ALCOVES CORRESPONDING TO AN AFFINE WEA YL GROUP

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ABSTRACT

In this paper, I study the alcoves of a Euclidean space $E$ corresponding to an affine Weyl group $W_a$. I give the coordinate form of an alcove of $E$ and establish an explicit correspondence between the elements of $W_a$ and the alcoves of $E$. In particular, I characterize an alcove by a $\Phi$-tuple over $\mathbb{Z}$ subject to certain conditions, where $\Phi$ is the root system determined by $W_a$.

In [3], I gave the coordinate form of alcoves in the Euclidean space $E$ spanned by a root system $\Phi$ of type $A_n$; these alcoves are in 1–1 correspondence with the elements of the affine Weyl group $W_a$ of type $A_n$. The coordinate form of an alcove of $E$ is a $\Phi$-tuple over $\mathbb{Z}$ subject to certain conditions. I gave necessary and sufficient conditions for a $\Phi$-tuple over $\mathbb{Z}$ to be the coordinate form of some alcove of $E$.

In the present paper, I shall generalize the above results on $\Phi$ from type $A$ to an arbitrary type, provided that $\Phi$ is indecomposable. Our main results are Theorems 3.3 and 5.2.

1. The alcoves of $E$

Let $\Phi$ be an indecomposable reduced root system. Let $E$ be the Euclidean space spanned by $\Phi$ with positive definite inner product $\langle \cdot, \cdot \rangle$ such that $|\alpha|^2 = \langle \alpha, \alpha \rangle = 1$ for any short root $\alpha$ of $\Phi$. Choose a simple root system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of $\Phi$. Then $\Phi^+, \Phi^-$ are the corresponding positive and negative root systems of $\Phi$. Define the fundamental weights $x_1, \ldots, x_l$ by $\langle \alpha_i, x_j \rangle = \delta_{ij}$ (the Kronecker delta), where for any $\alpha \in \Phi$, $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is called the coroot of $\alpha$. Let $-\alpha_0$ be the highest short root of $\Phi$. Then the set $\{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ has the property that $\langle \alpha_i, \alpha_j^\vee \rangle$ consists of non-positive integers for all pairs of distinct $i, j$ in $\{0, 1, \ldots, l\}$. The set $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ of coroots is again a root system such that the set $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ affords a choice of a simple root system in it. The root $(-\alpha_0)^\vee$ is the highest (co)root of $\Phi^\vee$. Let $h$ be the Coxeter number of $\Phi$. Then $h$ is also the Coxeter number of $\Phi^\vee$.

Let $W$ be the Weyl group of $\Phi$ generated by the reflections $s_\alpha$ on $E$ for $\alpha \in \Phi$, where $s_\alpha$ sends $x$ to $x - \langle x, \alpha^\vee \rangle \alpha$. Let $Q$ denote the root lattice $\mathbb{Z}\Phi$. Let $N$ denote the group consisting of all translations $T_\lambda$ operating on $E$ for $\lambda \in Q$, where $T_\lambda$ sends $x$ to $x + \lambda$. We denote by $W_a$ the group $NW$ of affine transformations of $E$ generated by $N$ and $W$. It is well known that $W_a$ is the semidirect extension of $W$ by the normal subgroup $N$ on which the action of $W$ is known.

For linear and affine transformations, we shall denote the operation on the right and compose them accordingly. With this convention, we define $s_0 = s_{\alpha_0}T_{-\alpha_0}, s_i = s_{\alpha_i}, 1 \leq i \leq l$. It is known that $W_a$ (respectively $W$) is a Coxeter group on generators $s_0, s_1, \ldots, s_l$ (respectively $s_1, \ldots, s_l$). We write $\Delta = \{s_0, s_1, \ldots, s_l\}$. The group $W_a$ will be called an affine Weyl group [1, 4].
Any \( w \in W_a \) can be written (being not necessarily unique) as a product of these generators. We define the length \( l(w) \) of \( w \) to be the smallest number \( r \) such that there exists an expression \( w = s_{i_1} s_{i_2} \ldots s_{i_r} \) with \( s_{i_j} \in \Delta \). An expression of \( w \) is called a reduced form if it is a product of \( l(w) \) generators.

The Bruhat order \( \leq \) on \( W_a \) is a partial order of \( W_a \) which is defined as follows. Say \( y \preceq w \) in \( W_a \) if there are two reduced forms \( w = s_{i_1} s_{i_2} \ldots s_{i_r} \) and \( y = s_{j_1} s_{j_2} \ldots s_{j_t} \) such that \( j_1, j_2, \ldots, j_t \) is a subsequence of \( i_1, i_2, \ldots, i_r \) [4].

Given any two sets \( S, R \), we call \( x = (x_i)_{i \in R} \) an \( R \)-tuple over \( S \) if \( x_i \in S \) for all \( i \in R \). Sometimes we simply call \( x \) an \( R \)-tuple when there is no danger of confusion. Two \( R \)-tuples \( x = (x_i)_{i \in R} \) and \( y = (y_i)_{i \in R} \) are said to be equal if \( x_i = y_i \) for all \( i \in R \).

For any \( \alpha \in \Phi^+ \), \( k \in \mathbb{Z} \) and a positive real number \( m \), we define a hyperplane

\[
H_{\alpha; k} = \{ v \in E | \langle v, \alpha^\vee \rangle = k \}
\]

and a stripe

\[
H^m_{\alpha; k} = H^{-m}_{\alpha; -k} = \{ v \in E | k < \langle v, \alpha^\vee \rangle < k + m \}.
\]

We call any non-empty connected simplex of

\[
E - \bigcup_{\alpha \in \Phi^+ \atop k \in \mathbb{Z}} H_{\alpha; k}
\]

an alcove of \( E \). Each alcove of \( E \) has the form \( \bigcap_{\alpha \in \Phi^+} H_{\alpha; k_\alpha} \) for a \( \Phi^+ \)-tuple \( (k_\alpha)_{\alpha \in \Phi^+} \) over \( \mathbb{Z} \). Since \( H_{\alpha; -k_\alpha} = H_{\alpha; k_\alpha} \), sometimes it is more convenient to denote the alcove \( \bigcap_{\alpha \in \Phi^+} H_{\alpha; k_\alpha} \) by \( \bigcap_{\alpha \in \Phi^+} H_{\alpha; k_\alpha} \) with the convention that \( k_{-\alpha} = -k_\alpha \) for \( \alpha \in \Phi^+ \).

Let \( (-\alpha_0)^\vee = \sum_{i=1}^l c_i \alpha_i^\vee \). Then \( c_i, 1 \leq i \leq l \), are all positive integers satisfying \( h = \sum_{i=1}^l c_i + 1 \). The following lemma gives an example of an alcove of \( E \) which can be shown directly by definition.

**Lemma 1.1.** Let \( A_1 = \bigcap_{\alpha \in \Phi^+} H_{\alpha; 0}^1 \). Then

(i) \( A_1 \) is an alcove of \( E \),

(ii) \( A_1 \) can also be expressed as the form \( \bigcap_{i=1}^l H_{\alpha_i; 0}^1 \cap H_{-\alpha_0; 0}^1 \),

(iii) \( \{ (1/c_i) \lambda_i : 1 \leq i \leq l; 0 \} \) is the set of vertices of the closure of \( A_1 \) in \( E \),

(iv) \( \{ H_{\alpha_i; 0}^1 : 1 \leq i \leq l ; H_{-\alpha_0; 1} \} \) is the set of facets of \( A_1 \) of codimension \( 1 \) in \( E \).

One should note that not every \( \Phi^+ \)-tuple \( (k_\alpha)_{\alpha \in \Phi^+} \) gives rise to an alcove of \( E \) as above. The following lemma gives a necessary condition on a \( \Phi^+ \)-tuple \( (k_\alpha)_{\alpha \in \Phi^+} \) over \( \mathbb{Z} \) such that \( \bigcap_{\alpha \in \Phi^+} H_{\alpha; k_\alpha} \) is an alcove. Later, we shall show that this is also a sufficient condition.

**Lemma 1.2.** Suppose that \( A_k = \bigcap_{\alpha \in \Phi^+} H_{\alpha; k_\alpha} \) is an alcove of \( E \). Then for any \( \alpha, \beta \in \Phi^+ \) with \( \alpha + \beta \in \Phi^+ \), we have

\[
|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leq |\alpha + \beta|^2 (k_{\alpha + \beta} + 1) \leq |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha + \beta|^2 - 1.
\]

**Proof.** Let \( v \in A_k \). Then \( k_\alpha < \langle v, \alpha^\vee \rangle < k_\alpha + 1 \), \( k_\beta < \langle v, \beta^\vee \rangle < k_\beta + 1 \) and \( k_{\alpha + \beta} < \langle v, (\alpha + \beta)^\vee \rangle < k_{\alpha + \beta} + 1 \). Hence

\[
|\alpha|^2 k_\alpha < 2\langle v, \alpha \rangle < |\alpha|^2 k_\alpha + |\alpha|^2, \quad |\beta|^2 k_\beta < 2\langle v, \beta \rangle < |\beta|^2 k_\beta + |\beta|^2
\]

and

\[
|\alpha + \beta|^2 k_{\alpha + \beta} < 2\langle v, \alpha + \beta \rangle < |\alpha + \beta|^2 k_{\alpha + \beta} + |\alpha + \beta|^2.
\]
This implies that

$$|\alpha|^2k_\alpha + |\beta|^2k_\beta < 2\langle \nu, \alpha \rangle + 2\langle \nu, \beta \rangle = 2\langle \nu, \alpha + \beta \rangle < |\alpha + \beta|^2(k_{\alpha + \beta} + 1)$$

and

$$|\alpha + \beta|^2k_{\alpha + \beta} < 2\langle \nu, \alpha + \beta \rangle = 2\langle \nu, \alpha \rangle + 2\langle \nu, \beta \rangle < |\alpha|^2k_\alpha + |\beta|^2k_\beta + |\alpha|^2 + |\beta|^2.$$  

So our conclusion follows immediately.

**Lemma 1.3.** Let $A_k = \bigcap_{\alpha \in \Phi^+} H_{\alpha; k_{\alpha}}$ be an alcove of $E$ satisfying $k_\alpha \geq 0$ for all $\alpha \in \Pi$. Then

(i) $k_\alpha \geq 0$ for all $\alpha \in \Phi^+$.

(ii) If there exists some $\gamma \in \Phi^+$ with $k_\gamma > 0$ then $k_{-\alpha_0} > 0$.

**Proof.** Take any $\nu \in A_k$. For any $\beta \in \Phi^+$, $k_\beta \geq 0$ if and only if $\langle \nu, \beta^\vee \rangle > 0$. Given any $\alpha \in \Phi^+$, we can write $\alpha^\vee = \sum_{i=1}^l a_i \alpha_i^\vee$ with each $a_i$ a non-negative integer, not all zero. By our condition, we have $\langle \nu, \alpha^\vee \rangle = \sum_{i=1}^l a_i \langle \nu, \alpha_i^\vee \rangle > 0$. So $k_\alpha > 0$ and (i) follows. For any $\beta \in \Phi^+$, $k_\beta > 0$ if and only if $\langle \nu, \beta^\vee \rangle > 1$. Thus we have $\langle \nu, \gamma^\vee \rangle > 1$. Since $(-\alpha_0)^\vee$ is the highest coroot of $\Phi^\vee$, we can write $(-\alpha_0)^\vee = \gamma^\vee + \sum_{i=1}^l a_i \alpha_i^\vee$ with the $a_i$ non-negative integers. By our condition, we get $\langle \nu, (-\alpha_0)^\vee \rangle > 1$. This implies that $k_{-\alpha_0} > 0$.

**Corollary 1.4.** Let $A_k = \bigcap_{\alpha \in \Phi^+} H_{\alpha; k_{\alpha}}$ be an alcove of $E$ with $k_\beta < 0$ for some $\beta \in \Phi^+$. Then there exists some $\gamma \in \Pi$ satisfying $k_\gamma < 0$.

**Proof.** This follows immediately from Lemma 1.3 (i).

**Examples 1.5.** The alcoves corresponding to the root systems $\Phi$ of types $A_2$, $B_2$ and $G_2$ are as in the following diagrams, where each small triangle in these diagrams represents an alcove. We label each alcove by its coordinate form. That is, let $\Delta$ be an alcove $\bigcap_{\alpha \in \Phi^+} H_{\alpha; k_{\alpha}}$. Then when $\Phi$ has type $A_2$, say $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$, we put

$$k_{\alpha + \beta} \quad k_\alpha \quad k_\beta$$

into this triangle. When $\Phi$ has type $B_2$, say $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$, we put

$$k_\beta \quad k_{\alpha + \beta} \quad k_\alpha$$

$$k_{2\alpha + \beta}$$

into this triangle. When $\Phi$ has type $G_2$, say $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$, we put

$$k_{3\alpha + 2\beta} \quad k_{\alpha + \beta}$$

$$k_{2\alpha + \beta} \quad k_\beta$$

$$k_{3\alpha + \beta} \quad k_\alpha$$

into this triangle.
2. Some properties of root systems

We shall study some properties of root systems which will be used later.

We say that a subset of $\Phi^+$ is a positive subsystem of $\Phi$ if it has the form $\Phi^+ \cap \Phi'$ for some subsystem $\Phi'$ of $\Phi$. We denote such a subset by $\Phi'^+$. We say that $\Phi'^+$ is indecomposable if $\Phi'$ is, and has type $X$ if $\Phi'$ does so.

**Lemma 2.2.** Assume that $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi^+$. Then one of the following cases must occur:

(i) $\alpha, \beta, \alpha + \beta$ have the same length and they span a subsystem of $\Phi$ of type $A_2$;
(ii) \( \alpha \) and \( \beta \) are short roots but \( \alpha + \beta \) is a long one. They span a subsystem of \( \Phi \) of type \( B_2 \) or \( G_2 \);

(iii) \( \alpha \) and \( \beta \) have different lengths and they form a simple root system of the subsystem of \( \Phi \) spanned by \( \alpha, \beta \). In that case, \( \alpha + \beta \) is always a short one.

**Proof.** This can be reduced to the case when \( \Phi \) has rank 2, and the results checked directly.

**Lemma 2.3.** Suppose that both \( \alpha \) and \( (\alpha) s_{a_0} \) are in \( \Phi^+ \). Then \( \alpha = (\alpha) s_{a_0} \).

**Proof.** Without loss of generality we may assume that \( \Phi \) is spanned by \( \alpha, -\alpha_0 \) over \( \mathbb{Z} \). We can then check the result case by case.

**Lemma 2.4.** Assume that \( \alpha, \beta \in \Phi^+ \) with \( \alpha + \beta \in \Phi^+ \) and \( -\alpha_0 \notin \{ \alpha, \beta, \alpha + \beta \} \). Assume that \( (\alpha) s_{a_0}, (\beta) s_{a_0} \in \Phi^- \). Then \( (\alpha + \beta) s_{a_0} \in \Phi^- \) and \( |\alpha + \beta|^2 = 2|\alpha|^2 = 2|\beta|^2 \).

**Proof.** Obviously, \( (\alpha + \beta) s_{a_0} = (\alpha) s_{a_0} + (\beta) s_{a_0} \in \Phi^- \). To show the rest, we may assume without loss of generality that \( \Phi \) is spanned by \( \alpha, \beta, -\alpha_0 \) over \( \mathbb{Z} \). The condition that \( (\alpha) s_{a_0}, (\beta) s_{a_0} \in \Phi^- \) implies that

\[
(\alpha + \beta) s_{a_0} = \alpha + \beta - \langle \alpha, (\alpha_0)^\vee \rangle + \langle \beta, (\alpha_0)^\vee \rangle (\alpha_0) \leq \gamma - 2(\alpha_0)
\]

with \( \gamma \) the highest root of \( \Phi \). Then \( -\alpha_0 \neq \alpha + \beta \) implies that \( \Phi \) has two different lengths of roots and that \( \alpha + \beta \) must be a long root. On the other hand, by Lemma 2.2, \( \alpha \) and \( \beta \) are either both short or both long roots. If they are both long, \( \langle \alpha, (\alpha_0)^\vee \rangle \geq 2 \), \( \langle \beta, (\alpha_0)^\vee \rangle \geq 2 \) and so \( (\alpha + \beta) s_{a_0} \leq \gamma - 4(\alpha_0) \). But there is no root \( \delta \) of \( \Phi \) satisfying \( \delta \leq \gamma - 4(\alpha_0) \). Thus both \( \alpha \) and \( \beta \) must be short roots. Since \( -\alpha_0 \notin \{ \alpha, \beta, \alpha + \beta \} \), it follows that \( \Phi \) cannot have type \( G_2 \). Thus \( |\alpha + \beta|^2 = 2|\alpha|^2 = 2|\beta|^2 \).
Lemma 2.5. Assume that \( \alpha, \beta \in \Phi^+ \) with \( \alpha + \beta \in \Phi^+ \). Assume that \( (\alpha) s_{\alpha_0} \in \Phi^+ \) and \( (\beta) s_{\alpha_0} \in \Phi^- \). Then \( (\alpha + \beta) s_{\alpha_0} \in \Phi^- \) and \( \beta, \alpha + \beta \) have the same length.

Proof. Suppose that \( (\alpha + \beta) s_{\alpha_0} \in \Phi^+ \). Then by Lemma 2.3,
\[
\alpha + \beta = (\alpha + \beta) s_{\alpha_0} = (\alpha) s_{\alpha_0} + (\beta) s_{\alpha_0} = \alpha + (\beta) s_{\alpha_0}.
\]
That is, \( (\beta) s_{\alpha_0} = \beta \in \Phi^+ \) which contradicts our condition. Since \( (\alpha) s_{\alpha_0} = \alpha \), we have \( \langle \alpha, (-\alpha_0) \rangle = 0 \). Thus
\[
\langle \beta, (-\alpha_0) \rangle = \langle \alpha + \beta, (-\alpha_0) \rangle > 0
\]
since \( (\beta) s_{\alpha_0} \in \Phi^- \). This implies that \( \beta, \alpha + \beta \) have the same length.

3. The correspondence between the alcoves of \( E \) and the affine Weyl group \( W_a \)

In this section, we shall establish the correspondence between the alcoves of \( E \) and the elements of the affine Weyl group \( W_a \). The main results of this section are Theorem 3.3 and Proposition 3.4.

It is well known that the right action of \( W_a \) on \( E \) gives rise to the permutations of the set
\[
\{ H_\alpha ; k | \alpha \in \Phi^+, k \in \mathbb{Z} \}.
\]
So it induces the permutations of the set \( \mathcal{A} \) of alcoves of \( E \). It is well known that \( \mathcal{A} \) is simply transitive under \( W_a \). Denote \( A_w = (A_1) w \) for any \( w \in W_a \). Thus any alcove of \( \mathcal{A} \) has the form \( A_w \), written
\[
A_w = \bigcap_{\alpha \in \Phi^+} H_\alpha^1 ; k(w, \alpha) \quad \text{or} \quad A_w = \bigcap_{\alpha \in \Phi} H_\alpha^1 ; k(w, \alpha)
\]
with the convention that \( k(w, -\alpha) = -k(w, \alpha) \) for any \( \alpha \in \Phi^+ \). We shall identify \( W_a \) with \( \mathcal{A} \) as a set under the correspondence \( w \mapsto A_w \). The integers \( k(w, \alpha) \) labelled by \( w \in W_a \) and \( \alpha \in \Phi \) always stand for the coordinates of the alcove \( A_w = \bigcap_{\alpha \in \Phi} H_\alpha^1 ; k(w, \alpha) \).

As \( W_a = W \times N \), any \( w \in W_a \) has a unique decomposition \( w = \bar{w} T_\lambda \) with \( \bar{w} \in W \) and \( \lambda \in Q \). We shall describe the integers \( k(w, \alpha) , \alpha \in \Phi^+ \) in terms of \( \bar{w} \) and \( \lambda \).

Lemma 3.1. For any \( w \in W \) and any \( \alpha \in \Phi^+ \), we have
\[
k(w, \alpha) = \begin{cases} 0 & \text{if } (\alpha) w^{-1} \in \Phi^+ , \\ -1 & \text{if } (\alpha) w^{-1} \in \Phi^- . \end{cases}
\]

Proof. Let \( v \in A_1 \) and \( \alpha \in \Phi^+ \). Then \( (v) w \in A_w \). It is well known that
\[
\langle (v) w, \alpha \rangle = \langle v, (\alpha) w^{-1} \rangle.
\]
If \( (\alpha) w^{-1} \in \Phi^+ \) then \( 0 < \langle v, ((\alpha) w^{-1}) \rangle < 1 \) and hence \( 0 < \langle (v) w, \alpha \rangle < 1 \). Thus \( k(w, \alpha) = 0 \). If \( (\alpha) w^{-1} \in \Phi^- \) then \( -1 < \langle v, ((\alpha) w^{-1}) \rangle < 1 \) and hence \( -1 < \langle (v) w, \alpha \rangle < 0 \). That is, \( -1 < \langle (v) w, \alpha \rangle < 0 \). Then \( k(w, \alpha) = -1 \).

Lemma 3.2. Assume that \( A_k = \bigcap_{\alpha \in \Phi} H_\alpha^1 ; k_\alpha \) is an alcove of \( E \). Let \( \lambda \in Q \). Then \( A_k' = (A_k) T_\lambda \) is also an alcove of \( E \), say \( A_k' = \bigcap_{\alpha \in \Phi} H_\alpha^1 ; k'_\alpha \). Hence for any \( \alpha \in \Phi, k'_\alpha = k_\alpha + \langle \lambda, \alpha \rangle \).
Proof. We know that there exists some \( w \in W_a \) with \( A_w = A_k \). Since \( T_\lambda \in W_a \), \( A_{wT_\lambda} = A_{\omega T_\lambda} \) is clearly an alcove of \( E \). For any \( \alpha \in \Phi^+ \) and \( v \in A_k \), we have \( k_\alpha < \langle v, \alpha^\vee \rangle < k_\alpha + 1 \).

Since \( (v) T_\lambda \in A_k \), this implies that \( k'_\alpha = k_\alpha + \langle \lambda, \alpha^\vee \rangle \). For \( \alpha \in \Phi^- \),

\[
k'_\alpha = -k'_{-\alpha} = -(k_{-\alpha} + \langle \lambda, (-\alpha)^\vee \rangle) = k_\alpha + \langle \lambda, \alpha^\vee \rangle
\]

and the result is proved.

From Lemmas 3.1 and 3.2, we get the following result immediately.

**Theorem 3.3.** For any \( w \in W \) and \( \lambda \in Q \), let \( w = \omega T_\lambda \). Then the equation

\[
k(w, \alpha) = \langle \lambda, \alpha^\vee \rangle + k(w, \alpha)
\]

holds for any \( \alpha \in \Phi \).

By Lemma 3.1 and Theorem 3.3, one can easily show that for any \( j, 0 \leq j \leq l \), and \( \alpha \in \Phi \),

\[
k(s_j, \alpha) = \begin{cases} 0 & \text{if } \alpha \neq \pm \alpha_j, \\ 1 & \text{if } \alpha = -\alpha_j, \\ -1 & \text{if } \alpha = \alpha_j. \end{cases} \tag{3.3.1}
\]

We can deduce the following result from Theorem 3.3.

**Proposition 3.4** Let \( w \in W_a \). Then for any \( \alpha \in \Phi \),

\[
k(w^{-1}, \alpha) = k(w, -\alpha w).
\]

Proof. Write \( w = \omega T_\lambda \) with \( \omega \in W \) and \( \lambda \in Q \). Then \( w^{-1} = \omega^{-1} T_{(-\lambda) \omega^{-1}} \). By Theorem 3.3, we have

\[
k(w, -\alpha w) = \langle \lambda, (-\alpha w)^\vee \rangle + k(\omega, -\alpha w),
\]

\[
k(w^{-1}, \alpha) = \langle (-\lambda) \omega^{-1}, \alpha^\vee \rangle + k(\omega^{-1}, \alpha)
\]

for any \( \alpha \in \Phi \). To show that \( k(w^{-1}, \alpha) = k(w, -\alpha w) \) is equivalent to showing that \( k(w^{-1}, \alpha) = k(\omega, -\alpha w) \). It is enough to show that \( k(\omega^{-1}, \alpha) = k(\omega, -\alpha w) \) for \( \alpha \in \Phi^+ \). If \( -\alpha w \in \Phi^+ \) then \( (\alpha (\omega^{-1}))^{-1} = (\alpha w) \in \Phi^- \) and so \( k(\omega^{-1}, \alpha) = -1 \). Also \( -\alpha w \in \Phi^- \) implies that \( k(\omega^{-1}, \alpha) = 0 \). Also, \( k(\omega, -\alpha w) = -k(\omega, \alpha w) \). But \( (\alpha w) \omega^{-1} = \alpha w \in \Phi^+ \) implies that \( k(\omega, \alpha w) = \alpha w \). This implies that we always have \( k(\omega^{-1}, \alpha) = k(\omega, -\alpha w) \) and the result follows.

**Examples 3.5.** Recall that in Examples 1.5 we drew the diagrams for the alcoves of \( E \) when \( \Phi \) has type \( A_2, B_2 \) or \( G_2 \). We labelled each alcove by the corresponding \( \Phi^+ \)-tuple there. Now we shall label them by the corresponding elements of \( W_a \) instead of \( \Phi^+ \)-tuples. We assume that \( s_1 = s_\alpha \) and \( s_2 = s_\beta \) and denote \( s_i \) by \( i \) for short.
4. The actions of $W_a$ on the alcoves of $E$

Let $w' = s_j w$ with $w \in W_a$ and $0 \leq j \leq l$. We wish to express the $k(w', \alpha)$ in terms of the $k(w, \beta)$.

Write $w = \overline{w} T_\lambda$ with $\overline{w} \in W$ and $\lambda \in Q$. First assume that $1 \leq j \leq l$. Then $w' = s_j \overline{w} T_\lambda$ with $s_j \overline{w} \in W$. By Theorem 3.3,

$$k(w, \alpha) = \langle \lambda, \alpha^\vee \rangle + k(\overline{w}, \alpha)$$

and

$$k(w', \alpha) = k(w, \alpha) + k(s_j \overline{w}, \alpha) - k(\overline{w}, \alpha)$$
for any $\alpha \in \Phi^+$. When $(\alpha)^{-1} \neq \pm \alpha_j$, we have $(\alpha)^{-1} \in \Phi^+$ if and only if $(\alpha)^{-1} s_j \in \Phi^+$. So $k(w, \alpha) = k(s_j, \alpha)$ and thus $k(w, \alpha) = k(w', \alpha)$ in this case. When $(\alpha)^{-1} = \alpha_j$ we have $(\alpha)^{-1} \in \Phi^+$ and $(\alpha)^{-1} s_j \in \Phi^-$. Thus $k(s_j, \alpha) = -1$ and $k(w', \alpha) = 0$ by Lemma 3.1. This implies that $k(w', \alpha) = k(w, \alpha) - 1$.

Next assume that $j = 0$. Then $w' = s_{\alpha_0} w T_{\lambda} (\pm \alpha_0) w$. By Theorem 3.3, we have, for any $\alpha \in \Phi^+$,

$$k(w', \alpha) = k(w, \alpha) + \langle -\alpha_0, ((\alpha)^{-1})^\vee \rangle + k(s_{\alpha_0} w, \alpha) - k(w, \alpha).$$

When $\langle -\alpha_0, ((\alpha)^{-1})^\vee \rangle = 0$, we have $(\alpha)^{-1} \in \Phi^+$ if and only if $(\alpha)^{-1} s_{\alpha_0} \in \Phi^+$. Thus $k(s_{\alpha_0} w, \alpha) = k(\alpha, \alpha)$. So in this case, $k(w', \alpha) = k(w, \alpha)$. When $\langle -\alpha_0, ((\alpha)^{-1})^\vee \rangle > 0$ and $(\alpha)^{-1} \neq \pm \alpha_0$ we have $(\alpha)^{-1} \in \Phi^+$ and $\langle -\alpha_0, ((\alpha)^{-1})^\vee \rangle = 1$ since $-\alpha_0$ is the highest short root of $\Phi$. We also have $(\alpha)^{-1} s_{\alpha_0} \in \Phi^-$. Thus $k(w', \alpha) = k(w, \alpha)$. When $\langle -\alpha_0, ((\alpha)^{-1})^\vee \rangle < 0$ and $(\alpha)^{-1} \neq \pm \alpha_0$, we have $(\alpha)^{-1} \in \Phi^-$ and $\langle -\alpha_0, ((\alpha)^{-1})^\vee \rangle = -1$. We also have $(\alpha)^{-1} s_{\alpha_0} \in \Phi^+$. Thus $k(w', \alpha) = k(w, \alpha)$. When $(\alpha)^{-1} = -\alpha_0$, we get $(\alpha)^{-1} \in \Phi^+$, $(\alpha)^{-1} s_{\alpha_0} \in \Phi^-$ and $\langle -\alpha_0, ((\alpha)^{-1})^\vee \rangle = 2$. Thus $k(w', \alpha) = k(w, \alpha) + 1$.

To sum up, we get the following result, by using (3.3.1).

**Proposition 4.1.** Let $w' = s_j w$ with $w \in W_\alpha$ and $0 \leq j \leq l$. Then for any $\alpha \in \Phi^+$, we have $k(w', \alpha) = k(w, \alpha) + k(s_j, (\alpha)^{-1})$.

Now assume that, $w' = w s_j$ instead of $w' = s_j w$ in the above. We shall find the relations between the $k(w, \alpha)$ and the $k(w', \beta)$.

First assume that $1 \leq j \leq l$. Then $w' = w s_j = w T_{\lambda} s_j = w s_j T_{(\lambda)} s_j$. By Theorem 3.3, $k(w, \alpha) = \langle \lambda, \alpha^\vee \rangle + k(\overline{w}, \alpha)$ and $k(w', \alpha) = \langle \lambda, (\alpha s_j)^\vee \rangle + k(s_j, \alpha)$ for any $\alpha \in \Phi^+$. This implies that

$$k(w', (\alpha) s_j) = k(w, \alpha) + k(\overline{w} s_j, (\alpha) s_j) - k(\overline{w}, \alpha).$$
When \( \alpha \neq \alpha_j \), we have \( (\alpha) s_j \in \Phi^+ \). We also have \( (\alpha) \bar{w}^{-1} \in \Phi^+ \) if and only if \( (\alpha) s_j \bar{w}^{-1} \in \Phi^+ \). So \( k(\bar{w} s_j, (\alpha) s_j) = k(\bar{w}, \alpha) \) and hence \( k(w', (\alpha) s_j) = k(w, \alpha) \) in that case. When \( \alpha = \alpha_j \), we have \( (\alpha_j) s_j = -\alpha_j \in \Phi^- \). Thus
\[
k(w', \alpha_j) = -k(w, \alpha_j) + k(\bar{w}, \alpha_j) + k(\bar{w} s_j, \alpha_j).
\]
Since \( (\alpha_j) \bar{w}^{-1} \in \Phi^+ \) if and only if \( (\alpha_j) s_j \bar{w}^{-1} \in \Phi^- \), we get \( k(\bar{w}, \alpha_j) + k(\bar{w} s_j, \alpha_j) = -1 \) and so \( k(w', \alpha_j) = -k(w, \alpha_j) - 1 \).

Next assume that \( j = 0 \). Then \( w' = w s_0 \). Let \( T_\lambda \) be a translation by a vector \( \lambda \). Then \( w' = w T_\lambda s_{\lambda 0} \). Thus
\[
k(w', \alpha) = \langle \lambda, ((\alpha) s_{\lambda 0}) \rangle + \langle -\alpha_0, \alpha \rangle + k(\bar{w} s_{\lambda 0}, \alpha)
\]
for any \( \alpha \in \Phi^+ \). When \( \langle -\alpha_0, \alpha \rangle = 0 \), we have \( (\alpha) s_{\lambda 0} = \alpha \), and \( (\alpha) \bar{w}^{-1} \in \Phi^+ \) if and only if \( (\alpha) s_{\lambda 0} \bar{w}^{-1} \in \Phi^+ \). In that case, \( k(w', \alpha) = k(w, \alpha) \). When \( \langle -\alpha_0, \alpha \rangle \neq 0 \) and \( \alpha \neq -\alpha_0 \), we have \( \langle -\alpha_0, \alpha \rangle = 1 \) and \( (\alpha) s_{\lambda 0} \in \Phi^- \) by Lemma 2.3. Thus
\[
k(w', -\alpha) s_{\lambda 0} = \langle \lambda, (-\alpha) \rangle + 1 + k(\bar{w} s_{\lambda 0}, -\alpha)\]
Since \( (\alpha) \bar{w}^{-1} \in \Phi^+ \) if and only if \( (-\alpha) s_{\lambda 0} \bar{w}^{-1} \in \Phi^- \), we get
\[
k(\bar{w}, \alpha) + k(\bar{w} s_{\lambda 0}, -\alpha) = -1.
\]
Thus \( k(w', -\alpha) s_{\lambda 0} = -k(w, \alpha) \). When \( \alpha = -\alpha_0 \), we have
\[
k(w', -\alpha_0) = -k(w, -\alpha_0) + 2 + k(\bar{w}, -\alpha_0) + k(\bar{w} s_{\lambda 0}, -\alpha).
\]
Since, \( (-\alpha_0) \bar{w}^{-1} \in \Phi^+ \) if and only if \( (-\alpha_0) s_{\lambda 0} \bar{w}^{-1} \in \Phi^- \), we have
\[
k(\bar{w}, -\alpha_0) + k(\bar{w} s_{\lambda 0}, -\alpha_0) = -1
\]
and hence \( k(w', -\alpha_0) = -k(w, -\alpha_0) + 1 \).

To sum up, we get the following, by using (3.3.1).

**Proposition 4.2.** Let \( w' = w s_j \) with \( w \in W_\alpha \) and \( s_j \in \Delta \). Then for any \( \alpha \in \Phi \), we have
\[
k(w', \alpha) = k(w, (\alpha) s_j) + k(s_j, \alpha).
\]

For any \( w \in W_\alpha \), we associate two subsets of \( \Delta \):
\[
\mathcal{L}(w) = \{ s \in \Delta \mid s w < w \},
\]
\[
\mathcal{R}(w) = \{ s \in \Delta \mid w s < w \}.
\]

Now we can describe an element \( w \) of \( W_\alpha \) in terms of the \( k(w, \alpha) \).

**Proposition 4.3.** Suppose that \( A_w \), [3, Chapter 7].
5. The map $\phi: W_a \to F_0$

In this section, we shall characterize any element of $W_a$ in terms of a $\Phi$-tuple over $\mathbb{Z}$.

Let $F$ be the set of $\Phi$-tuples $(k_a)_{a \in \Phi}$ satisfying $k_a = -k_{-a}$ for any $a \in \Phi$. Let $F_0$ be the subset of $F$ consisting of all $(k_a)_{a \in \Phi}$ such that for any $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi^+$, the inequality

$$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leq |\alpha + \beta|^2 (k_{\alpha + \beta} + 1) \leq |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha + \beta|^2 - 1$$

holds. Any $\Phi$-tuple in $F_0$ is called a special $\Phi$-tuple.

Define a map $\phi: W_a \to F$ sending $A_w = \bigcap_{a \in \Phi} H_{k_a, (k(w, \alpha))_{a \in \Phi}}$ to $(k(w, \alpha))_{a \in \Phi}$. Then from Lemma 1.2, we see that the image of $\phi$ is in $F_0$ and hence $\phi$ can be regarded as a map from $W_a$ to $F_0$. Now we shall show that $\phi$ is bijective. It is obvious that $\phi$ is injective. So it is enough to show that $\phi$ is also surjective.

Define a right action of $W_a$ on $F$ as follows: for $k = (k_a)_{a \in \Phi} \in F$, $s_t \in \Delta$ and $x, y \in W_a$,

(i) $(k) s_t = (k'_a)_{a \in \Phi}$ with $k'_a = k(a) s_t + e_{a, t}$, where

$$e_{a, t} = \begin{cases} 0 & \text{if } \alpha \neq \pm a, \\ -1 & \text{if } \alpha = a, \\ 1 & \text{if } \alpha = -a. \end{cases}$$

(ii) $(k) x y = ((k) x) y$.

One can easily check that this action of $W_a$ on $F$ is well defined. By noting that $k(s_t, \alpha) = e_{a, t}$ for $0 \leq t \leq l$ and $\alpha \in \Phi$, we see that the map $\phi: W_a \to F$ is $W_a$-equivariant and so $\phi(W_a)$ is a $W_a$-orbit of $F$ in $F_0$.

Now we are ready to show the following.

**Proposition 5.1.** The map $\phi: W_a \to F_0$ is bijective.

**Proof.** It is sufficient to show that $F_0$ is a single $W_a$-orbit. Call $k = (k_a)_{a \in \Phi} \in F$ a minimal element if, for any $s \in \Delta$,

$$\sum_{a \in \Phi^+} |k_a| < \sum_{a \in \Phi^+} |k'_a|,$$

where $k' = (k) s = (k'_a)_{a \in \Phi}$. It is clear that $k$ is minimal if and only if $k_{a_t} \geq 0$ for all $t$, with $0 \leq t \leq l$. It is not difficult to show that $F_0$ contains a unique minimal element, namely $\phi(A_l)$. So it suffices to show that if $k' = (k) s_t$, for some $k \in F_0$ and $s \in \Delta$ then $k' \in F_0$; that is, we must show that for any $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi^+$, the inequality

$$|\alpha|^2 k'_\alpha + |\beta|^2 k'_\beta + 1 \leq |\alpha + \beta|^2 (k'_{\alpha + \beta} + 1) \leq |\alpha|^2 k'_\alpha + |\beta|^2 k'_\beta + |\alpha|^2 + |\beta|^2 + |\alpha + \beta|^2 - 1$$

holds, or equivalently, the inequality

$$|\alpha|^2 k_{(a)} s_t + |\beta|^2 k_{(b)} s_t + |\alpha|^2 e_{a, r} + |\beta|^2 e_{b, r} + 1 \leq |\alpha + \beta|^2 (k_{(a + b)} s_t + e_{a + b, r} + 1)$$

$$\leq |\alpha|^2 k_{(a)} s_t + |\beta|^2 k_{(b)} s_t + |\alpha|^2 e_{a, r} + |\beta|^2 e_{b, r} + |\alpha|^2 + |\beta|^2 + |\alpha + \beta|^2 - 1 \quad (1)$$

holds.
When $r \neq 0$ and $\alpha, \neq \alpha, \beta$, the result is obvious. When $r \neq 0$ and $\alpha, \in \{\alpha, \beta\}$, say $\alpha, = \alpha, (1)$ becomes

$$|\alpha|^2 k_\alpha + |(\beta) s_\alpha - \alpha|^2 k_{(\beta) s_\alpha - \alpha} + 1 \leq |(\beta) s_\alpha|^2 (k_{(\beta) s_\alpha - \alpha} + 1)$$

$$\leq |\alpha|^2 k_\alpha + |(\beta) s_\alpha - \alpha|^2 k_{(\beta) s_\alpha - \alpha} + |\alpha|^2 + |(\beta) s_\alpha - \alpha|^2 + |(\beta) s_\alpha|^2 - 1,$$

which holds since $k \in F_0$ and $\alpha, (\beta) s_\alpha, (\beta) s_\alpha - \alpha \in \Phi^+$.  

Now assume that $r = 0$. Then one of the following cases must occur:

(i) $-\alpha_0 = \alpha + \beta$; (ii) $-\alpha_0 \in \{\alpha, \beta\}$; (iii) $-\alpha_0 \notin \{\alpha, \beta, \alpha + \beta\}$.  

First assume that we are in case (i). Then $e_{\alpha, 0} = e_{\beta, 0} = 0$, $e_{\alpha + \beta, 0} = 1$ and $\alpha + \beta$ is a short root. By Lemma 2.2, one of $\alpha, \beta$, say $\alpha$, must be a short root. If $\beta$ is also a short root then (1) becomes

$$|-(\alpha) s_0|^2 k_{-(\alpha) s_0} + |-(\beta) s_0|^2 k_{-(\beta) s_0} + 1 \leq |-(\alpha) s_0|^2 (k_{-(\alpha) s_0} + 1)$$

$$\leq |-(\alpha) s_0|^2 k_{-(\alpha) s_0} + |-(\beta) s_0|^2 k_{-(\beta) s_0} + |-(\alpha) s_0|^2 + |-(\beta) s_0|^2 + |\alpha_0|^2 - 1$$

which holds since $k \in F_0$ and $-(\alpha) s_0, -(\beta) s_0, -\alpha_0 \in \Phi^+$. If $\beta$ is a long root, (1) becomes

$$|\alpha|^2 k_\alpha + |\alpha_0|^2 k_{-\alpha_0} + 1 \leq |-(\beta) s_0|^2 (k_{-(\beta) s_0} + 1)$$

$$\leq |\alpha|^2 k_\alpha + |\alpha_0|^2 k_{-\alpha_0} + |\alpha|^2 + |\alpha_0|^2 + |-(\beta) s_0|^2 - 1$$

which holds because $k \in F_0$ and $-(\beta) s_0 = \alpha + (-\alpha_0)$.  

Next assume that we are in case (ii), say $\alpha = -\alpha_0$. Then $e_{\alpha, 0} = 1$, $e_{\beta, 0} = e_{\alpha + \beta, 0} = 0$. Let $\Phi^+$ be the positive subsystem of $\Phi$ spanned by $\alpha, \beta$ over $\mathbb{Z}$. Then $\Phi^+$ is either of type $B_2$ or $G_2$ and $\alpha$ is the highest short root of $\Phi^+$. By the condition that $\alpha + \beta \in \Phi^+$ and by Lemma 2.2, $\beta$ is a positive short root of $\Phi^+$ and $\alpha + \beta$ is a positive long root of $\Phi^+$. If $\Phi^+$ has type $B_2$ then (1) becomes

$$|\beta|^2 k_\beta + |-(\alpha) - \beta|^2 k_{-(\alpha) - \beta} + 1 \leq |\alpha|^2 (k_{-(\alpha) s_0} + 1)$$

$$\leq |\beta|^2 k_\beta + |-(\alpha) - \beta|^2 k_{-(\alpha) - \beta} + |\beta|^2 + |\alpha|^2 - |\alpha|^2 - 1.$$

This holds because $\beta, -(\alpha) - \beta, -\alpha_0 \in \Phi^+$ and $k \in F_0$. If $\Phi^+$ has type $G_2$ then (1) becomes

$$|\alpha_0|^2 k_{-\alpha_0} + |-(\beta) s_0|^2 k_{-(\beta) s_0} + 1 \leq |-(\beta) s_0|^2 (k_{-(\beta) s_0} + 1)$$

$$\leq |\alpha_0|^2 k_{-\alpha_0} + |-(\beta) s_0|^2 k_{-(\beta) s_0} + |\alpha|^2 + |-(\beta) s_0|^2 + |\alpha_0| - -(\beta) s_0|^2 - 1.$$

This holds because $-\alpha_0, -(\beta) s_0, -\alpha_0 - (\beta) s_0 \in \Phi^+$ and $k \in F_0$. So (1) holds when one of $\alpha, \beta$ is equal to $-\alpha_0$.  

Finally assume that we are in case (iii). Then $e_{\alpha, 0} = e_{\beta, 0} = e_{\alpha + \beta, 0} = 0$ and one of the following cases must occur:

(a) $(\alpha) s_0, (\beta) s_0 \in \Phi^+$; (b) $(\alpha) s_0, (\beta) s_0 \in \Phi^-$;

(c) $(\alpha) s_0 \in \Phi^+, (\beta) s_0 \in \Phi^-$; (d) $(\alpha) s_0 \in \Phi^-, (\beta) s_0 \in \Phi^+$.

One can verify (1) in case (a) by Lemma 2.3, in case (b) by Lemma 2.4 and in cases (c), (d) by Lemma 2.5.  

An immediate consequence of the above proposition is the following.
Theorem 5.2. Let \( A_k = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha; k} \) with \( k_\alpha \in \mathbb{Z} \). Then \( A_k \) is an alcove of \( E \) if and only if for any \( \alpha, \beta \in \Phi^+ \) with \( \alpha + \beta \in \Phi^+ \), the inequality
\[
|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leq (k_\alpha + k_\beta + 1) \leq |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha| + |\beta| - 1
\]
holds.

This theorem characterizes an alcove of \( E \) by a special \( \Phi \)-tuple.

6. The facets of an alcove

We know that each alcove of \( E \) has the form \( (A_i)_w \) for some \( w \in W_a \). So by Lemma 1.1, any alcove of \( E \) has \( l+1 \) facets. We know that the right action of \( W_a \) on \( E \) induces a permutation on the set of facets of all alcoves of \( E \). It is well known that each \( W^- \)-orbit of such facets intersects the closure of any alcove in a unique facet. So we can label any facet of an alcove by an element \( s \in \Delta \) if it is in the \( W^- \)-orbit of facets containing the common facet of \( A_1 \) and \( A_s \).

Lemma 6.1. If \( w, w' \in W_a \) have the relation \( w' = s_t w \) for some \( s_t \in \Delta \) then the alcoves \( A_w \) and \( A_{w'} \) share the common \( s_t \)-facet. Conversely, if \( A_w \) and \( A_{w'} \) are two alcoves of \( E \) which share a common facet then the labelling of this facet for \( A_w \) is the same as for \( A_{w'} \), say \( s_t \)-facet. We have \( w' = s_t w \).

Proof. First assume that \( w' = s_t w \). We have \( A_w = (A_i)_w \), \( A_{w'} = (A_{s_t})_w \), and \( A_s \) and \( A_{s_t} \) share the common \( s_t \)-facet, this implies that \( A_w \) and \( A_{w'} \) share the common \( s_t \)-facet.

Conversely, assume that \( A_w \) and \( A_{w'} \) share a common facet. Then by the definition, the labelling of this facet for \( A_w \) and for \( A_{w'} \) must be the same, say \( s_t \)-facet. Let \( y = s_t w \). Then by the above argument, \( A_y \) and \( A_w \) share the common \( s_t \)-facet. This forces \( A_{s_t} = A_{w'} \) and hence \( w' = s_t w \).

Now we can give another description of the length function \( l(w) \) on \( W_a \) which is a direct consequence of Lemma 6.1.

Corollary 6.2. For any \( w \in W_a \), \( l(w) \) is the minimum number of facets of alcoves of \( E \) which separate the alcove \( A_w \) from \( A_1 \). In other words, \( l(w) \) is the smallest number \( r \) such that there exists a sequence of alcoves \( A_0 = A_w, A_1, \ldots, A_r = A_1 \) where any two consecutive alcoves in this sequence share a common facet.

Recall that in §1 we assumed that
\[
H^1_{\alpha; -k} = H^1_{\alpha; k} = \{v \in E | -k < \langle v, \alpha \rangle < -k+1 \} \quad \text{for} \quad \alpha \in \Phi^+.
\]
So \( H^1_{\alpha; k} \) (respectively \( H^1_{\alpha; -k} \)) is bounded by two parallel hyperplanes \( H_{\alpha; k} \) and \( H_{\alpha; k+1} \) (respectively \( H_{\alpha; -k} \) and \( H_{\alpha; -k+1} \)). We define \( H_{\alpha; h} \) by \( H_{\alpha; h+1} \) for any integer \( h \) and any positive root \( \alpha \in \Phi^+ \). Then \( H^1_{\alpha; k} \) is also bounded by \( H_{\alpha; k} \) and \( H_{\alpha; k+1} \). So we can say that for any integer \( k \) and any root \( \alpha \in \Phi^+ \), \( H^1_{\alpha; k} \) is bounded by \( H_{\alpha; k} \) and \( H_{\alpha; k+1} \).

For any \( w \in W_a \) and integer \( t, 0 \leq t \leq l \), we denote \( k_t = k(w, (\alpha_t)_w) \) in the remainder of this section. Let \( H_t(w) \) be the hyperplane of \( E \) supporting the \( s_t \)-facet.
of the alcove $A_w$. Then we have $H_t(w) = (H_t(1))w = (H_{a_t;0})w$. So $H_t(w)$ has the form $H_{(a_t)\,\omega;\,k}$ for some $k \in \{k_t, k_t \pm 1\}$.

Now we wish to decide $H_t(w)$.

By Corollary 6.2 and Lemma 6.1, we see that $s_t \in \mathcal{L}(w)$ if and only if $A_w$ and $A_1$ are on different sides of $H_t(w)$, where $\mathcal{L}(x) = \{s \in \Delta \,|\, sx \prec x\}$ for any $x \in W_a$. On the other hand, by Proposition 4.3 (iii), we see that $s_t \in \mathcal{L}(w)$ if and only if $k_t > 0$, or equivalently, $s_t \notin \mathcal{L}(w)$ if and only if $k_t \leq 0$.

First assume that $s_t \in \mathcal{L}(w)$. Then $H_{1(a_t)}\omega;\,k_t$ and $A_1$ are on different sides of $H_t(w)$.

So $H_t(w) = H_{1(a_t)}\omega;\,k_t$ by the fact that $k_t > 0$.

Next assume that $s_t \notin \mathcal{L}(w)$. Then $H_{1(a_t)}\omega;\,k_t$ and $A_1$ are on the same side of $H_t(w)$.

In that case we have $k_t \leq 0$. If $k_t < 0$ then $H_t(w) = H_{(a_t)\,\omega;\,k_t}$. If $k_t = 0$ then $H_t(w) = H_{(a_t)\,\omega;\,1}$.

So we can summarize the above results as follows.

**Proposition 6.3.** For any $w \in W_a$ and integer $t$, $0 \leq t \leq l$, let $H_t(w)$ be the hyperplane of $E$ supporting the $s_t$-facet of the alcove $A_w$. Let $k_t = k(w, (a_t)\,\omega)$. Then

$$ H_t(w) = \begin{cases} H_{(a_t)\,\omega;\,k_t} & \text{if } k_t \neq 0, \\ H_{(a_t)\,\omega;\,1} & \text{if } k_t = 0, \end{cases} $$

and so $A_w = \bigcap_{0 \leq t \leq l} H_{(a_t)\,\omega;\,k_t}$.

**References**


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