s(1)s(2)\ldots s(r)\) with \(s(i) \in \Delta\). An expression of \(w\) is called a reduced form if it is a product of \(l(w)\) generators.

The symbol \(\leq\) denotes the Bruhat order on \(W_a\) (defined, for example, in [5]). For any \(w \in W_a\), we associate two subsets of \(\Delta\) as follows:

\[
\mathcal{L}(w) = \{s \in \Delta | sw < w\}, \\
\mathcal{R}(w) = \{s \in \Delta | ws < w\}.
\]

Given any two sets \(S, R\), we call \(x = (x_i)_{i \in R}\) an \(R\)-tuple over \(S\) if \(x_i \in S\) for all \(i \in R\). Sometimes we simply call \(x\) an \(R\)-tuple when there is no danger of confusion. Two \(R\)-tuples \(x = (x_i)_{i \in R}\) and \(y = (y_i)_{i \in R}\) are said to be equal if \(x_i = y_i\) for all \(i \in R\).

For any \(\alpha \in \Phi^+, k \in \mathbb{Z}\), and a positive real number \(m\), we define a hyperplane

\[
H_{\alpha, k} = \{v \in E | \langle v, \alpha' \rangle = k\}
\]

and a stripe

\[
H_{\alpha, k}^m = H_{-\alpha, -k}^m = \{v \in E | k - \langle v, \alpha' \rangle < k + m\}.
\]

We call any non-empty connected simplex of

\[
E - \bigcup_{\alpha \in \Phi} H_{\alpha, k},
\]

an alcove of \(E\). Each alcove of \(E\) has the form \(\bigcap_{x \in \Phi^+} H_{\alpha, k_x}\) for a \(\Phi^+\)-tuple \((k_x)_{\alpha \in \Phi^+}\) over \(\mathbb{Z}\). The following results are well known.

**Lemma 1.1 [4, Lemma 1.1].** Let \(A_1 = \bigcap_{\alpha \in \Phi^+} H_{\alpha, k}\). Then \(A_1\) is an alcove of \(E\) which can also be expressed in the form \(\bigcap_{\alpha \in \Phi} H_{\alpha, c_\alpha}\) where the \(c_\alpha\) satisfy the equation

\[
(-\alpha_0)' = \sum_{\alpha \in \Pi} c_\alpha \alpha'.
\]

**Theorem 1.2 [4, Theorem 5.2].** Let \(A_k = \bigcap_{\alpha \in \Phi^+} H_{\alpha, k}\) with \(k_\alpha \in \mathbb{Z}\). Then \(A_k\) is an alcove of \(E\) if and only if for any \(\alpha, \beta \in \Phi^+\) with \(\alpha + \beta \in \Phi^+\), the inequality

\[
|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leq |\alpha + \beta|^2 (k_{\alpha + \beta} + 1) \leq |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha + \beta|^2 - 1
\]

holds.

It is well known that the right action of \(W_a\) on \(E\) induces a bijective map \(w \mapsto (A_w)w = A_{w}\) from the set of elements \(W_a\) to the set \(\mathcal{A}\) of alcoves of \(E\). Thus any alcove of \(E\) has the form \(A_w = \bigcap_{\alpha \in \Phi^+} H_{\alpha, k(w, \alpha)}\) or \(A_w = \bigcap_{\alpha \in \Phi} H_{\alpha, k(w, \alpha)}\).
with the convention that $k(w, -\alpha) = -k(w, \alpha)$ for any $\alpha \in \Phi^+$. We shall identify $W_a$ with $\Phi$ as a set under the correspondence $w \mapsto A_w$. Later the integers $k(w, \alpha)$ indexed by $w \in W_a$ and $\alpha \in \Phi$ always stand for the coordinates of the alcove $A_w$. The following result is known.

**Proposition 1.3** [4, Proposition 4.2]. Let $w' = ws_j$ with $w \in W_a$ and $s_j \in \Delta$. Then for any $\alpha \in \Phi$, we have $k(w', \alpha) = k(w, (\alpha)s_j) + k(s_j, \alpha)$.

### 2. Admissible sign types

A $\Phi$-tuple $X = (X_\alpha)_{\alpha \in \Phi}$ over the set $\{+, -, 0\}$ is called a sign type of type $\Phi$ if the set $\{X_\alpha, X_{-\alpha}\}$ is either $\{0, 0\}$ or $\{+, -\}$ for any $\alpha \in \Phi$. We see that a sign type $(X_\alpha)_{\alpha \in \Phi}$ is entirely determined by the $\Phi^+$-tuple $(X_\alpha)_{\alpha \in \Phi^+}$. So sometimes we can identify $(X_\alpha)_{\alpha \in \Phi^+}$ with $(X_\alpha)_{\alpha \in \Phi}$ and call $(X_\alpha)_{\alpha \in \Phi^+}$ a sign type.

Let $\mathcal{F} = \mathcal{F}(\Phi)$ be the set of all sign types of type $\Phi$. Let

$$G_1 = \{ +, +, +, O, - + + + +, O, - + + +, O, - + + +, O, - + + +, O, - + + +, O \}.$$  

$$G_2 = \{ O, O, O, O, - - - + +, O, O, O, O, - - - + +, O, O, O, O, - - - + +, O, O, O, O, - - - + +, O, O, O, O, - - - + + \}.$$  

$$G_3 = \{ O, O, O, O, - - - + +, O, O, O, O, - - - + +, O, O, O, O, - - - + +, O, O, O, O, - - - + +, O, O, O, O, - - - + + \}.$$
Given an indecomposable positive subsystem $\Phi^+$ of $\Phi$ of rank 2, we say that a sign type $(X_\alpha)_\alpha \in \Phi^+$ is admissible if one of the following conditions is satisfied:

1. $\Phi^+$ has type $A_2$, say $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$. Then $X_\alpha, X_\beta \in G_1$.

2. $\Phi^+$ has type $B_2$, say $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$. Then $X_\alpha, X_\beta, X_{2\alpha + \beta} \in G_2$.

3. $\Phi^+$ has type $G_2$, say $\Phi^+ = \{\alpha, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$. Then $X_\alpha, X_\beta, X_{3\alpha + \beta}, X_{3\alpha + 2\beta} \in G_3$.

We say that a sign type $(X_\alpha)_\alpha \in \Phi$ is admissible if for any indecomposable positive subsystem $\Phi^+$ of $\Phi$ of rank 2, the sign type $(X_\alpha)_\alpha \in \Phi^+$ is admissible.

Let $\mathcal{S} = \mathcal{S}(\Phi)$ be the set of all admissible sign types of $\mathcal{S}$.

Define a map

$$\zeta: W_a \rightarrow \mathcal{S}$$

by sending $A_w = \bigcap_{\alpha \in \Phi} H^1_\alpha; k(w, \alpha)$ to $X_w = (X(w, \alpha))_\alpha \in \Phi$ such that for any $\alpha \in \Phi$,

$$k(w, \alpha) > 0 \iff X(w, \alpha) = +,$$

$$k(w, \alpha) = 0 \iff X(w, \alpha) = 0,$$

$$k(w, \alpha) < 0 \iff X(w, \alpha) = -.$$

By Theorem 1.2, one can check that $\zeta(W_a) \subseteq \mathcal{S}$. Thus $\zeta$ induces a map from $W_a$ to $\mathcal{S}$ which we still denote by $\zeta$. In particular, one can check directly that $\zeta(W_a) = \mathcal{S}$ when $\Phi$ has rank 2. In the following sections we shall go further and show the following.

**Theorem 2.1.** $\zeta(W_a) = \mathcal{S}(\Phi)$ for any indecomposable root system $\Phi$.

We denote $\Pi \cup \{-\alpha_0\}$ by $\tilde{\Pi}$. 

3. Some results on $X_\beta, \beta \in \Pi$

Sections 3 and 4 will be reserved mainly for the proof of Theorem 2.1. We assume that rank $\Phi > 2$ in these two sections.

For any $X = (X_\alpha)_{\alpha \in \Phi} \in \mathcal{J}$, we define $m_X = \#\{\alpha \in \Phi^+ | X_\alpha = -\}.

**Lemma 3.1.** Assume that $X = (X_\alpha)_{\alpha \in \Phi} \in \mathcal{J}$ and $m_X > 0$. Then there exists some $\beta \in \Pi$ satisfying $X_\beta = -$.

**Proof.** It is enough to show that if $X_\alpha \in \{+, 0\}$ for all $\alpha \in \Pi$ then $m_X = 0$, that is, for any $\beta \in \Phi^+$, we have $X_\beta \in \{+, 0\}$. Now we assume that $X_\alpha \in \{+, 0\}$ for all $\alpha \in \Pi$. We apply induction on $ht(\beta) > 1$, the height of $\beta \in \Phi^+$. The result is obviously true when $ht(\beta) = 1$, by our assumption. Now assume that $ht(\beta) > 1$. Then we can write $\beta = \gamma + \delta$ for some $\gamma, \delta \in \Phi^+$. By the inductive hypothesis, $X_\gamma, X_\delta \in \{+, 0\}$. By symmetry, we need only to consider the following cases.

(i) $\{\gamma, \delta, \beta\}$ forms a positive subsystem of $\Phi$ of type $A_2$. By the hypothesis that $X \in \mathcal{J}$ and $X_\gamma, X_\delta \in \{+, 0\}$, we have

$$X_\beta \in \{+, 0\}.$$

(ii) $\{\gamma, \delta, \beta, \beta + \gamma\}$ forms a positive subsystem of $\Phi$ of type $B_2$. Then by the same reasoning as in (i), we have

$$X_\beta \in \{+, 0\}.$$

(iii) $\{\delta - \gamma, \gamma, \delta, \beta\}$ forms a positive subsystem of $\Phi$ of type $B_2$. Then by the same reasoning as above, we have

$$X_\beta \in \{+, 0\}.$$

Therefore, our result follows by induction.

Call a sign type $X = (X_\alpha)_{\alpha \in \Phi} \in \mathcal{J}$ dominant if $m_X = 0$. 
**Lemma 3.2.** Assume that $X = (X_\alpha)_{\alpha \in \Phi} \in \mathcal{S}$ is dominant. Assume that not all $X_\alpha$, $\alpha \in \Phi$, are equal to 0. Then $X_{-\alpha_0} = +$.

*Proof.* One can check the result directly when rank $\Phi = 2$. Now assume that rank $\Phi > 2$. By our condition, there exists some $\alpha \in \Phi^*$ with $X_\alpha = +$.

(i) First assume that $\alpha$ is a short root of $\Phi$ (including the case when the roots in $\Phi$ all have the same length). Then $\alpha = -\alpha_0$. By a well-known result, there exists a sequence of roots

$$\beta_0 = \alpha, \beta_1, \ldots, \beta_r = -\alpha_0$$

in $\Phi^+$ such that for every $i$, $1 \leq i \leq r$, $\beta_i < \beta_{i-1}$ and $\beta_i = (\beta_{i-1})_{\gamma_i}$ with some $\gamma_i \in \Pi$. Clearly, all $\beta_j$, $0 \leq j \leq r$, are short roots. Now it is enough to show that if $X_{\beta_{i-1}} = +$ for some $i$, $1 \leq i \leq r$, then $X_{\beta_i} = -$. Our conditions on $\beta_{i-1}$, $\beta_i$ clearly imply that $\langle \beta_{i-1}, \gamma_i \rangle = -1$ and hence $\beta_i = \beta_{i-1} + \gamma_i$. If $\gamma_i$ is short then $\{\beta_{i-1}, \gamma_i, \beta_i\}$ forms a positive subsystem of $\Phi$ of type $A_2$. Then by the assumption that $X \in \mathcal{S}$, $X_{\beta_{i-1}} = +$ and $X_{\gamma_i} \in \{+, 0\}$, we have

$$X_{\beta_i} \in \{+, +, +\} \setminus \{0\}$$

which implies that $X_{\beta_i} = +$. If $\gamma_i$ is long then $\{\beta_{i-1}, \gamma_i, \beta_i, \beta_{i-1} + \beta_i\}$ forms a positive subsystem of $\Phi$ of type $B_2$. Then by the assumption that $\beta_{i-1} = +$, $\gamma_i \in \{+, 0\}$ and $X \in \mathcal{S}$, we have

$$X_{\gamma_i} \in \{+, +, +\}$$

which also implies that $X_{\beta_i} = +$. As $i$ runs over $1, 2, \ldots, r$ in turn, we can show that $X_{\alpha_0} = +$ from $X_\alpha$ by repeatedly using the above argument.

(ii) Now assume that the roots in $\Phi$ have two different lengths and that $\alpha$ is a long root. Let $\beta$ be the highest (long) root of $\Phi$. Then there exists a sequence of long roots

$$\beta_0 = \alpha, \beta_1, \ldots, \beta_r = \beta$$

in $\Phi^+$ such that for every $i$, $1 \leq i \leq r$, $\beta_i < \beta_{i-1}$ and $\beta_i = (\beta_{i-1})_{\gamma_i}$ with some $\gamma_i \in \Pi$. By a similar argument to that in (i), we can show that $X_{\beta} = +$ from $X_\alpha = +$.

We see that $\Phi$ has type $B_1$, $C_1$ or $F_1$ according to our assumption. In any of these cases, $\{-2\alpha_0 - \beta, \alpha_0 + \beta, -\alpha_0, \beta\}$ forms a positive subsystem of $\Phi$ of type $B_2$. So by the hypothesis that $X \in \mathcal{S}$, $X_\beta = +$ and $X_{-2\alpha_0 - \beta}, X_{\alpha_0 + \beta}, X_{-\alpha_0} \in \{+, 0\}$, we have

$$X_{-2\alpha_0 - \beta} \in \{0, +, +, +\} \setminus \{+, +, +, +\}$$

which implies that $X_{-\alpha_0} = +$.

Putting (i) and (ii) together we conclude that $X_{-\alpha_0} = +$. 
4. The sign types $X'$ and $X''$ for $X \in \mathcal{S}$

Here we shall give three key lemmas for the proof of Theorem 2.1. We assume that rank $\Phi > 2$ in this section.

**Lemma 4.1.** Assume that $X = (X_\alpha)_{\alpha \in \Phi} \in \mathcal{S}$ is dominant and assume that not all $X_\alpha$, $\alpha \in \Phi$, are zero. Then we have $X_{-\alpha_0} = \pm$ by Lemma 3.2. Let $X' = (X'_\alpha)_{\alpha \in \Phi}$ be in $\mathcal{S}$ satisfying $X'_\alpha = X_{\alpha_0} \epsilon_0$ for any $\alpha \in \Phi^+$. Let $X'' = (X''_\alpha)_{\alpha \in \Phi}$ be obtained from $X'$ by replacing $X''_{\alpha_0} = \epsilon$ by $X''_{\alpha_0} = \circ, \epsilon = \pm$. Then $\{X', X''\} \not\in \mathcal{S} \neq \emptyset$.

**Sketch of the proof.** We must show either that $(X'_\alpha)_{\alpha \in \Phi^+}$ are admissible for all indecomposable positive subsystems $\Phi^+$ of $\Phi$ of rank 2, or that $(X''_\alpha)_{\alpha \in \Phi^+}$ are admissible for all these subsystems $\Phi^+$ of $\Phi$. To do this, we need only show that if $\Phi^{'+}, \Phi^{''+}$ are two indecomposable positive subsystems of $\Phi$ of rank 2 then either $(X'_\alpha)_{\alpha \in \Phi^+}$ or $(X''_\alpha)_{\alpha \in \Phi^+}$ must be admissible. If $-\alpha_0 \notin \Phi^{'+} \cap \Phi^{''+}$ then either

$$(X'_\alpha)_{\alpha \in \Phi^{'+}} = (X''_\alpha)_{\alpha \in \Phi^{''+}}$$

with $-\alpha_0 \notin \Phi^{'+}$ or

$$(X''_\alpha)_{\alpha \in \Phi^{'+}} = (X'_\alpha)_{\alpha \in \Phi^{''+}}$$

with $-\alpha_0 \notin \Phi^{''+}$. So our result follows by the assumption that $X \in \mathcal{S}$. If $-\alpha_0 \in \Phi^{'+} \cap \Phi^{''+}$ then we have one of the following cases.

(a) $\Phi^{'+} = \Phi^{''+}$ and both have type $A_2$ or $B_2$.

(b) $\Phi^{'+} \neq \Phi^{''+}$ and they are both in some indecomposable positive subsystem of $\Phi$ of type $A_3, B_3$ or $C_3$.

We can verify our result case by case. For example, in case (a) with $\Phi^{'+}$ of type $A_2$, we assume that $(X'_\alpha)_{\alpha \in \Phi} \notin \mathcal{S}(\Phi')$. Say $\Phi^{'+} = \{\alpha, \beta, \alpha + \beta\}$. Then $-\alpha_0 = \alpha + \beta$. Since $X \in \mathcal{S}$ is dominant and $X_{-\alpha_0} = \pm$, we have

$$X_{\alpha + \beta} \in \{+ + + + + \}$$

Thus

$$X''_{\alpha + \beta} \in \{- - - - - \}$$

By the assumption of $(X'_\alpha)_{\alpha \in \Phi} \notin \mathcal{S}(\Phi')$, we get

$$X'_{\alpha + \beta} = \circ \circ$$
But then

is admissible.

**Lemma 4.2.** Assume that \( X = (X_\alpha)_{\alpha \in \Phi} \in \mathcal{S} \) and \( X_\beta = - \) for some \( \beta \in \Pi \). Let \( X' = (X'_\alpha)_{\alpha \in \Phi} \) be such that \( X'_\alpha = X_\alpha \delta_\beta \) for all \( \alpha \in \Phi \). Let \( X'' = (X''_\alpha)_{\alpha \in \Phi} \) be obtained from \( X' \) by replacing \( X'_{\epsilon \beta} = \epsilon \) by \( X''_{\epsilon \beta} = \circ \), \( \epsilon = \pm \). Then either \( X' \) or \( X'' \) (or both) must be in \( \mathcal{S} \).

The strategy of the proof for this lemma is similar to that for Lemma 4.1 but is more complicated. We omit the detail.

**Lemma 4.3.** Assume that \( X = (X_\alpha)_{\alpha \in \Phi} \in \mathcal{S} \) and \( m_X = 1 \). Then by Lemma 3.1, we have \( X_\beta = - \) for some \( \beta \in \Pi \). Let \( X'' = (X''_\alpha)_{\alpha \in \Phi} \) be a sign type of \( \mathcal{S} \) satisfying

\[
X''_\alpha = \begin{cases} 
X_\alpha \delta_\beta & \text{if } \alpha \neq \beta, \\
\circ & \text{if } \alpha = \beta,
\end{cases}
\]

for any \( \alpha \in \Phi^+ \). Then \( X'' \in \mathcal{S} \).

**Proof.** Let \( \Phi'^+ \) be any positive subsystem of \( \Phi \) of rank 2. We must show that \( (X''_\alpha)_{\alpha \in \Phi'^+} \) is admissible.

If \( \beta \notin \Phi'^+ \) then \( (X'_\alpha)_{\alpha \in \Phi'^+} = (X''_\alpha)_{\alpha \in (\Phi'^+)} \delta_\beta \). Since \( (\Phi'^+) \delta_\beta \) is also a positive subsystem of \( \Phi \) of rank 2, the admissibility of \( (X''_\alpha)_{\alpha \in \Phi'^+} \) follows from \( X \in \mathcal{S} \).

Now assume that \( \beta \in \Phi'^+ \). We know that \( \Phi'^+ \) either has type \( A_2 \) or \( B_2 \). First suppose that \( \Phi'^+ \) has type \( A_2 \) with \( \Phi'^+ = \{\beta, \gamma, \beta + \gamma\} \). By the assumption that \( X \in \mathcal{S} \), \( m_X = 1 \) and \( X_\beta = - \), we have

Thus

which is admissible. Next suppose that \( \Phi'^+ \) has type \( B_2 \) with \( \Phi'^+ = \{\beta, \gamma, \beta + \gamma, \beta + 2\gamma\} \). By the same reasoning as above, we have

\[
\epsilon \left\{ \begin{array}{c}
- - - - \\
\circ \circ \circ \circ + + \\
\circ \circ \circ \circ + + \\
\circ \circ \circ \circ + + \\
\circ \circ \circ \circ + + \\
\circ \circ \circ \circ + + \\
\end{array} \right. 
\]
So

\[
\begin{array}{c}
\begin{array}{c}
X''_\beta \\
X''_{\beta+\gamma} \\
X''_{\beta+2\gamma}
\end{array}
\end{array}
\in \left\{ \begin{array}{c}
\begin{array}{cccc}
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & +
\end{array}
\end{array}\right\
\]

which is admissible also. Finally, suppose that \( \Phi^+ \) has type \( B_2 \) with \( \Phi^+ = (\beta, \gamma, \beta+\gamma, 2\beta+\gamma) \). We have

\[
\begin{array}{c}
\begin{array}{c}
X'_{\gamma} \\
X'_{\beta} \\
X'_{\alpha\beta+\gamma}
\end{array}
\end{array}
\in \left\{ \begin{array}{c}
\begin{array}{cccc}
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & +
\end{array}
\end{array}\right\
\]

So

\[
\begin{array}{c}
\begin{array}{c}
X''_\gamma \\
X''_{\beta} \\
X''_{\alpha\beta+\gamma}
\end{array}
\end{array}
\in \left\{ \begin{array}{c}
\begin{array}{cccc}
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & + & +
\end{array}
\end{array}\right\
\]

which is again admissible.

Therefore we have \( X'' \in \mathcal{F} \).

5. The proof of Theorem 2.1

It is known that the result is true when \( \Phi \) has rank 2. So we may assume that rank \( \Phi > 2 \). It is also known that the inclusion \( \zeta(W_\alpha) \subseteq \mathcal{F} \) holds in general. Now we define \( n_X = \# \{ \alpha \in \Phi^+ | X_\alpha \neq O \} \) and \( m_X = \# \{ \alpha \in \Phi^+ | X_\alpha = - \} \) for any \( X \in \mathcal{F} \).

Assume that we are given a sign type \( X = (X_\alpha)_{\alpha \in \Phi} \) in \( \mathcal{F} \). We must find an element \( w \) of \( W_\alpha \) such that \( \zeta(w) = X \). We apply induction on \( n_X \geq 0 \). It is clear that \( \zeta(1) = X \) in the case when \( n_X = 0 \). Now assume that \( n_X > 0 \).

(a) If \( m_X = 1 \) then by Lemma 3.1 there exists some \( \beta \in \Pi \) such that \( X_\beta = - \). Let \( X'' = (X''_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be defined by

\[
X''_\alpha = \begin{cases} 
X_\alpha & \text{if } \alpha \neq \beta, \\
O & \text{if } \alpha = \beta,
\end{cases}
\]

for any \( \alpha \in \Phi^+ \). Then by Lemma 4.3 we have \( X'' \in \mathcal{F} \) with \( n_{X''} < n_X \). By the inductive hypothesis, there exists some \( w' \in \zeta^{-1}(X'') \). Let \( w = w's_\beta \). Then \( \zeta(w) = X \) by Proposition 1.3.

(b) If \( m_X > 1 \) then there exists some \( \beta(1) \in \Pi \) with \( X_{\beta(1)} = - \) by Lemma 3.1. Let \( X' = (X'_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be defined by \( X'_\alpha = X_\alpha \) for any \( \alpha \in \Phi \). Let \( X'' = (X''_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be obtained from \( X' \) by replacing \( X'_\beta(1) = \varepsilon \) by \( X''_\beta(1) = O, \varepsilon = \pm \). Then by Lemma 4.2, one of \( X' \) and \( X'' \) must be in \( \mathcal{F} \). We denote this sign type by \( X(1) \) (note that when both \( X' \) and \( X'' \) are in \( \mathcal{F} \) we can freely choose one of them and call it \( X(1) \)). Clearly \( m_{X(1)} = m_X - 1 \). If \( m_{X(1)} \) is still greater than 1 then the same process can be carried
on and we get a sequence of sign types $X(0) = X, X(1), \ldots, X(m)$ in $\mathcal{S}$ with $m = m_X$ such that for every $i$, $1 \leq i \leq m$, $X(i-1)_{\beta(i)} = -$ with some $\beta(i) \in \Pi$, and either $X(i)_{\alpha} = X(i-1)_{\alpha} s_{\beta(i)}$ for all $\alpha \in \Phi$ or $X(i)_{\alpha} = X(i-1)_{\alpha} s_{\beta(i)}$ for all $\alpha \in \Phi - \{\epsilon \beta(i)\}$ and $X(i)_{\beta(i)} = \emptyset$. In particular, $X(m)_{\beta(m)} = \emptyset$. Since $m_X(i) = m_X(i-1) - 1$, we have $m_X(i) = m_X - i$ and in particular $m_X(m-1) = 1$. Hence such a sequence $X(0), X(1), \ldots, X(m)$ does exist in $\mathcal{S}$. Clearly $n_X(m) < n_X$. Thus by the inductive hypothesis, there exists some $x \in \zeta^{-1}(X(m))$. Let $w = x s_{\beta(m)} s_{\beta(m-1)} \cdots s_{\beta(1)}$. Then by Proposition 1.3, we have $\zeta(w) = X$.

(c) If $m_X = 0$ then $X$ is dominant. Since $n_X > 0$, we see by Lemma 3.2 that $X_{-a_0} = +$. Let $X' = (X'_{\alpha})_{\alpha \in \Phi}$ be in $\mathcal{S}$ satisfying $X'_{\alpha} = X_{(\alpha)} s_0$ for all $\alpha \in \Phi$ and let $X'' = (X''_{\alpha})_{\alpha \in \Phi} \in \mathcal{S}$ be obtained from $X'$ by replacing $X''_{a_0} = \epsilon$ by $X''_{a_0} = \emptyset$. Then Lemma 4.1 asserts that $\{X', X''\} \cap \mathcal{S} \neq \emptyset$. Denote one of $\{X', X''\}$ by $X(1)$. Then either $n_{X(1)} < n_X$, or $n_{X(1)} = n_X$ and $m_{X(1)} > 0$. So either by the inductive hypothesis or by (b) we can find some $x \in \zeta^{-1}(X(1))$. Let $w = x s_0$. Then by Proposition 1.3, we get $\zeta(w) = X$, again.

Therefore our result follows by induction.

6. The geometrical interpretation of admissible sign types

Let $\mathcal{S} = \{H_{\alpha, \tau} | \alpha \in \Phi^+, \tau = 0, 1\}$. Then the connected components of $E - \bigcup_{H \in \mathcal{S}} H$ are open simplices. We see that any alcove of $E$ lies in some connected component of $E - \bigcup_{H \in \mathcal{S}} H$ and that two alcoves correspond to the same sign type if and only if they are in the same connected component of $E - \bigcup_{H \in \mathcal{S}} H$. So by Theorem 2.1, the map $\zeta$ induces a bijection between the set of connected components of $E - \bigcup_{H \in \mathcal{S}} H$ and the set $\mathcal{S}$. Then we can identify these two sets.

Examples 6.1. (1) When $\Phi$ has type $A_2$, say $\Phi^+ = \{\lambda, \mu, \lambda + \mu\}$, the number of connected components of $E - \bigcup_{H \in \mathcal{S}} H$ is 16. Each of these components determines a sign type

$$X = \begin{array}{ccc} X_{\lambda + \mu} & & \\
\Lambda & X_{\lambda} & X_{\mu} \\
& & \\
& & \\
\end{array}$$

as in Figure 1.
(2) When \( \Phi \) has type \( E_7 \), say \( \Phi^+ = \{ \lambda, \mu, \lambda + \mu, \lambda + 2\mu \} \), then \( E - \bigcup_{H \in \mathcal{F}} H \) has 25 connected components each of which determines a sign type as in Figure 2.

\[
X = \begin{array}{c c}
X_{\lambda} & X_{\lambda + \mu} \\
X_{\lambda + 2\mu} & X_{\mu}
\end{array}
\]

as in Figure 2.

(3) Let \( \Phi^+ = \{ \lambda, \mu, \lambda + \mu, \lambda + 2\mu, \lambda + 3\mu, 2\lambda + 3\mu \} \) be the positive system of \( \Phi \) of type \( G_2 \). Then \( E - \bigcup_{H \in \mathcal{F}} H \) has 49 connected components each of which determines a sign type as in Figure 3.

\[
X = \begin{array}{c c}
X_{2\lambda + 3\mu} & X_{\lambda + \mu} \\
X_{\lambda + 2\mu} & X_{\lambda}
\end{array}
\]

as in Figure 3.

It is well known that any alcove of \( E \) has \( l + 1 \) facets. In [4], we labelled any facet of an alcove by an element \( s \in \Delta \) such that the following result holds.

**Lemma 6.2** [4, Lemma 6.1]. If \( w, w' \in W_\alpha \) have the relation \( w' = s_t w \) for some \( s_t \in \Delta \) then the alcoves \( A_w \) and \( A_{w'} \) share the common \( s_t \)-facet. Conversely, if \( A_w \) and \( A_{w'} \) are
two alcoves of $E$ which share a common facet then the labelling of this facet for $A_{w'}$ is the same as for $A_{w''}$, say $s_i$-facet. We have $w' = s_i w$.

The following result is due to the convexity of an admissible sign type.

**Proposition 6.3.** For any $X \in \mathcal{S}$ and $w, y \in \zeta^{-1}(X)$, there exists a sequence of elements $w_0 = w, w_1, \ldots, w_r = y$ in $W_a$ such that for every $h, j$ with $0 \leq h \leq r$ and $1 \leq j \leq r$, $w_h \in \zeta^{-1}(X)$ and $w_j = s_j w_{j-1}$ for some $s_j \in \Delta$.

**Proof.** We see that each connected component of $E - \bigcup_{H \in \mathcal{S}} H$ is convex. Our condition means that $w, y$ are in the same connected component $X$ of $E - \bigcup_{H \in \mathcal{S}} H$. So there exists a sequence of alcoves $A_0 = A_w, A_1, \ldots, A_r = A_y$ in $X$ such that for every $j, 1 \leq j \leq r$, $A_j$ and $A_{j-1}$ share a common wall. Hence our result follows by Lemma 6.2.

Recall that $\mathcal{R}(w) = \{ s \in \Delta \mid ws < w \}$ for any $w \in W_a$. Now we need the following.

**Lemma 6.4** [4, Proposition 4.3(ii)].

$$\mathcal{R}(w) = \{ s_j \in \Delta \mid k(w, \alpha_j) < 0 \}.$$
By the definition of the map \( \zeta \) and Lemma 6.4, the function \( \mathcal{R}(w) \) on \( \zeta^{-1}(X) \) for any \( X \in \mathcal{S} \) is constant. So we can define \( \mathcal{R}(X) = \mathcal{R}(w) \) for any \( w \in \zeta^{-1}(X) \). We see from Lemma 6.4 that \( \mathcal{R}(X) = \{ \sigma \in \Delta \mid X_{\sigma} = - \} \).

7. The shortest elements of \( \zeta^{-1}(X), X \in \mathcal{S} \)

The shortest elements of \( \zeta^{-1}(X), X \in \mathcal{S} \), have very nice properties. They will play a crucial role in the calculation of the cardinality of \( \mathcal{S} \).

**Proposition 7.1.** Let \( U \) be a set of sign types of \( \mathcal{S} \) such that there exists some \( Y \in U \) which can be obtained from any \( Z \in U \) by substituting some non-zero signs by zero signs. Then there exists an element \( y \in \zeta^{-1}(Y) \) such that \( k(y, \alpha) = \min \{ k(x, \alpha) \mid x \in \zeta^{-1}(U) \} \) for all \( \alpha \in \Phi^+ \).

**Proof.** We apply induction on \( l = n_Y \geq 0 \). The result is trivial in the case when \( l = 0 \).

(1) First assume that \( m_Y > 0 \). Then by Lemma 3.1 there must exist some \( \lambda = \eta(1) \in \Pi \) with \( Y_\lambda = - \). Then we also have \( Z_\lambda = - \) for any \( Z \in U \). Let \( Y' = (Y'_\alpha)_{\alpha \in \Phi} = \mathcal{F} \) be defined by \( Y'_\alpha = Y(\alpha) \varepsilon_i \) for all \( \alpha \in \Phi \). Let \( Y'' = (Y''_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be obtained from \( Y' \) by replacing \( Y_{\varepsilon_i} \) with \( Y''_{\varepsilon_i} = 0, \varepsilon = \pm \). We define \( Z', Z'' \in \mathcal{F} \) from any \( Z \in U \) in the same way as \( Y', Y'' \) from \( Y \). Then by our assumption on \( U \) we see that \( Y'' \) can be obtained from \( Z'' \) (respectively \( Z' \)) with \( Z \in U \) by replacing some non-zero signs by zero signs. We also see that \( Y' \) can be obtained from \( Z' \) with \( Z \in U \) by replacing some non-zero signs by zero signs. We can show that if \( Z'' \in \mathcal{S} \) for some \( Z \in U \) then \( Y'' \in \mathcal{S} \). Thus by Lemma 4.2, we have either \( Y'' \in \mathcal{S} \) or \( \{ Z', Z'' \mid Z \in U \} \cap \mathcal{S} = \{ Z' \mid Z \in U \} \).

Let \( U^1 = \{ Z', Z'' \mid Z \in U \} \cap \mathcal{S} \) and let

\[
Y(1) = \begin{cases} Y'' & \text{if } Y'' \in \mathcal{S}, \\ Y' & \text{if } Y'' \notin \mathcal{S}. \end{cases}
\]

Then \( Y(1) \) can be obtained from any sign type of \( U^1 \) by replacing some non-zero signs by zero signs.

If \( m_Y(1) = m_Y - 1 > 0 \) then the same process can be carried on by substituting \( U \) and \( Y(1) \) for \( Y \) and \( Y(1) \) in \( \mathcal{S} \), a sequence of sign types \( Y(0) = Y, Y(1), ..., Y(m) \) in \( \mathcal{S} \) and a sequence of simple roots \( \eta(1), \eta(2), ..., \eta(m) \) in \( \Pi \) with \( m = m_Y \). These roots are such that for every \( i, 1 \leq i \leq m \), and for any \( X = (X_\alpha)_{\alpha \in \Phi} \in U^{i-1} \), the following conditions hold.

(a) \( X(\eta(i)) = - \).

(b) Let \( X' = (X'_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be defined by \( X'_\alpha = X(\alpha) \varepsilon_i \) for all \( \alpha \in \Phi \) and let \( X'' = (X''_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be obtained from \( X' \) by replacing \( X'_{\varepsilon_i} \) by \( X''_{\varepsilon_i} = 0 \). Then \( U^i = \{ X', X'' \mid X \in U^{i-1} \} \cap \mathcal{S} \) and \( Y(i) \in U^i \) is defined to be \( Y(i-1)'' \) if \( Y(i-1)'' \in \mathcal{S} \) or to be \( Y(i-1)' \) otherwise. Then by Lemma 4.2 and the above result we see that for every \( i, 0 \leq i \leq m, Y(i) \) can be obtained from any \( X \in U^i \) by substituting zero signs for some non-zero signs. In particular, we see from Lemma 4.3 that \( Y(m) = Y(m-1)' \) and so \( n_{Y(m)} < n_Y \).

Let \( \mathcal{M} = \{ w \in W_\alpha \mid \zeta(w) \in U^m \} \) and \( \mathcal{Y} = \zeta^{-1}(U) \). Then by Proposition 1.3, the map
\[ \phi: w \mapsto ws_{n(m)}s_{n(m-1)} \cdots s_{n(1)} \] gives a bijection from \( \mathcal{M} \) to \( \mathcal{Y} \) which satisfies \( l(w) + m = l(\phi(w)) \) for any \( w \in \mathcal{M} \).

By the inductive hypothesis, there exists an element \( y \) of \( \zeta^{-1}(Y(m)) \) satisfying \( |k(y, x)| = \min\{|k(x, \alpha)| \mid x \in \mathcal{M} \} \) for all \( \alpha \in \Phi^+ \). By the rule of the right action of \( W_a \) on \( \mathcal{Y} \), we see that for any \( \alpha \in \Phi \), the difference \( |k(\phi(w), \alpha)| - |k(w, \alpha)| \) is a non-negative constant on \( w \in \mathcal{M} \). Thus we have
\[
|k(\phi(y), \alpha)| = \min\{|k(\phi(x), \alpha)| \mid x \in \mathcal{M} \} = \min\{|k(x, \alpha)| \mid x \in \mathcal{Y} \},
\]
for all \( \alpha \in \Phi^+ \). Clearly, \( \phi(y) \in \zeta^{-1}(Y) \). So our result follows in this case.

(2) Next assume that \( m_Y = 0 \). Then by Lemma 3.2, we have \( Y_{-\alpha_0} = + \) and also \( Z_{\alpha_0} = + \) for all \( Z = (Z_\alpha)_{\alpha \in \Phi} \in \mathcal{U} \). Let \( Y = (Y_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be defined by \( Y_\alpha = Y_{(\alpha)} s_\alpha \) for all \( \alpha \in \Phi \). Let \( Y'' = (Y''_\alpha)_{\alpha \in \Phi} \in \mathcal{F} \) be obtained from \( Y \) by replacing \( Y_{\alpha_0} \) by \( Y''_{\alpha_0} = 0 \). We define \( Z', Z'' \in \mathcal{F} \) from any \( Z \in \mathcal{U} \) in the same way as \( Y', Y'' \) were defined from \( Y \). Then \( Y'' \) can be obtained from \( Z' \) (respectively \( Z'' \)) with \( Z \in \mathcal{U} \) by replacing some non-zero signs by zero signs. Also, \( Y'' \) can be obtained from \( Z' \) with \( Z \in \mathcal{U} \) by replacing some non-zero signs by zero signs.

We claim that if \( Z'' \in \mathcal{F} \) for some \( Z \in \mathcal{U} \) then \( Y'' \in \mathcal{F} \), since otherwise, this would imply \( Y'' \notin \mathcal{F} \) which contradicts the fact that \( \{Y', Y''\} \cap \mathcal{F} \neq \emptyset \). Thus we have either \( Y'' \in \mathcal{F} \) or
\[
\{Z', Z'' \mid Z \in \mathcal{U} \} \cap \mathcal{F} = \{Z' \mid Z \in \mathcal{U} \}.
\]
We define
\[
\tilde{\mathcal{Y}} = \begin{cases} 
Y'' & \text{if } Y'' \in \mathcal{F}; \\
Y' & \text{otherwise}.
\end{cases}
\]
Let \( \mathcal{U}' = \{Z', Z'' \mid Z \in \mathcal{U} \} \cap \mathcal{F} \). Then \( \tilde{\mathcal{Y}} \) can be obtained from any \( X \in \mathcal{U}' \) by substituting zero signs for some non-zero signs. We also have \( n_{\tilde{\mathcal{Y}}} \leq n_Y \).

If \( \tilde{\mathcal{Y}} = Y'' \) then \( n_{\tilde{\mathcal{Y}}} < n_Y \). By the inductive hypothesis, we have
\[
|k(y, \alpha)| = \min\{|k(x, \alpha)| \mid x \in \zeta^{-1}(U)\}
\]
for any \( \alpha \in \Phi \) and some \( y \in \zeta^{-1}(\tilde{Y}) \). If \( \tilde{\mathcal{Y}} = Y' \) then \( m_{\tilde{\mathcal{Y}}} > 0 \). By the case which we have discussed with \( Y' \) and \( U' \) instead of \( Y \) and \( U \), we also have
\[
|k(y, \alpha)| = \min\{|k(x, \alpha)| \mid x \in \zeta^{-1}(U')\}
\]
for any \( \alpha \in \Phi \) and some \( y \in \zeta^{-1}(\tilde{Y}) \). Then in either case, \( \phi: w \mapsto ws_0 \) gives a bijection from \( \zeta^{-1}(U') \) to \( \zeta^{-1}(U) \) which satisfies
\[
|k(w, \alpha)| = \begin{cases} 
|k(\phi(w), \alpha)| & \text{if } \alpha \in \Phi^+ \setminus \{-\alpha_0\}, \\
|k(\phi(w), \alpha)| - 1 & \text{if } \alpha = -\alpha_0.
\end{cases}
\]
for all \( w \in \zeta^{-1}(U') \). So
\[
|k(\phi(y), \alpha)| = \min\{|k(\phi(x), \alpha)| \mid x \in \zeta^{-1}(U')\} = \min\{|k(x, \alpha)| \mid x \in \zeta^{-1}(U)\}.
\]
Clearly, \( \phi(y) \) is in \( \zeta^{-1}(Y) \). So our result is also true in this case. By induction, we reach our goal.

The element \( y \in \zeta^{-1}(Y) \) in the above proposition is clearly the shortest element of \( \zeta^{-1}(U) \) which is unique. In particular, when \( U \) consists of a single sign type, we get the following.
PROPOSITION 7.2. For any \( X \in \mathcal{S} \), there exists a unique shortest element, say \( y \), of \( C, \zeta^{-1}(X) \) which is characterized by the requirement that
\[
|k(y, \alpha)| = \min \{|k(x, \alpha)| \mid x \in \zeta^{-1}(X)\}
\]
for all \( \alpha \in \Phi \), where the \( k(x, \alpha) \) are as in Proposition 7.1.

Now we shall give another criterion for an element to be the shortest element of \( \zeta^{-1}(X) \) for any \( X \in \mathcal{S} \).

Let \( w', w \in W_a \) and \( s_j \in \Delta \) satisfy \( w' = s_j w \) and \( l(w') = l(w) - 1 \). Then by the definition of the left action of \( W_a \) on the alcove set \( \mathfrak{U} \) we have \( k(w', \alpha) = k(w, \alpha) \) for all \( \alpha \in \Phi^+ \) but one. Let the exceptional one be \( \beta \in \Phi^+ \). Then we have \( |k(w', \beta)| = |k(w, \beta)| - 1 \) and \( k(w', \beta) = k(w, \beta) \pm 1 \). Now assume that \( w \) is the shortest element of \( \zeta^{-1}(\zeta(w)) \). Then by Proposition 7.2, we must have \( k(w, \beta) = \pm 1 \) and \( k(w', \beta) = 0 \). In particular, \( w' \notin \zeta^{-1}(\zeta(w)) \).

PROPOSITION 7.3. Let \( X \in \mathcal{S} \) and \( w \in \zeta^{-1}(X) \). Then \( w \) is the shortest element of \( \zeta^{-1}(X) \) if and only if, for any \( s \in \mathcal{L}(w) \), we have \( sw \notin \zeta^{-1}(X) \).

Proof. Let \( w \) be the shortest element of \( \zeta^{-1}(X) \). By the above discussion, it is sufficient to show that if \( y \in \zeta^{-1}(X) \) with \( y \neq w \) then there must exist some \( s \in \mathcal{L}(y) \) such that \( sy \in \zeta^{-1}(X) \). Now assume that \( y \in \zeta^{-1}(X) \) with \( y \neq w \). Then by Proposition 7.2, we have, for any \( \alpha \in \Phi^+ \),
\[
k(y, \alpha) \begin{cases} 
  \geq k(w, \alpha) & \text{if } X_\alpha = +, \\
  \leq k(w, \alpha) & \text{if } X_\alpha = -, \\
  = k(w, \alpha) & \text{if } X_\alpha = \emptyset,
\end{cases}
\]
and the set \( D = \{ \alpha \in \Phi^+ | k(y, \alpha) \neq k(w, \alpha) \} \) is non-empty. Let \( D^+ = \{ \alpha \in D | X_\alpha = + \} \) and \( D^- = \{ \alpha \in D | X_\alpha = - \} \).

Set \( a = k(y, \alpha) + 1 - k(w, \alpha) \) and \( b = 2 - a \). Let
\[
K_1 = \bigcap_{\alpha \in \Phi^+ \setminus D^-} H^a_{\alpha; k(w, \alpha)}, \quad K_2 = \bigcap_{\alpha \in D^+} H^a_{\alpha; k(w, \alpha)}, \quad K_3 = \bigcap_{\alpha \in D^-} H^b_{\alpha; k(y, \alpha)}
\]
with the convention that \( K_1 = E \) if the set of indices for the corresponding intersection is empty. Let \( K = K_1 \cap K_2 \cap K_3 \). We have \( A_w \cup A_y \subset K \subset X \) regarded as sets of vectors of \( E \). On the other hand, we see that for any alcove \( A \in \mathfrak{U} \), either \( A \subset K \) or \( A \cap K = \emptyset \). So \( K \) can be regarded as a set of all elements \( x \) of \( W_a \) with \( A_x \subset K \). Thus \( w \) (respectively \( y \)) is the shortest (respectively longest) element in \( K \). Since \( K \) is a convex set of \( E \), which contains more than one alcove of \( E \), there must exist some alcove \( A_x \) in \( K \) with \( x \neq y \) such that \( A_x \) and \( A_y \) share a common facet. That is, \( x = sy \) for some \( s \in \Delta \) by Lemma 6.2. Clearly, we have \( s \in \mathcal{L}(y) \) and \( x \in \zeta^{-1}(X) \). Thus our result follows.

8. The cardinality of \( \mathcal{S}(\Phi) \)

In this section, our aim is to prove Carter’s conjecture.

THEOREM 8.1 (Carter’s conjecture). \(|\mathcal{S}(\Phi)| = (h + 1)^l\), where \( l = \operatorname{rank} \Phi \) and \( h \) is the Coxeter number of \( \Phi \).

To prove our result, we need an earlier result.
Proposition 8.2 [4, Proposition 3.4]. Let \( w \in W_a \). Then for any \( \alpha \in \Phi \), \( k(w^{-1}, \alpha) = k(w, -\alpha) \).

Let
\[
E(S) = \{ w \in W_a \mid w \text{ is the shortest element of } \zeta^{-1}(\zeta(w)) \}.
\]

Let \( E(S)^{-1} = \{ w \mid w^{-1} \in E(S) \} \). For any \( w \in W_a \), we see from Propositions 8.2, 7.2 and 7.3 that \( w \in E(S)^{-1} \) if and only if \( k(w, \lambda) = -1 \) for any \( s_i \in \mathcal{R}(w) \) and this is the case if and only if \( k(w, \lambda) \geq -1 \) for any \( s_i \in \Delta \), where we assume that \( A_w = \bigcap_{\alpha \in \Phi} \mathcal{H}_\alpha^{+}; k(w, \alpha) \).

We define
\[
H_\alpha^{+}; k = \{ v \in E \mid \langle v, \alpha^\vee \rangle < k \}, \quad H_\alpha^{-}; k = \{ v \in E \mid \langle v, \alpha^\vee \rangle < k \}
\]
for any \( \alpha \in \Phi^+ \) and \( k \in \mathbb{Z} \). Then regarded as a set of alcoves of \( E \), \( E(S)^{-1} \) is the set of all alcoves of \( E \) contained in \( H = (\bigcap_{\alpha \in \Phi} \mathcal{H}_\alpha^{+}; -1) \cap H_{-\alpha_2; -2} \).

Define \( Z \subset W_a \) to be a left connected set of \( W_a \) if for any \( x, y \in Z \), there exists a sequence of elements \( x_0 = x, x_1, \ldots, x_r = y \) in \( Z \) such that for every \( i, 1 \leq i \leq r, x_{j-1} x_j^{-1} \in \Delta \). Then by the convexity of \( H \) we see that \( E(S)^{-1} \) is a left connected set of \( W_a \).

By the way, any admissible sign type, regarded as a set of elements of \( W_a \), is a left connected set of \( W_a \) by the convexity of the sign type in \( E \).

Examples 8.3. When the rank of \( \Phi \) is 2, \( E(S)^{-1} \) is the set of all alcoves in the fully shaded area of each of Figures 4, 5 and 6. From these figures, we see that

\[
H = \mathcal{H}_{\alpha_4; -1} \cap \mathcal{H}_{\alpha_4; -1} \cap \mathcal{H}_{-\alpha_2; -2} \text{ are all triangles (the fully shaded areas) similar to the corresponding alcoves } A_1 \text{ (the alcoves labelled by 1). The scale of } A_1 \text{ to } H = 1: h+1 \text{ and so the area of } H \text{ is } (h+1)^{\dim E} \text{ times that of } A_1, \text{ where } h \text{ is the Coxeter number of } \Phi. \text{ Then } H \text{ contains } (h+1)^{\dim E} \text{ alcoves altogether.}
\]

Recall from §1 that \( H_\alpha^m; k = \{ v \in E \mid k < \langle v, \alpha^\vee \rangle < k+m \} \) for any \( \alpha \in \Phi^+, k \in \mathbb{Z} \) and \( m > 0 \) in \( \mathbb{R} \).
FIG. 5. Type $B_2$

FIG. 6. Type $G_2$
Lemma 8.4. Let $H^{h+1} = (\bigcap_{\alpha \in \Pi} H^{h+1}_{\alpha} \cap H_{\alpha}^{-1} \cap H_{\alpha}^{-h})$. Then $H = H^{h+1}$, where $(-\alpha_0)^{\vee} = \sum_{\alpha \in \Pi} c_\alpha \alpha^{\vee}$.

Proof. Since $H^{h+1}_{\alpha} \cap H_{\alpha}^{-1}$ for all $\alpha \in \Pi$, and $H_{\alpha}^{-1} \cap H_{\alpha}^{-h} \cap H_{\alpha}^{-1} \subset H_{\alpha}^{-h}$, the inclusion $H \supseteq H^{h+1}$ is obvious.

Now let $v \in H$. To prove that $v \in H^{h+1}$, it is enough to show that
\[ \langle v, \alpha^{\vee} \rangle < (h+1)/c_\alpha - 1 \]
for $\alpha \in \Pi$ and that $\langle v, (-\alpha_0)^{\vee} \rangle > 1 - h$.

For $\alpha \in \Pi$, we have $\alpha^{\vee} = (-\alpha_0)^{\vee} - \sum_{\alpha \neq \beta \in \Pi} c_\beta \beta^{\vee} / c_\alpha$. So
\[
\langle v, \alpha^{\vee} \rangle = (\langle v, (-\alpha_0)^{\vee} \rangle - \sum_{\alpha \neq \beta \in \Pi} c_\beta \langle v, \beta^{\vee} \rangle) / c_\alpha < (2 + h - 1 - c_\alpha)/c_\alpha
\]
\[ = (h+1)/c_\alpha - 1. \]

We also have
\[
\langle v, (-\alpha_0)^{\vee} \rangle = \langle v, \sum_{\alpha \in \Pi} c_\alpha \alpha^{\vee} \rangle = \sum_{\alpha \in \Pi} c_\alpha \langle v, \alpha^{\vee} \rangle > - \sum_{\alpha \in \Pi} c_\alpha = 1 - h.
\]
So $H \supseteq H^{h+1}$ and hence $H = H^{h+1}$.

Recall that $\Pi = \Pi \cup \{-\alpha_0\}$.

Lemma 8.5. Let $m > 0$ be an integer and let the $\Pi$-tuple $k = (k_\alpha)_{\alpha \in \Pi}$ over $\mathbb{Z}$ satisfy the condition that $k_{-x_0} = \sum_{\alpha \in \Pi} c_\alpha k_\alpha$. Let
\[
H^m_k = (\bigcap_{\alpha \in \Pi} H^m_{\alpha, k_\alpha}) \cap H^m_{\alpha} \cap H^m_{-\alpha_0, k_{-x_0}}.
\]
Then $H^m_k$ contains exactly $m \dim E$ alcoves of $E$.

Proof. We have $A_1 = H^m_{k_0}$ by Lemma 1.1, where $k_0 = (k_\alpha)_{\alpha \in \Pi}$, with $k_\alpha = 0$ for all $\alpha$, satisfies the condition $k_{-x_0} = \sum_{\alpha \in \Pi} c_\alpha k_\alpha$. Then for any integer $m > 0$, $H^m_{k_0}$ is similar to $A_1$ in geometrical shape and the scale of $A_1$ is one $m$th part of that of $H^m_{k_0}$. So the volume of $H^m_{k_0}$ is $m \dim E$ times that of $A_1$. This implies that $H^m_{k_0}$ contains exactly $m \dim E$ alcoves of $E$. Now we take any other $\Pi$-tuple $k = (k_\alpha)_{\alpha \in \Pi}$ with $k_{-x_0} = \sum_{\alpha \in \Pi} c_\alpha k_\alpha$. Then there exists a unique vector $v \in E$ satisfying $\langle v, \alpha^{\vee} \rangle = k_\alpha$ for all $\alpha \in \Pi$ and $\langle v, (-\alpha_0)^{\vee} \rangle = k_{-x_0}$. Let $T_v$ be the translation on $E$ which sends the origin to $v$. Then $T_v$ also sends $H^m_{k_0}$ to $H^m_k$. Hence $H^m_k$ contains $m \dim E$ alcoves of $E$ by the condition that the $k_\alpha, \alpha \in \Pi$, are all integers.

Corollary 8.6. Let $H^{h+1}$ be defined as in Lemma 8.5. Then $H^{h+1}$ contains exactly $(h+1)^l$ alcoves of $E$, where $h$ is the Coxeter number of $\Phi$ and $l$ is the rank of $\Phi$.

Proof. Let $k = (k_\alpha)_{\alpha \in \Pi}$ with $k_\alpha = -1$ for all $\alpha \in \Pi$ and $k_{-x_0} = 1 - h$. Then $H^{h+1} = H^{h+1}_k$. Since $\dim E = l$, the result follows by Lemma 8.5.

Proof of Theorem 8.1. By Proposition 7.2, it is enough to show that $|E(S)| = (h+1)^l$ or, equivalently, to show that $|E(S)^{-1}| = (h+1)^l$. But this follows by Lemma 8.5 and Corollary 8.6.
References


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